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FOCK REPRESENTATIONS OF THE HEISENBERG GROUP $H_{\mathbb{R}}^{(g,h)}$

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ABSTRACT. In this paper, we introduce the Fock representation $U^{F,\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ associated with a positive definite symmetric half-integral matrix $\mathcal{M}$ of degree h and prove that $U^{F,\mathcal{M}}$ is unitarily equivalent to the Schrödinger representation of index $\mathcal{M}$.

1. Introduction

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} := \left\{ (\lambda,\mu,\kappa) \mid \lambda,\mu\in\mathbb{R}^{(h,g)},\ \kappa\in\mathbb{R}^{(h,h)},\ \kappa+\mu\,{}^t\lambda \text{ symmetric} \right\}$$

endowed with the following multiplication law

$$(\lambda,\mu,\kappa)\circ(\lambda',\mu',\kappa') := (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda'). \tag{1.1}$$

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded in the symplectic group $Sp(g+h,\mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)} \ni (\lambda,\mu,\kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g+h,\mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactification of Siegel moduli spaces. In

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fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g+h,\mathbb{R})$ associated with the rational boundary component F_g (cf. [4] p. 21).

The purpose of this article is to study the Fock representation of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ associated with a positive definite symmetric half-integral matrix of degree h. This paper is organized as follows. In section two, we review the Schrödinger representation $U(\sigma_c)$ of $H_{\mathbb{R}}^{(g,h)}$ associated with a real symmetric matrix c of degree h. In section three, we construct the *Fock representation* $U^{F,\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ associated with a positive definite symmetric half-integral matrix $\mathcal{M}$ of degree h and prove that $U^{F,\mathcal{M}}$ is unitarily equivalent to the Schrödinger representation $U(\sigma_{\mathcal{M}})$ of index $\mathcal{M}$. For more results on the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$, we refer to [5]-[11].

NOTATIONS. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\mathbb{C}_1^{\times}$ denotes the multiplicative group consisting of all complex numbers z with $|z|=1$. $Sp(g,\mathbb{R})$ denotes the symplectic group of degree g. The symbol ":=" means that the expression on the right is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A\in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. E_k denotes the identity matrix of degree k. For a positive integer n, $Sym\,(n,K)$ denotes the vector space consisting of all symmetric $n\times n$ matrices with entries in a field K.

$$\begin{aligned}
\mathbb{Z}_{\geq 0}^{(h,g)} &= \left\{ J=(J_{ka})\in\mathbb{Z}^{(h,g)} \mid J_{ka}\geq 0 \ \text{ for all } k,a \right\},\\
|J| &= \sum_{k,a} J_{k,a},\\
J\pm\epsilon_{kl} &= (J_{11},\cdots,J_{ka}\pm 1,\cdots,J_{hg}),\\
J! &= J_{11}!\cdots J_{ka}!\cdots J_{hg}!.
\end{aligned}$$

For $\xi=(\xi_{ka})\in\mathbb{R}^{(h,g)}$ or $\mathbb{C}^{(h,g)}$ and $J=(J_{ka})\in\mathbb{Z}_{\geq 0}^{(h,g)}$, we denote

$$\xi^J=\xi_{11}^{J_{11}}\,\xi_{12}^{J_{12}}\cdots\xi_{ka}^{J_{ka}}\cdots\xi_{hg}^{J_{hg}}.$$

2. Schrödinger representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ is given by

$$(\lambda, \mu, \kappa)^{-1} = (-\lambda, -\mu, -\kappa + \lambda\,{}^t\mu - \mu\,{}^t\lambda).$$

Now we put

$$(2.1) \qquad [\lambda, \mu, \kappa] := (0, \mu, \kappa) \circ (\lambda, 0, 0) = (\lambda, \mu, \kappa - \mu\,{}^t\lambda).$$

Then $H_{\mathbb{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

$$(2.2) \quad [\lambda, \mu, \kappa] \diamond [\lambda_0, \mu_0, \kappa_0] := [\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda\,{}^t\mu_0 + \mu_0\,{}^t\lambda].$$

The inverse of $[\lambda, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)}$ is given by

$$[\lambda, \mu, \kappa]^{-1} = [-\lambda, -\mu, -\kappa + \lambda\,{}^t\mu + \mu\,{}^t\lambda].$$

We set

$$(2.3) \qquad K := \left\{ [0, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \mu \in \mathbb{R}^{(h,g)},\ \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\}.$$

Then K is a commutative normal subgroup of $H_{\mathbb{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbb{R}^{(h,g)} \times Sym\,(h, \mathbb{R})$ via

$$(2.4) \quad < a, \hat{a} >:= e^{2\pi i \sigma(\hat{\mu}\,{}^t\mu + \hat{\kappa}\kappa)}, \quad a = [0, \mu, \kappa] \in K,\ \hat{a} = (\hat{\mu}, \hat{\kappa}) \in \hat{K}.$$

We put

$$(2.5) \qquad S := \left\{ [\lambda, 0, 0] \in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

Then S acts on K as follows:

$$(2.6) \qquad \alpha_\lambda([0, \mu, \kappa]) := [0, \mu, \kappa + \lambda\,{}^t\mu + \mu\,{}^t\lambda], \quad [\lambda, 0, 0] \in S.$$

It is easy to see that the Heisenberg group $\left(H_{\mathbb{R}}^{(g,h)}, \diamond\right)$ is isomorphic to the semidirect product $S \ltimes K$ of S and K whose multiplication is given by

$$(\lambda, a) \cdot (\lambda_0, a_0) := (\lambda + \lambda_0, a + \alpha_\lambda(a_0)), \quad \lambda, \lambda_0 \in S, \ a, a_0 \in K.$$

On the other hand, S acts on $\hat{K}$ by

$$\alpha_\lambda^*(\hat{a}) := (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda, 0, 0] \in S, \quad a = (\hat{\mu}, \hat{\kappa}) \in \hat{K}. \tag{2.7}$$

Then we have the relation $< \alpha_\lambda(a), \hat{a} >=< a, \alpha_\lambda^*(\hat{a}) >$ for all $a \in K$ and $\hat{a} \in \hat{K}$.

We have two types of S-orbits in $\hat{K}$.

