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MAXIMAL MONOTONE OPERATORS IN THE ONE DIMENSIONAL CASE

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ABSTRACT. Our basic concern in this paper is to investigate some geometric properties of the graph of a maximal monotone operator in the one dimensional case. Using a well-known theorem of Minty, we answer S. Simons' questions affirmatively in the one dimensional case. Further developments of these results are also treated. In addition, we provide a new proof of Rockafellar's characterization of maximal monotone operators on R: every maximal monotone operator from R to 2^R is the subdifferential of a proper convex lower semicontinuous function.

1. Introduction and Preliminaries

The maximal monotonicity of the subdifferential of a proper convex lower semicontinuous function due to Rockafellar [6] is one of the fundamental theorems in convex analysis. Many authors [1, 11] have provided independent proofs for this theorem. Recently, Simons [8-10] has generalized this result in several directions and has given very elementary and simple proofs which do not depend on any difficult results. One of the generalizations [9, Theorem 6.1] is as follows; Let E be a real Banach space and E^* its dual space. Let $\psi : E \to R \cup \{\infty\}$ be a proper convex lower semicontinuous function and $\partial\psi$ be the subdifferential of ψ. If Q is a nonempty weakly compact convex subset of E, $q^* \in E^*$ and, for all $(z, z^*) \in Gr(\partial\psi)$, there exists $q \in Q$ such that $\langle z - q,\ z^* - q^* \rangle \geq 0$ then

$$(Q \times \{q^*\}) \cap Gr(\partial\psi) \neq \emptyset.$$

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At the same time, he asked whether the corresponding result is true for general maximal monotone operators: let M be a maximal monotone opreator from E to 2^{E^*}, and $Gr(M)$ be the graph of M in $E \times E^*$. Let Q be a nonempty weakly compact convex subset of E, $q^* \in E^*$ such that $(Q \times \{q^*\}) \cap Gr(M) = \emptyset$. Does there necessarily exist $(z, z^*) \in Gr(M)$ such that for all $q \in Q$, $\langle z - q,\ z^* - q^* \rangle < 0$.

Our basic concern in this paper is to investigate some geometric properties of the graph of a maximal monotone operator in the one dimensional case. In section 2, we answer Simons' question affirmatively in the one dimensional case $E = R$. Moreover, we verify that this result remains true for the case where the point q^* is replaced by a nonempty compact convex set Q^*. This also answers in the one dimensional case another question posed by Simons [9] for general Banach space. After this, we give a characterization of maximal monotone operators in R: every maximal monotone operator from R to 2^R is cyclically monotone, hence is the subdifferential of a proper convex lower semicontinuous function by means of Rockafellar [6]. Actually, this characterization was first proved by Rockafellar [7, Theorem 24.3, p.232]. His proof was based upon some properties of the one-sided directional derivative of a proper convex lower semicontinuous function and a characterization of the graph of a maximal monotone operator as a complete non-decreasing curve [7, Sections 23 and 24]. However, we will provide a simpler approach to that characterization. Section 3 deals with a refinement of the two theorems in the previous section.

Let $M : R \to 2^R$ be maximal monotone and $D(M)$ be the effective domain of M. If $D(M)$ is a singleton $\{a\}$, then $M(x) = (-\infty, \infty)$ if $x = a$, $M(x) = \emptyset$ otherwise. Recall that Minty's theorem [3] says: let H be a finite dimensional Hilbert space, with real or complex scalars, and $M : H \to 2^H$ be maximal monotone. Then $D(M)$ is *almost convex* in the sense that $D(M)$ contains the relative interior of the convex hull co$D(M)$ of $D(M)$. So, in the case where $D(M)$ contains at least two points, using Minty's result [3], we know that $D(M)$ is a convex subset of R. Indeed, let $a,\ b \in D(M)$. Clearly, $(a,\ b)$ is contained in the relative interior of co$D(M)$. Since $D(M)$ is almost convex, the open interval (a, b) is contained in $D(M)$, so is the closed interval [a,b] as desired. Hence $D(M)$ is of one of the forms $[a, b]$ (a, b), $(a, b]$, $[a, b)$. Moreover, $Gr(M)$ is path-connected by virtue of another theorem of

Minty [4]. Since M is monotone, we have

$$\sup M(x_1) \leq \inf M(x_2) \text{ whenever } x_1 < x_2,\ x_1,\ x_2 \in D(M). \tag{1.1}$$

It is easily seen that M can be represented by

$$M(x) = \begin{cases} [\inf M(x),\ \sup M(x)] & \text{if } a < x < b \\ (-\infty,\ \sup M(a)] & \text{if } x = a \\ [\inf M(b),\ \infty) & \text{if } x = b \\ \emptyset & \text{otherwise.} \end{cases}$$

In the above representation, we follow the usual convention

$$\sup \emptyset = -\infty, \qquad \inf \emptyset = \infty.$$

Recall that a set-valued map $M : E \to 2^{E^*}$ is said to be *cyclically monotone* provided

$$\sum_{k=1}^{n} \langle x_k - x_{k-1}, x_k^* \rangle \geq 0$$

whenever $n \geq 2$ and $x_0, x_1, \cdots, x_n \in E$, $x_n = x_0$, and $x_k^* \in M(x_k)$, $k = 1, 2, \cdots, n$.

2. Applications of a theorem of Minty

THEOREM 1. *Let $M : R \to 2^R$ be maximal monotone and Q be a nonempty compact convex subset of R, $q^* \in R$ such that $(Q \times \{q^*\}) \cap Gr(M) = \emptyset$. Then there exists $(z, z^*) \in Gr(M)$ such that*

$$\langle z - q,\ z^* - q^* \rangle < 0 \text{ for all } q \in Q. \tag{2.1}$$

Proof. Since $Q = [a, b]$ for some $a, b \in R$, $a \leq b$, and $q^* \notin M(Q)$, there exist (z_1, z_1^*) and $(z_2, z_2^*) \in Gr(M)$ such that

$$\langle z_1 - a,\ z_1^* - q^* \rangle < 0 \text{ and } \langle z_2 - b,\ z_2^* - q^* \rangle < 0. \tag{2.2}$$

We shall now prove that

$$\text{(2.3)} \qquad \text{either } z_1 < a,\ z_1^* > q^* \quad \text{or} \quad z_2 > b,\ z_2^* < q^*.$$

Suppose the contrary. By (2.2), we have $z_1 > a$, $z_1^* < q^*$ and $z_2 < b$, $z_2^* > q^*$, so $z_1^* < q^* < z_2^*$. By (1.1), we have $a < z_1 \leq z_2 < b$. Since M is maximal monotone, the range $R(M)$ is convex by Minty's theorem [3], hence it must contain $[z_1^*, z_2^*]$. Thus there exists $z_0 \in R$ such that $q^* \in M(z_0)$ and $a < z_1 \leq z_0 \leq z_2 < b$ by (1.1), thus $z_0 \in Q$. This contradicts the assumption. Therefore (2.3) is true. Moreover, each case of the two possibilities in (2.3) clearly implies (2.1). This completes the proof. □

In a similar way, we get the following generalization of Theorem 1.

THEOREM 2. *Let $M : R \to 2^R$ be maximal monotone and $Q_1 = [a, b]$, $Q_2 = [c, d]$ be two closed intervals such that*

$$(Q_1 \times Q_2) \cap Gr(M) = \emptyset.$$

Then there exists $(z, z^*) \in Gr(M)$ such that

$$\langle z - q,\ z^* - q^* \rangle < 0 \quad \text{for all} \quad (q, q^*) \in Q_1 \times Q_2.$$

Proof. Since $(Q_1 \times \{c\}) \cap Gr(M) = \emptyset$, it follows from the proof of Theorem 1 that there exists $(z_1, z_1^*) \in Gr(M)$ such that either $z_1 < a$, $z_1^* > c$ or $z_1 > b$, $z_1^* < c$. Similarly, because $(Q_1 \times \{d\}) \cap Gr(M) = \emptyset$, we see that there exists $(z_2, z_2^*) \in Gr(M)$ such that either $z_2 < a$, $z_2^* > d$ or $z_2 > b$, $z_2^* < d$. We claim that at least one of the two possibilities

$$\text{(2.4)} \qquad z_1 > b,\ z_1^* < c \quad \text{or} \quad z_2 < a,\ z_2^* > d$$