Type I. Let $\hat{\kappa} \in Sym\,(h, \mathbb{R})$ with $\hat{\kappa} \neq 0$. The S-orbit of $\hat{a}(\hat{\kappa}) := (0, \hat{\kappa}) \in \hat{K}$ is given by

$$\hat{\mathcal{O}}_{\hat{\kappa}} := \left\{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \hat{K} \;\middle|\; \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}. \tag{2.8}$$

Type II. Let $\hat{y} \in \mathbb{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}) := (\hat{y}, 0)$ is given by

$$\hat{\mathcal{O}}_{\hat{y}} := \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}). \tag{2.9}$$

We have

$$\hat{K} = \left(\bigcup_{\hat{\kappa} \in Sym(h,\mathbb{R})} \hat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(h,g)}} \hat{\mathcal{O}}_{\hat{y}} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}. \tag{2.10}$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0, 0] \;\middle|\; \lambda \in \mathbb{R}^{(h,g)} \right\} = S \cong \mathbb{R}^{(h,g)}. \tag{2.11}$$

From now on, we set $G := H_{\mathbb{R}}^{(g,h)}$ for brevity. K is a closed, commutative normal subgroup of G. Since $(\lambda, \mu, \kappa) = (0, \mu, \kappa + \mu\, {}^t\lambda) \circ (\lambda, 0, 0)$

for $(\lambda, \mu, \kappa) \in G$, the homogeneous space $X := K\backslash G$ is identified with $\mathbb{R}^{(h,g)}$ via

$$Kg = K \circ (\lambda, 0, 0) \longmapsto \lambda, \quad g = (\lambda, \mu, \kappa) \in G.$$

We observe that G acts on X by

$$(Kg) \cdot g_0 := K(\lambda + \lambda_0, 0, 0) = \lambda + \lambda_0, \tag{2.12}$$

where $g = (\lambda, \mu, \kappa) \in G$ and $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$.

If $g = (\lambda, \mu, \kappa) \in G$, we have

$$k_g = (0, \mu, \kappa + \mu\, {}^t\lambda), \quad s_g = (\lambda, 0, 0) \tag{2.13}$$

in the Mackey decomposition of $g = k_g \circ s_g$ (cf. [3]). Thus if $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$, then we have

$$s_g \circ g_0 = (\lambda, 0, 0) \circ (\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu_0, \kappa_0 + \lambda\, {}^t\mu_0) \tag{2.14}$$

and so

$$k_{s_g \circ g_0} = (0, \mu_0, \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda). \tag{2.15}$$

For a real symmetric matrix $c = {}^tc \in \mathbb{R}^{(h,h)}$ with $c \neq 0$, we consider the one-dimensional unitary representation σ_c of K defined by

$$\sigma_c((0, \mu, \kappa)) := e^{2\pi i \sigma(c\kappa)} I, \quad (0, \mu, \kappa) \in K, \tag{2.16}$$

where I denotes the identity mapping. Then the induced representation $U(\sigma_c) := \mathrm{Ind}_K^G \sigma_c$ of G induced from σ_c is realized in the Hilbert space $\mathcal{H}_{\sigma_c} = L^2(X, d\dot{g}, \mathbb{C}) \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ as follows. If $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$ and $x = Kg \in X$ with $g = (\lambda, \mu, \kappa) \in G$, we have

$$\left(U_{g_0}(\sigma_c) f\right)(x) = \sigma_c\left(k_{s_g \circ g_0}\right)\left(f(x g_0)\right), \quad f \in \mathcal{H}_{\sigma_c}. \tag{2.17}$$

It follows from (2.15) that

$$\left(U_{g_0}(\sigma_c) f\right)(\lambda) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}} f(\lambda + \lambda_0). \tag{2.18}$$

Here we identified $x = Kg$ (resp. $xg_0 = Kgg_0$) with λ (resp. $\lambda + \lambda_0$). The induced representation $U(\sigma_c)$ is called the *Schrödinger representation* of G associated with σ_c. Thus $U(\sigma_c)$ is a monomial representation.

Now we denote by $\mathcal{H}^{\sigma_c}$ the Hilbert space consisting of all functions $\phi : G \longrightarrow \mathbb{C}$ which satisfy the following conditions:

(1) $\phi(g)$ is measurable with respect to dg.

(2) $\phi((0,\mu,\kappa) \circ g)) = e^{2\pi i\sigma(c\kappa)}\phi(g)$ for all $g \in G$.

(3) $\| \phi \|^2 := \int_X |\phi(g)|^2 \, d\dot{g} < \infty, \quad \dot{g} = Kg,$

where dg (resp. $d\dot{g}$) is a G-invariant measure on G (resp. $X = K\backslash G$). The inner product $(\ ,\ )$ on $\mathcal{H}^{\sigma_c}$ is given by

$$(\phi_1, \phi_2) := \int_G \phi_1(g)\, \overline{\phi_2(g)}\, dg, \quad \phi_1,\ \phi_2 \in \mathcal{H}^{\sigma_c}.$$

We observe that the mapping $\Phi_c : \mathcal{H}_{\sigma_c} \longrightarrow \mathcal{H}^{\sigma_c}$ defined by

(2.19)
$$(\Phi_c(f))(g) := e^{2\pi i\sigma\{c(\kappa + \mu\, {}^t\lambda)\}} f(\lambda), \quad f \in \mathcal{H}_{\sigma_c},\ g = (\lambda,\mu,\kappa) \in G$$

is an isomorphism of Hilbert spaces. The inverse $\Psi_c : \mathcal{H}^{\sigma_c} \longrightarrow \mathcal{H}_{\sigma_c}$ of Φ_c is given by

$$(\Psi_c(\phi))(\lambda) := \phi((\lambda,0,0)), \quad \phi \in \mathcal{H}^{\sigma_c},\ \lambda \in \mathbb{R}^{(h,g)}. \tag{2.20}$$

The Schrödinger representation $U(\sigma_c)$ of G on $\mathcal{H}^{\sigma_c}$ is given by

$$(U_{g_0}(\sigma_c)\phi)(g) = e^{2\pi i\sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 - \lambda_0\, {}^t\mu)\}}\, \phi((\lambda_0,0,0) \circ g), \tag{2.21}$$

where $g_0 = (\lambda_0, \mu_0, \kappa_0)$, $g = (\lambda,\mu,\kappa) \in G$ and $\phi \in \mathcal{H}^{\sigma_c}$. (2.21) can be expressed as follows.