must be true. If not, $z_1 < a$, $z_1^* > c$ and $z_2 > b$, $z_2^* < d$. Since $z_1 < a \leq b < z_2$, we have

$$\text{(2.5)} \qquad c < z_1^* \leq z_2^* < d$$

by (1.1). Applying Minty's theorem [3] again to the domain $D(M)$, we know that $D(M)$ contains $[z_1, z_2]$, hence contains $[a, b] = Q_1$. Moreover,

$$\text{(2.6)} \qquad M(q) \subset [z_1^*, z_2^*] \subset [c, d] = Q_2 \quad \text{for all} \quad q \in Q_1.$$

(The first inclusion comes from (1.1) and the second one from (2.5).) But the relation (2.6) obviously contradicts the assumption $(Q_1 \times Q_2) \cap Gr(M) = \emptyset$. Thus (2.4) is true. Each case of the two possibilities in (2.4) clearly implies the theorem □

REMARK. As mentioned in the introduction, Theorem 2 answers in the one dimensional case another question posed by Simons [9, Remarks and Problems, p.1387] for general Banach space.

THEOREM 3. *Let $M : R \to 2^R$ be maximal monotone. Then M is cyclically monotone, hence is the subdifferential of a proper convex lower semicontinuous function.*

Proof. If $D(M)$ is a singleton $\{a\}$, M is represented by

$$M(x) = \begin{cases} (-\infty, \infty) & \text{if } x = a \\ \emptyset & \text{otherwise.} \end{cases}$$

In this case, M is clearly cyclically monotone, and the subdifferential of the proper convex lower semicontinuous function $\psi : R \to R \cup \{\infty\}$ defined by

$$\psi(x) = \begin{cases} k & \text{if } x = a \\ \infty & \text{otherwise,} \end{cases}$$

where k is a real number. Now we assume that $D(M)$ contains at least two points. Let $n \geq 2$ and $x_0,\ x_1, \cdots,\ x_n \in R,\ x_n = x_0$, and $x_k^* \in M(x_k),\ k = 1,\ 2, \cdots,\ n$. Since the x_k s are not necessarily distinct, it may be true that for some $0 \leq i < j \leq n,\ x_i = x_j$ and $x_{i+1}, \cdots,\ x_j$ are distinct. In this case, $x_i,\ x_{i+1}, \cdots,\ x_j$ is called *a cycle.* We put $S_0 = \sum_{k=1}^{n} \langle x_k - x_{k-1}, x_k^* \rangle$. Then $S_0 = S_1 + R_1$ where $S_1 = \sum_{k=i+1}^{j} \langle x_k - x_{k-1}, x_k^* \rangle$ and R_1 is the remainder part. Observe that R_1 is the same type of summation as S_0 except R_1 may be a cycle. Anyway, by recursion, we have

$$S_0 = \sum_{k=1}^{m} S_k$$

where each S_k is the corresponding sum of a cycle. So we may assume without loss of generality that $x_0,\ x_1, \cdots,\ x_n$ is a cycle. Then we can select a monotone function $f : D(M) \to R$ such that $f(x) \in M(x)$ for all $x \in D(M)$, and $f(x_k) = x_k^*$, $k = 1, 2, \cdots, n$. Since $D(M)$ is an interval, f is integrable on $D(M)$. For each k, we have

$$f(x_k)(x_k - x_{k-1}) \geq \int_{x_{k-1}}^{x_k} f(t)dt$$

since f is monotone. Hence

$$\begin{aligned}\sum_{k=1}^{n} \langle x_k - x_{k-1}, x_k^* \rangle &= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) \\ &\geq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)dt \\ &= \int_{x_0}^{x_0} f(t)dt \\ &= 0.\end{aligned}$$

The second part of the theorem directly follows from Rockafellar's result [6, Theorem B]. □

As a direct consequence of Theorem 3, we get the following.

COROLLARY. *Let $M : R \to 2^R$ be monotone. Then M is cyclically monotone.*

Proof. By Zorn's lemma, M can be extended to a maximal monotone operator $\overline{M}$. It follows from Theorem 3 that $\overline{M}$ is cyclically monotone, hence M is cyclically monotone.