(2.22)
$$(U_{g_0}(\sigma_c)\phi)(g) = e^{2\pi i\sigma\{c(\kappa_0 + \kappa + \mu_0\, {}^t\lambda_0 + \mu\, {}^t\lambda + 2\lambda\, {}^t\mu_0)\}}\, \phi((\lambda_0 + \lambda, 0, 0)).$$

THEOREM 2.1. *Let c be a positive symmetric half-integral matrix of degree h. Then the Schrödinger representation $U(\sigma_c)$ of G is irreducible.*

Proof. The proof can be found in [5], Theorem 3. □

3. Fock representations

We consider the vector space $V^{(h,g)} := \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$. We put

$$P_{ka} := (E_{ka}, 0), \quad Q_{lb} := (0, E_{lb}), \tag{3.1}$$

where $1 \le k, l \le h$ and $1 \le a, b \le g$. Then the set $\{P_{ka},\, Q_{ka}\}$ forms a basis for $V^{(h,g)}$. We define the alternating bilinear form $\mathbf{A} : V^{(h,g)} \times V^{(h,g)} \longrightarrow \mathbb{R}$ by

$$\mathbf{A}\left((\lambda_0, \mu_0), (\lambda, \mu)\right) := \sigma(\lambda_0\, {}^t\mu - \mu_0\, {}^t\lambda), \quad (\lambda_0, \mu_0),\, (\lambda, \mu) \in V^{(h,g)}. \tag{3.2}$$

Then we have

$$\mathbf{A}(P_{ka}, P_{lb}) = \mathbf{A}(Q_{ka}, Q_{lb}) = 0, \quad \mathbf{A}(P_{ka}, Q_{lb}) = \delta_{ab}\, \delta_{kl}, \tag{3.3}$$

where $1 \le k, l \le h$ and $1 \le a, b \le g$. Any element $v \in V^{(h,g)}$ can be written uniquely as

$$v = \sum_{k,a} x_{ka} P_{ka} + \sum_{l,b} y_{lb} Q_{lb}, \quad x_{ka},\, y_{lb} \in \mathbb{R}. \tag{3.4}$$

From now on, for brevity, we write $V := V^{(h,g)}$ and $v = xP + yQ$ instead of (3.4). Then it is easy to see that the endomorphism $J : V \longrightarrow V$ defined by

$$J(xP + yQ) := -yP + xQ, \quad xP + yQ \in V \tag{3.5}$$

is a complex structure on V which is compatible with the alternating bilinear form $\mathbf{A}$. This means that J is an endomorphism of V satisfying the following conditions:

(J1) $J^2 = -I$ on V.

(J2) $\mathbf{A}(Jv_0, Jv) = \mathbf{A}(v_0, v)$ for all $v_0, v \in V$.

(J3) $\mathbf{A}(v, Jv) > 0$ for all $v \in V$ with $v \neq 0$.

Now we let $V_{\mathbb{C}} = V + iV$ be the complexification of V, where $i = \sqrt{-1}$. For an element $w = v_1 + iv_2 \in V_{\mathbb{C}}$ with $v_1, v_2 \in V$, we put

$$\bar{w} := v_1 - iv_2. \tag{3.6}$$

Let $\mathbf{A}_{\mathbb{C}}$ be the complex bilinear form on $V_{\mathbb{C}}$ extending $\mathbf{A}$ and let $J_{\mathbb{C}}$ be the complex linear map of $V_{\mathbb{C}}$ extending J. Since $J_{\mathbb{C}}^2 = -I$, $J_{\mathbb{C}}$ has the only eigenvalues $\pm i$. We denote by V^+ (resp. V^-) the eigenspace of $V_{\mathbb{C}}$ corresponding to the eigenvalues i (resp. $-i$). Thus $V_{\mathbb{C}} = V^+ + V^-$. Since

$$J_{\mathbb{C}}(P_{ka} \pm iQ_{ka}) = \mp i(P_{ka} \pm iQ_{ka}),$$

we have

$$V^+ = \sum_{k,a} \mathbb{C}\,(P_{ka} - iQ_{ka}), \quad V^- = \sum_{k,a} \mathbb{C}\,(P_{ka} + iQ_{ka}). \tag{3.7}$$

Let

$$V_* := \sum_{k,a} \mathbb{C}\,P_{ka}, \quad 1 \le k \le h, \ \ 1 \le a \le g \tag{3.8}$$

be the subspace of $V_{\mathbb{C}}$ as a $\mathbb{C}$-vector space. It is easy to see that V_* is isomorphic to V as $\mathbb{R}$-vector spaces via the isomorphism $T : V \longrightarrow V_*$ defined by

$$T(P_{ka}) := P_{ka}, \quad T(Q_{lb}) := iP_{lb}. \tag{3.9}$$

We define the complex linear map $J_* : V_* \longrightarrow V_*$ by $J_*(P_{ka}) = iP_{ka}$ for $1 \le k \le h$, $1 \le a \le g$. Then J_* is compatible with J, that is, $T \circ J = J_* \circ T$. It is easily seen that there exists a unique hermitian form $\mathbf{H}$ on V_* with $\operatorname{Im} \mathbf{H} = \mathbf{A}$. Indeed, $\mathbf{H}$ is given by

$$\mathbf{H}(v, w) = \mathbf{A}(v, J_* w) + i\mathbf{A}(v, w), \quad v, w \in V_*. \tag{3.10}$$

For $v = \sum_{k,a} z_{ka} P_{ka} \in V_*$ with $z_{ka} = x_{ka} + iy_{ka}$ $(x_{ka}, y_{ka} \in \mathbb{R})$, for brevity we write $v = zP$. For two elements $v = zP$ and $v' = z'P$ in V_*, $\mathbf{H}(v, v') = \sum_{k,a} \overline{z_{ka}}\, z'_{ka}$.
We observe that

$$V_{\mathbb{C}} = \sum_{k,a} \mathbb{C}\,P_{ka} + \sum_{l,b} \mathbb{C}\,Q_{lb} = V^+ + V^- \supset V^{\pm}.$$

For $w = z^0 P + z^1 Q \in V_{\mathbb{C}}$, we put

$$w = w^+ + w^-, \quad w^+ := z^+(P - iQ), \quad w^- := z^-(P + iQ).$$

The relations among z^0, z^1, z^+, z^- are given by

$$(3.11) \qquad z^{\pm} = \frac{1}{2}(z^0 \pm iz^1), \quad z^0 = z^+ + z^-, \quad z^1 = i(z^- - z^+).$$

Precisely, (3.11) implies that

$$z^{\pm}_{ka} = \frac{1}{2}(z^0_{ka} \pm iz^1_{ka}), \quad z^0_{ka} = z^+_{ka} + z^-_{ka}, \quad z^1_{ka} = i(z^-_{ka} - z^+_{ka}),$$

where $1 \leq k \leq h$ and $1 \leq a \leq g$. It is easy to see that

$$(3.12) \quad \mathbf{A}_{\mathbb{C}}(w^-, w^+) = -2i \sum_{k,a} z^-_{ka} z^+_{ka} = -\frac{i}{2} \sum_{k,a} \left\{ (z^0_{ka})^2 + (z^1_{ka})^2 \right\}.$$

Let

$$G_{\mathbb{C}} := \left\{ (z^0, z^1, a) \;\middle|\; z^0, z^1 \in \mathbb{C}^{(h,g)}, \ a \in \mathbb{C}^{(h,h)}, \ a + z^1\, {}^t z^0 \text{ symmetric} \right\}$$

be the complexification of the real Heisenberg group $G := H_{\mathbb{R}}^{(h,g)}$. Analogously in the real case, the multiplication on $G_{\mathbb{C}}$ is given by (1.1). If $w = z^0 P + z^1 Q := \sum_{k,a} z^0_{ka} P_{ka} + \sum_{l,b} z^1_{lb} Q_{lb}$, we identify z^0, z^1 with the $h \times g$ matrices respectively:

$$z^0 := \begin{pmatrix} z^0_{11} & z^0_{12} & \cdots & z^0_{1g} \\ z^0_{21} & z^0_{22} & \cdots & z^0_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ z^0_{h1} & z^0_{h2} & \cdots & z^0_{hg} \end{pmatrix}, \quad z^1 := \begin{pmatrix} z^1_{11} & z^1_{12} & \cdots & z^1_{1g} \\ z^1_{21} & z^1_{22} & \cdots & z^1_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ z^1_{h1} & z^1_{h2} & \cdots & z^1_{hg} \end{pmatrix}.$$

That is, we identify $w = z^0 P + z^1 Q \in V_{\mathbb{C}}$ with $(z^0. z^1) \in \mathbb{C}^{(h,g)} \times \mathbb{C}^{(h,g)}$. If $w = z^0 P + z^1 Q$, $\hat{w} = \hat{z^0} P + \hat{z^1} Q \in V_{\mathbb{C}}$, then

$$(3.13) \quad (w, a) \circ (\hat{w}, \hat{a}) = (w + \hat{w}, a + \hat{a} + z^0\, {}^t\hat{z}^1 - z^1\, {}^t\hat{z}^0), \quad a, \hat{a} \in \mathbb{C}^{(h,h)}.$$

From now on, for brevity we put

$$(3.14) \qquad R^+ := P - iQ, \qquad R^- := P + iQ.$$

If $w = z^+R^+ + z^-R^-,\ \hat{w} = \hat{z}^+R^+ + \hat{z}^-R^- \in V_{\mathbb{C}}$, by an easy computation, we have

$$(3.15)\qquad (w,a)\circ(\hat{w},\hat{a}) = (\tilde{w}, a+\hat{a}+2i(z^+\,{}^t\hat{z}^- - z^-\,{}^t\hat{z}^+))$$

with

$$\tilde{w} = (z^+ + \hat{z}^+)R^+ + (z^- + \hat{z}^-)R^-.$$

Here we identified z^+, z^- with $h\times g$ matrices

$$z^+ := \begin{pmatrix} z_{11}^+ & z_{12}^+ & \dots & z_{1g}^+ \\ z_{21}^+ & z_{22}^+ & \dots & z_{2g}^+ \\ \vdots & \vdots & \ddots & \vdots \\ z_{h1}^+ & z_{h2}^+ & \dots & z_{hg}^+ \end{pmatrix}, \quad z^- := \begin{pmatrix} z_{11}^- & z_{12}^- & \dots & z_{1g}^- \\ z_{21}^- & z_{22}^- & \dots & z_{2g}^- \\ \vdots & \vdots & \ddots & \vdots \\ z_{h1}^- & z_{h2}^- & \dots & z_{hg}^- \end{pmatrix}.$$

It is easy to see that

$$(3.16)\qquad P_{\mathbb{C}} := \left\{ (w^-,a)\in G_{\mathbb{C}} \mid w^-\in V^-,\quad a\in\mathbb{C}^{(h,h)} \right\}$$

is a commutative subgroup of $G_{\mathbb{C}}$ and

$$G\cap P_{\mathbb{C}} = \mathcal{Z},\quad G_{\mathbb{C}} = G\circ P_{\mathbb{C}},$$

where $\mathcal{Z} := \left\{ (0,0,\kappa)\in G \mid \kappa = {}^t\kappa\in\mathbb{R}^{(h,h)} \right\} \cong Sym\,(h,\mathbb{R})$ is the center of G. Moreover,

$$(3.17)\qquad P_{\mathbb{C}}\backslash G_{\mathbb{C}} \cong V^+ \cong \mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)} \cong \mathcal{Z}\backslash G.$$

For $c = {}^tc \in \mathrm{Sym}\,(h,\mathbb{R})$ with $c>0$, we let $\delta_c : P_{\mathbb{C}} \longrightarrow \mathbb{C}^\times$ be a quasi-character of $P_{\mathbb{C}}$ defined by

$$(3.18)\qquad \delta_c((w^-,a)) := e^{2\pi i\sigma(ca)},\quad (w^-,a)\in P_{\mathbb{C}}.$$

Let

$$U^{F,c} := \mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}\,\delta_c$$

be the representation of $G_{\mathbb{C}}$ induced from a quasi-character δ_c of $P_{\mathbb{C}}$. Then $U^{F,c}$ is realized in the Hilbert space $\mathcal{H}^{F,c}$ consisting of all holomorphic functions $\psi : G_{\mathbb{C}} \longrightarrow \mathbb{C}$ satisfying the following conditions:

(F1) $\psi((w^-,a)\circ g) = \delta_c((w^-,a))\psi(g) = e^{2\pi i\sigma(ca)}\,\psi(g)$ for all $(w^-,a) \in P_{\mathbb{C}}$ and $g \in G_{\mathbb{C}}$.

(F2) $\int_{\mathcal{Z}\backslash G} |\psi(\dot g)|^2\, d\dot g < \infty$.