REMARKS. 1. Theorem 3 implies that there is only one kind of maximal monotone operators, namely, the subdifferential of a proper convex lower semicontinuous function in R. However, Theorem 3 does not hold even in R^2. As shown in Phelps [5, Examples 2.21, p.26], the linear map in R^2 defined by $T(x_1, x_2) = (x_2, -x_1)$ is maximal monotone, but not cyclically monotone.
2. Due to Theorem 3, we observe that Simons' result [9, Theorem 6.1] already contains the answer to his own question [9, Remarks and Problems, p.1387] in the one dimensional case. To get the result in general Banach spaces, Simons used the Ekeland variational principle [2] and Mazur-Orlicz version of the Hahn-Banach theorem.

3. Further developments

We begin with the following result which holds for general Banach spaces.

PROPOSITION 4. *Let f and g be proper convex lower semicontinuous functions on a Banach space E such that $D(\partial g) = E$. Let Q be a nonempty weakly compact convex subset of E. If for each $(z, z^*) \in Gr(\partial f)$, there exists $(q, q^*) \in Gr(-\partial g)$ such that*

$$q \in Q \quad \text{and} \quad \langle z - q, z^* - q^* \rangle \geq 0.$$

Then

$$(Q \times E^*) \cap Gr(\partial f) \cap Gr(-\partial g) \neq \emptyset.$$

Proof. Let $(x, x^*) \in Gr(\partial f + \partial g)$. Then $x^* = u^* + v^*$, where $(x, u^*) \in Gr(\partial f)$ and $(x, v^*) \in Gr(\partial g)$. By assumption there exists $(q, q^*) \in Gr(-\partial g)$ such that

$$q \in Q \quad \text{and} \quad \langle x - q, u^* - q^* \rangle \geq 0. \tag{4.1}$$

Since (x, v^*), $(q, -q^*) \in Gr(\partial g)$, we have

$$\langle x - q, v^* + q^* \rangle = \langle x - q, v^* - (-q^*) \rangle \geq 0.$$

Combining with (4.1), we obtain

$$\langle x - q, x^* \rangle = \langle x - q, u^* + v^* \rangle = \langle x - q, u^* - q^* \rangle + \langle x - q, v^* + q^* \rangle \geq 0.$$

So we have proved: for $(x, x^*) \in Gr(\partial f + \partial g)$, there exists $q \in Q$ such that $\langle x - q, x^* \rangle \geq 0$.
From the sum formula [5, Theorem 3.16, p.47], $\partial(f + g) = \partial f + \partial g$. Now apply Simons' theorem [9, Theorem 6.1] to the subdifferential $\partial(f + g) = \partial f + \partial g$ (with $q^* = 0$): there exists $q \in Q$ such that $(q, 0) \in Gr(\partial(f + g))$. This gives the desired result. □

Actually, we can recover Simons' theorem [9, Theorem 6.1] from Proposition 4, which implies that two results are equivalent. For the sake of completeness, we state and prove Simons' theorem.

THEOREM. (Simons [9, Theorem 6.1]) *Let $f : E \to R \cup \{\infty\}$ be a proper convex lower semicontinuous function, and ∂f be the subdifferential of f. If Q is a nonempty weakly compact convex subset of E, $q^* \in E^*$ and, for all $(z, z^*) \in Gr(\partial f)$, there exists $q \in Q$ such that $\langle z - q,\ z^* - q^* \rangle \geq 0$ then*

$$(Q \times \{q^*\}) \cap Gr(\partial f) \neq \emptyset.$$

Proof. Take the function $g \equiv -q^*$ in Proposition 4. Then it follows from the assumption that for each $(z, z^*) \in Gr(\partial f)$, there exists $(q, q^*) \in Gr(-\partial g)$ such that

$$q \in Q \quad \text{and} \quad \langle z - q, z^* - q^* \rangle \geq 0$$

because $-\partial g(x) = q^*$ for every $x \in E$. By Proposition 4

$$(Q \times \{q^*\}) \cap Gr(\partial f) = (Q \times E^*) \cap Gr(\partial f) \cap Gr(-\partial g) \neq \emptyset,$$

which completes the proof. □

In one dimensional case, we can rewrite Proposition 4 as follows.