The inner product $<\,,\,>_{F,c}$ on $\mathcal{H}^{F,c}$ is given by

$$< \psi_1, \psi_2 >_{F,c} := \int_{\mathcal{Z}\backslash G} \psi_1(\dot g)\,\overline{\psi_2(\dot g)}\, d\dot g, \quad \psi_1,\psi_2 \in \mathcal{H}^{F,c},\ \dot g = \mathcal{Z}g.$$

$U^{F,c}$ is realized by the right regular representation of $G_{\mathbb{C}}$ on $\mathcal{H}^{F,c}$:

$$\text{(3.19)} \qquad \left(U^{F,c}(g_0)\psi\right)(g) = \psi(gg_0), \quad \psi \in \mathcal{H}^{F,c},\ g_0, g \in G_{\mathbb{C}}.$$

Now we will show that $U^{F,c}$ is realized as a representation of G in the Fock space. The Fock space $\mathcal{H}_{F,c}$ is the Hilbert space consisting of all holomorphic functions $f : \mathbb{C}^{(h,g)} \cong V_* \longrightarrow \mathbb{C}$ satisfying the condition

$$\| f \|^2_{F,c} := \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, e^{-2\pi\sigma(cW\,{}^t\overline{W})}\, dW < \infty.$$

The inner product $(\,,\,)_{F,c}$ on $\mathcal{H}_{F,c}$ is given by

$$(f_1, f_2)_{F,c} := \int_{\mathbb{C}^{(h,g)}} f_1(W)\,\overline{f_2(W)}\, e^{-2\pi\sigma(cW\,{}^t\overline{W})}\, dW, \quad f_1, f_2 \in \mathcal{H}_{F,c}.$$

LEMMA 3.1. *The mapping $\Lambda : \mathcal{H}_{F,c} \longrightarrow \mathcal{H}^{F,c}$, $\Lambda_f := \Lambda(f)\,(f \in \mathcal{H}_{F,c})$ defined by*

$$\text{(3.20)} \qquad \Lambda_f((z^0P + z^1Q, a)) := e^{2\pi i\sigma\{c(a+2iz^-\,{}^tz^+)\}}\, f(2z^+)$$

is an isometry of $\mathcal{H}_{F,c}$ onto $\mathcal{H}^{F,c}$, where $2z^\pm = z^0 \pm iz^1$ (cf. (3.11)). The inverse $\Delta : \mathcal{H}^{F,c} \longrightarrow \mathcal{H}_{F,c}$, $\Delta_\psi := \Delta(\psi)\,(\psi \in \mathcal{H}^{F,c})$ is given by

$$\text{(3.21)} \qquad \Delta_\psi(W) := \psi\left(\frac{1}{2}WR^+\right), \quad W \in \mathbb{C}^{(h,g)},$$

where $R^\pm = P \mp iQ$ (cf. (3.14)).

Proof. First we observe that for $w = z^0 P + z^1 Q = z^+ R^+ + z^- R^- \in V_{\mathbb{C}}$,

$$(w, a) = (z^- R^-, a + 2iz^-\,{}^t z^+) \circ (z^+ R^+, 0).$$

Thus if $\psi \in \mathcal{H}^{F,c}$ and $w = z^0 P + z^1 Q = z^+ R^+ + z^- R^-$, by (F1),

$$\psi((w,a)) = e^{2\pi i \sigma\{c(a+2iz^-\,{}^t z^+)\}}\, \psi((z^+ R^+, 0)). \tag{3.22}$$

Let $W = x + iy \in \mathbb{C}^{(h,g)}$ with $x, y \in \mathbb{R}^{(h,g)}$. Then

$$xP + yQ = z^+ R^+ + z^- R^-, \quad 2z^{\pm} = x \pm iy.$$

So $z^-\,{}^t z^+ = \frac{1}{4} W\,{}^t\overline{W}$. According to (3.22), if $\psi \in \mathcal{H}^{F,c}$, we have

$$\psi((xP + yQ, 0)) = e^{-\pi\sigma(cW\,{}^t\overline{W})}\, \psi((\frac{1}{2} W R^+, 0)).$$

Thus we get

$$|\psi((xP + yQ, 0))|^2 = e^{-2\pi\sigma(cW\,{}^t\overline{W})} \left| \psi((\frac{1}{2} W R^+, 0)) \right|^2.$$

Therefore

$$\int_{\mathcal{Z}\backslash G} |\psi(\dot{g})|^2\, d\dot{g} = \int_{\mathbb{C}^{(h,g)}} e^{-2\pi\sigma(cW\,{}^t\overline{W})}\, \left|\Delta_{\psi}(W)\right|^2 dW < \infty.$$

It is easy to see that Δ is the inverse of Λ. Hence we obtain the desired results. □

LEMMA 3.2. *The representation $U^{F,c}$ is realized as a representation of G in the Fock space $\mathcal{H}_{F,c}$ as follows. If $g = (\lambda P + \mu Q, \kappa) = (\lambda, \mu, \kappa) \in G$ and $f \in \mathcal{H}_{F,c}$, then*

$$\left(U^{F,c}(g) f\right)(W) = e^{2\pi i \sigma(c\kappa)}\, e^{-\pi\sigma\{c(\zeta\,{}^t\bar{\zeta} + 2W\,{}^t\bar{\zeta})\}}\, f(W + \zeta), \quad W \in \mathbb{C}^{(h,g)}, \tag{3.23}$$

where $\zeta = \lambda + i\mu$.

Proof.

$$\begin{aligned}
\left(U^{F,c}(g)f\right)(W) &= \left(\Delta(U^{F,c}(g)(\Lambda_f))\right)(W)\\
&= \left(U^{F,c}(g)(\Lambda_f)\right)\left(\frac{1}{2}WR^+\right)\\
&= \Lambda_f\left(\left(\frac{1}{2}WR^+,0\right)\circ g\right)\\
&= \Lambda_f\left(\left(\frac{1}{2}W,-\frac{i}{2}W,0\right)\circ(\lambda,\mu,\kappa)\right)\\
&= \Lambda_f\left((\lambda+\frac{1}{2}W)P+(\mu-\frac{i}{2}W)Q,\kappa+\frac{1}{2}W\,{}^t\mu+\frac{i}{2}W\,{}^t\lambda\right)\\
&= e^{2\pi i\sigma\{c(\kappa+\frac{i}{2}W\,{}^t\bar\zeta+\frac{i}{2}\bar\zeta\,{}^tW+\frac{i}{2}\bar\zeta\,{}^t\zeta)\}}\,f(W+\zeta) \qquad (*)\\
&= e^{2\pi i\sigma(c\kappa)}\cdot e^{-\pi\sigma\{c(\zeta\,{}^t\bar\zeta+W\,{}^t\bar\zeta)\}}\,f(W+\zeta),
\end{aligned}$$

where $\zeta=\lambda+i\mu$. In (*), we used (3.20) and the facts that $2iz^-\,{}^tz^+=\frac{i}{2}(\overline{W}\,{}^t\zeta+\overline{W}\,{}^tW)$ and $2z^+=W+\zeta$. □

DEFINITION 3.3. The induced representation $U^{F,c}$ of G in the Fock space $\mathcal{H}_{F,c}$ is called the *Fock representation* of G.