THEOREM 5. *Let M and $N : R \to 2^R$ be maximal monotone operators such that $D(N) = R$. Let Q be a nonempty compact convex subset of R. If for each $(z, z^*) \in Gr(M)$, there exists $(q, q^*) \in Gr(-N)$ such that*

$$q \in Q \quad \text{and} \quad \langle z - q, z^* - q^* \rangle \geq 0.$$

Then

$$(Q \times R) \cap Gr(M) \cap Gr(-N) \neq \emptyset.$$

Proof. Directly from Theorem 3. □

As a special case of Theorem 5, we have the following.

COROLLARY. *Let $M : R \to 2^R$ be maximal monotone, and L be a line segment joining two points (a, b) and (c, d) in R^2 whose slope $(d-b)/(c-a)$ is less than 0. Assume that $L \cap Gr(M) = \emptyset$. Then there exists $(z, z^*) \in Gr(M)$ such that*

$$\langle z - q, z^* - q^* \rangle < 0 \quad \textit{for all} \quad (q, q^*) \in L.$$

Proof. Taking $-N$ the straight line through two points (a,b) and (c,d) in Theorem 5, we get the result. □

LEMMA 6. *Let L be a line segment joining two points (a,b) and (c,d) in R^2 whose slope $(d-b)/(c-a)$ is real and greater than 0. (We may assume $a < c$.) Then there exists a maximal monotone operator $M : R \to 2^R$ such that $L \cap Gr(M) = \emptyset$ and any point $(z,z^*) \in Gr(M)$ does not satisfy the inequality*

$$\langle z-q,\ z^*-q^*\rangle < 0 \text{ for all } (q,q^*) \in L.$$

Proof. Define $M : R \to 2^R$ by

$$M(x) = \begin{cases} \frac{d-b}{c-a}(x-a)+d & \text{if } 2a-c < x < a \\ (-\infty,\ b] & \text{if } x = 2a-c \\ [d,\ \infty) & \text{if } x = a \\ \emptyset & \text{otherwise.} \end{cases}$$

Then M is a desired maximal monotone operator. □

REMARK. We can take another $M : R \to 2^R$ as follows;

$$M(x) = \begin{cases} \frac{d-b}{c-a}(x-c)+b & \text{if } c < x < 2c-a \\ (-\infty,\ b] & \text{if } x = c \\ [d,\ \infty) & \text{if } x = 2c-a \\ \emptyset & \text{otherwise.} \end{cases}$$

Combining Theorem 1, the Corollary of Theorem 5 and Lemma 6, we have the following.

THEOREM 7. *Let L be a line segment joining two points (a,b) and (c,d) in R^2. The slope of L is ≤ 0 if and only if for each maximal monotone operator $M : R \to 2^R$ with $L \cap Gr(M) = \emptyset$, there exists $(z,z^*) \in Gr(M)$ such that*

$$\langle z-q,\ z^*-q^*\rangle < 0 \text{ for all } (q,q^*) \in L.$$

As a direct consequence of Lemma 6 and the remark following, we get the following.

THEOREM 8. *Let P be the convex hull of a finite subset $\{x_1, x_2, \cdots, x_n\}$ in R^2. (We may assume that each x_i is an extreme point of P and the set of line segments $\{\overline{x_i x_{i+1}}\}_{i=1}^n$ consist of the edges of P. Here, $x_{n+1} = x_1$.) If we have at least one $1 \leq i \leq n$ such that the slope of $\overline{x_i x_{i+1}}$ is real and greater than 0, then there exists a maximal monotone operator $M : R \to 2^R$ such that $P \cap Gr(M) = \emptyset$ and for any $(z, z^*) \in Gr(M)$,*

$$\langle z - q,\ z^* - q^* \rangle < 0 \text{ for all } (q, q^*) \in P$$

is not true.

REMARK. Theorem 8 tells us that the conclusion "there exists $(z, z^*) \in Gr(M)$ such that

$$\langle z - q,\ z^* - q^* \rangle < 0 \text{ for all } (q, q^*) \in P."$$

does depend on the shape of the given compact convex set P. Hence it is not unreasonable to guess that Simons' question [9] for general maximal monotone operators on a Banach space may have a negative answer.

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