Let $W=U+iV\in\mathbb{C}^{(h,g)}$ with $U,V\in\mathbb{R}^{(h,g)}$. If $U=(u_{ka})$, $V=(v_{lb})$ are coordinates in $\mathbb{C}^{(h,g)}$, we put

$$dU=du_{11}du_{12}\cdots du_{hg},\quad dV=dv_{11}dv_{12}\cdots dv_{hg}$$

and $dW=dUdV$. And we set

$$d\mu(W):=e^{-\pi\sigma(W\,{}^t\overline{W})}\,dW. \tag{3.24}$$

Let f be a holomorphic function on $\mathbb{C}^{(h,g)}$. Then $f(W)$ has the Taylor expansion

$$f(z)=\sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}}a_JW^J,\quad W=(w_{ka})\in\mathbb{C}^{(h,g)},$$

where $J=(J_{ka})\in J\in\mathbb{Z}_{\geq 0}^{(h,g)}$ and $W^J:=w_{11}^{J_{11}}w_{12}^{J_{12}}\cdots w_{hg}^{J_{hg}}$.

We set $|W|_\infty := \max_{k,a}(|w_{ka}|)$. Then by an easy computation, we have

$$\begin{aligned}\int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu(W) &= \lim_{r\longrightarrow\infty} \int_{|W|_\infty\le r} |f(W)|^2 d\mu(W)\\ &= \lim_{r\longrightarrow\infty} \sum_{J,K} a_J\overline{a_K} \int_{|W|_\infty\le r} W^J\overline{W^K}\, d\mu(W)\\ &= \sum_J |a_J|^2\pi^{-|J|}J!,\end{aligned}$$

where J runs over $J\in\mathbb{Z}_{\ge 0}^{(h,g)}$.

Let $\mathcal{H}_{h,g}$ be the Hilbert space consisting of all holomorphic functions $f:\mathbb{C}^{(h,g)}\longrightarrow\mathbb{C}$ satisfying the condition

$$\|f\|^2 := \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu(W) < \infty. \tag{3.25}$$

The inner product $(\ ,\ )$ on $\mathcal{H}_{h,g}$ is given by

$$(f_1,f_2) := \int_{\mathbb{C}^{(h,g)}} f_1(W)\,\overline{f_2(W)}\, d\mu(W),\quad f_1,f_2\in\mathcal{H}_{h,g}.$$

Thus we have

Lemma 3.4. *Let $f\in\mathcal{H}_{h,g}$ and let $f(W)=\sum_J a_JW^J$ be the Taylor expansion of f. Then*

$$\|f\|^2 = \sum_{J\in\mathbb{Z}_{\ge 0}^{(h,g)}} |a_J|^2\pi^{-|J|}J!.$$

For each $J\in\mathbb{Z}_{\ge 0}^{(h,g)}$, we define the holomorphic function $\Phi_J(W)$ on $\mathbb{C}^{(h,g)}$ by

$$\Phi_J(W) := (J!)^{-\frac{1}{2}}\left(\pi^{\frac{1}{2}}W\right)^J,\quad W\in\mathbb{C}^{(h,g)}. \tag{3.26}$$

Then

$$(\Phi_J,\Phi_K) = \begin{cases} 1 & \text{if } J=K\\ 0 & \text{otherwise.}\end{cases} \tag{3.27}$$

It is easy to see that the set $\left\{ \Phi_J \,|\, J \in \mathbb{Z}_{\geq 0}^{(h,g)} \right\}$ forms a complete orthonormal system in $\mathcal{H}_{h,g}$. By the Schwarz inequality, for any $f \in \mathcal{H}_{h,g}$, we have

$$(3.28) \qquad |f(W)| \leq e^{\frac{\pi}{2}\sigma(W\,{}^t\overline{W})}\,\|f\|, \quad W \in \mathbb{C}^{(h,g)}.$$

Consequently, the norm convergence in $\mathcal{H}_{h,g}$ implies the uniform convergence on any bounded subset of $\mathbb{C}^{(h,g)}$. We observe that for a fixed $W' \in \mathbb{C}^{(h,g)}$, the holomorphic function $W \longrightarrow e^{\pi\sigma(W\,{}^t\overline{W'})}$ admits the following Taylor expansion

$$(3.29) \qquad e^{\pi\sigma(W\,{}^t\overline{W'})} = \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} \Phi_J(W)\,\Phi_J(\overline{W'}).$$

From (3.29), we obtain

$$(3.30) \qquad \Phi_J(\overline{W'}) = (J!)^{-\frac{1}{2}} \int_{\mathbb{C}^{(h,g)}} e^{\pi\sigma(W\,{}^t\overline{W'})} \left(\pi^{\frac{1}{2}}\overline{W}\right)^J d\mu(W).$$

Thus if $f \in \mathcal{H}_{h,g}$, we get

$$\begin{aligned} \left(f(W),\, e^{\pi\sigma(W\,{}^t\overline{W'})}\right) &= \left(f,\, \sum_J \Phi_J(\overline{W'})\,\Phi_J(\cdot)\right) \\ &= \sum_J \Phi_J(W')\,(f, \Phi_J) \\ &= f(W'). \end{aligned}$$

Hence $e^{\pi\sigma(W\,{}^t\overline{W'})}$ is the reproducing kernel for $\mathcal{H}_{h,g}$ in the sense that for any $f \in \mathcal{H}_{h,g}$,

$$(3.31) \qquad f(W) = \int_{\mathbb{C}^{(h,g)}} e^{\pi\sigma(W\,{}^t\overline{W'})}\, f(W')\, d\mu(W').$$

We set

$$(3.32) \qquad \kappa(W, W') := e^{\pi\sigma(W\,{}^t\overline{W'})}, \quad W, W' \in \mathbb{C}^{(h,g)}.$$

Obviously $\kappa(W,W') = \overline{\kappa(W',W)}$. (3.31) may be written as

$$f(W) = \int_{\mathbb{C}^{(h,g)}} \kappa(W,W')\, f(W')\, d\mu(W'), \quad f \in \mathcal{H}_{h,g}. \tag{3.33}$$

Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree h. We define the measure

$$d\mu_{\mathcal{M}}(W) := e^{-2\pi\sigma(\mathcal{M}W\,{}^t\overline{W})}\, dW. \tag{3.34}$$

We recall the *Fock* space $\mathcal{H}_{F,\mathcal{M}}$ consisting of all holomorphic functions $f : \mathbb{C}^{(h,g)} \longrightarrow \mathbb{C}$ that satisfy the condition

$$\|f\|^2_{\mathcal{M}} := \|f\|^2_{F,\mathcal{M}} := \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu_{\mathcal{M}}(W) < \infty. \tag{3.35}$$

The inner product $(\ ,\ )_{\mathcal{M}} := (\ ,\ )_{F,\mathcal{M}}$ on $\mathcal{H}_{F,\mathcal{M}}$ is given by

$$(f_1, f_2)_{\mathcal{M}} := \int_{\mathbb{C}^{(h,g)}} f_1(W)\, \overline{f_2(W)}\, d\mu_{\mathcal{M}}(W), \quad f_1, f_2 \in \mathcal{H}_{F,\mathcal{M}}.$$

LEMMA 3.5. *Let $f \in \mathcal{H}_{F,\mathcal{M}}$ and let $g(W) := f\left((2\mathcal{M})^{-\frac{1}{2}}W\right)$ be the holomorphic function on $\mathbb{C}^{(h,g)}$. We let*

$$g(W) = \sum_{J \in \mathbb{Z}^{(h,g)}_{\geq 0}} a_{\mathcal{M},J}\, W^J$$

be the Taylor expansion of $g(W)$. Then we have

$$\|f\|^2_{\mathcal{M}} = (f,f)_{\mathcal{M}} = 2^{-g}(\det \mathcal{M})^{-g} \sum_{J \in \mathbb{Z}^{(h,g)}_{\geq 0}} |a_{\mathcal{M},J}|^2\, \pi^{-|J|} J!.$$

Proof. Let $\mathcal{M}^{\frac{1}{2}}$ be the unique positive definite symmetric matrix of degree h such that $\left(\mathcal{M}^{\frac{1}{2}}\right)^2 = \mathcal{M}$. We put $\tilde{W} := \sqrt{2}\mathcal{M}^{\frac{1}{2}}W$. Obviously $d\tilde{W} = 2^g\,(\det \mathcal{M})^g dW$. Thus for $f \in \mathcal{H}_{F,\mathcal{M}}$, we have

$$\begin{aligned}
(f,f)_{\mathcal{M}} &= \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu_{\mathcal{M}}(W) \\
&= 2^{-g}(\det \mathcal{M})^{-g} \int_{\mathbb{C}^{(h,g)}} |g(W)|^2\, d\mu(W) \\
&= 2^{-g}(\det \mathcal{M})^{-g} \sum_{J \in \mathbb{Z}^{(h,g)}_{\geq 0}} |a_{\mathcal{M},J}|^2 \pi^{-|J|} J! \quad \text{(by Lemma 3.4)}
\end{aligned}$$

Proof. Let $\mathcal{M}^{\frac{1}{2}}$ be the unique positive definite symmetric matrix of degree h such that $\left(\mathcal{M}^{\frac{1}{2}}\right)^2 = \mathcal{M}$. We put $\tilde{W} := \sqrt{2}\mathcal{M}^{\frac{1}{2}}W$. Obviously $d\tilde{W} = 2^g\,(det\,\mathcal{M})^g dW$. Thus for $f \in \mathcal{H}_{F,\mathcal{M}}$, we have

$$\begin{aligned}(f,f)_{\mathcal{M}} &= \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu_{\mathcal{M}}(W)\\ &= 2^{-g}(det\,\mathcal{M})^{-g}\int_{\mathbb{C}^{(h,g)}} |g(W)|^2\, d\mu(W)\\ &= 2^{-g}(det\,\mathcal{M})^{-g}\sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}} |a_{\mathcal{M},J}|^2\pi^{-|J|}J! \quad \text{(by Lemma 3.4)}\end{aligned}$$

□

For each $J \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we put
(3.36)

$$\Phi_{\mathcal{M},J}(W) := 2^{\frac{g}{2}}\,(\,det\,\mathcal{M}\,)^{\frac{g}{2}}\,(J!)^{-\frac{1}{2}}\left((2\pi\mathcal{M})^{\frac{1}{2}}W\right)^J, \quad W \in \mathbb{C}^{(h,g)}.$$

LEMMA 3.6. *The set* $\left\{\Phi_{\mathcal{M},J} \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)}\right\}$ *is a complete orthonormal system in* $\mathcal{H}_{F,\mathcal{M}}$.

Proof. For $J, K \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we have

$$\begin{aligned}(\Phi_{\mathcal{M},J},\Phi_{\mathcal{M},K})_{\mathcal{M}} &= 2^g(det\,\mathcal{M})^g(J!)^{-\frac{1}{2}}(K!)^{-\frac{1}{2}}\\ &\quad\times\int_{\mathbb{C}^{(h,g)}}\left((2\pi\mathcal{M})^{\frac{1}{2}}W\right)^J\left((2\pi\mathcal{M})^{\frac{1}{2}}\overline{W}\right)^K d\mu_{\mathcal{M}}(W)\\ &= (J!)^{-\frac{1}{2}}(K!)^{-\frac{1}{2}}\int_{\mathbb{C}^{(h,g)}}(\pi^{\frac{1}{2}}W)^J\overline{(\pi^{\frac{1}{2}}W)^K}\,d\mu(W)\\ &= (\Phi_J,\Phi_K).\end{aligned}$$

By (3.27), we have

$$(3.37)\qquad (\Phi_{\mathcal{M},J},\Phi_{\mathcal{M},K})_{\mathcal{M}} = \begin{cases} 1 & \text{if } J = K\\ 0 & \text{otherwise.}\end{cases}$$

We leave the proof of the completeness to the reader. □

We observe that for a fixed $W' \in \mathbb{C}^{(h,g)}$, the holomorphic function $W \longrightarrow e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}$ admits the following Taylor expansion

$$e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})} = \sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}} \Phi_{\mathcal{M},J}(W)\,\Phi_{\mathcal{M},J}(\overline{W'}). \tag{3.38}$$

If $f \in \mathcal{H}_{F,\mathcal{M}}$, we have

$$\begin{aligned}\left(f(W),\, e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}\right)_{\mathcal{M}} &= \sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}} (f,\Phi_{\mathcal{M},J})_{\mathcal{M}}\,\Phi_{\mathcal{M},J}(W') \\ &= f(W').\end{aligned}$$

Hence $e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}$ is the reproducing kernel for $\mathcal{H}_{F,\mathcal{M}}$ in the sense that

$$f(W) = \int_{\mathbb{C}^{(h,g)}} f(W')\,e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}\,d\mu_{\mathcal{M}}(W'). \tag{3.39}$$

For $U \in \mathbb{R}^{(h,g)}$ and $W \in \mathbb{C}^{(h,g)}$, we put

$$k(U,W) := e^{2\pi\sigma(-U\,{}^tU + \frac{1}{2}W\,{}^tW + 2iU\,{}^tW)}. \tag{3.40}$$

Then we have the following lemma.

LEMMA 3.7.

$$\int_{\mathbb{R}^{(h,g)}} k(U,W)\,\overline{k(U,W')}\,dU = e^{2\pi\sigma(W'\,{}^tW')}.$$

Proof. We put

$$\mathcal{I}(W,W') := \int_{\mathbb{R}^{(h,g)}} k(U,W)\,\overline{k(U,W')}\,dU.$$

Then we have

$$\begin{aligned}\mathcal{I}(W,W') &= e^{\pi\sigma(W\,{}^tW + \overline{W'}\,{}^t\overline{W'})} \int_{\mathbb{R}^{(h,g)}} e^{-4\pi\sigma(U\,{}^tU)}\,e^{4\pi i\sigma\{U\,{}^t(W-\overline{W'})\}}\,dU \\ &= e^{\pi\sigma(W\,{}^tW + \overline{W'}\,{}^t\overline{W'})}\cdot\prod_{k,a}\int_{\mathbb{R}} e^{-4\pi\{u_{ka}^2 - iu_{ka}(w_{ka}-\overline{w'_{ka}})\}}\,du_{ka},\end{aligned}$$

where $W = (w_{ka}),\ W' = (w'_{ka}) \in \mathbb{C}^{(h,g)}$ and $U = (u_{ka}) \in \mathbb{R}^{(h,g)}$. It is easy to show that

$$\int_{\mathbb{R}} e^{-4\pi\{u_{ka}^2 - iu_{ka}(w_{ka}-\overline{w'_{ka}})\}}\, du_{ka} = e^{-\pi(w_{ka}-\overline{w'_{ka}})^2}.$$

Thus we get

$$\begin{aligned}\mathcal{I}(W,W') &= e^{\pi\sigma(W\,{}^tW+\overline{W'}\,{}^t\overline{W'})}\cdot e^{-\pi\sum_{k,a}(w_{ka}-\overline{w'_{ka}})^2}\\ &= e^{2\pi\sum_{k,a} w_{ka}\overline{w'_{ka}}}\\ &= e^{2\pi\sigma(W\,{}^t\overline{W'})}.\end{aligned}$$

For $U \in \mathbb{R}^{(h,g)}$ and $W \in \mathbb{C}^{(h,g)}$, we put

$$(3.41)\qquad k_{\mathcal{M}}(U,W) := e^{2\pi\sigma\{\mathcal{M}(-U\,{}^tU-\frac{1}{2}W\,{}^tW+2U\,{}^tW)\}}. \qquad \square$$

LEMMA 3.8. *Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree h. Then we have*

$$(3.42)\qquad k_{\mathcal{M}}(U,W) = k(\mathcal{M}^{\frac{1}{2}}U,\ -i\mathcal{M}^{\frac{1}{2}}W)$$

and

$$(3.43)\quad \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U,W)\,\overline{k_{\mathcal{M}}(U,W')}\,dU = (\det\mathcal{M})^{-\frac{g}{2}}\cdot e^{2\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}.$$

Proof. The formula (3.42) follows immediately from a straightforward computation. We put

$$\mathcal{I}_{\mathcal{M}}(W,W') := \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U,W)\,\overline{k_{\mathcal{M}}(U,W')}\,dU.$$

Using (3.42), we have

$$\begin{aligned}\mathcal{I}_{\mathcal{M}}(W,W') &= \int_{\mathbb{R}^{(h,g)}} k\left(\mathcal{M}^{\frac{1}{2}}U,\ -i\mathcal{M}^{\frac{1}{2}}W\right)\cdot\overline{k\left(\mathcal{M}^{\frac{1}{2}}U, -i\mathcal{M}^{\frac{1}{2}}W'\right)}\,dU\\ &= (\det\mathcal{M})^{-\frac{g}{2}}\int_{\mathbb{R}^{(h,g)}} k\left(U, -i\mathcal{M}^{\frac{1}{2}}W\right)\cdot\overline{k\left(U, -i\mathcal{M}^{\frac{1}{2}}W'\right)}\,dU\\ &= (\det\mathcal{M})^{-\frac{g}{2}}\cdot e^{2\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})} \qquad \text{(by Lemma 3.7)}\end{aligned}$$

$\square$

We recall that the Fock representation $U^{F,\mathcal{M}}$ of the real Heisenberg group G in $\mathcal{H}_{F,\mathcal{M}}$ (cf. (3.23)) is given by

$$(3.44)\quad \left(U^{F,\mathcal{M}}(g)f\right)(W) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\cdot e^{-\pi\sigma\{\mathcal{M}(\zeta\,{}^t\bar{\zeta}+2W\,{}^t\bar{\zeta})\}}\,f(W+\zeta),$$

where $g = (\lambda,\mu,\kappa) \in G,\ f \in \mathcal{H}_{F,\mathcal{M}}$ and $\zeta = \lambda + i\mu \in \mathbb{C}^{(h,g)}$.

LEMMA 3.9. *The Fock representation $U^{F,\mathcal{M}}$ of G in $\mathcal{H}_{F,\mathcal{M}}$ is unitary.*

Proof. For brevity, we put $U_{g,f}(W) := \left(U^{F,\mathcal{M}}(g)f\right)(W)$ for $g = (\lambda, \mu, \kappa) \in G$ and $f \in \mathcal{H}_{F,\mathcal{M}}$. Then we have

$$\begin{aligned}
(U_{g,f}, U_{g,f})_{\mathcal{M}} &= \|U_{g,f}\|_{\mathcal{M}}^2 \\
&= \int_{\mathbb{C}^{(h,g)}} U_{g,f}(W)\, \overline{U_{g,f}(W)}\, d\mu_{\mathcal{M}}(W) \\
&= \int_{\mathbb{C}^{(h,g)}} e^{-\pi\sigma\{\mathcal{M}(\zeta\, {}^t\bar{\zeta}+2W\, {}^t\bar{\zeta}+\bar{\zeta}\, {}^t\zeta+2\overline{W}\, {}^tW+2W\, {}^t\overline{W})\}}\, |f(W+\zeta)|^2\, dW \\
&= \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu_{\mathcal{M}}(W) \\
&= (f, f)_{\mathcal{M}} = \|f\|_{\mathcal{M}}^2.
\end{aligned}$$

□

We recall that the Schrödinger representation $U^{S,\mathcal{M}} := U(\sigma_{\mathcal{M}})$ of the real Heisenberg group G in the Hilbert space $\mathcal{H}_{S,\mathcal{M}} \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ (cf. (2.17) or (2.18)) is given by

$$\left(U^{S,\mathcal{M}}(g)f\right)(\xi) = e^{2\pi i\sigma\{\mathcal{M}(\kappa+\mu\, {}^t\lambda+2\mu\, {}^t\xi)\}}\, f(\xi+\lambda), \tag{3.45}$$

where $g = (\lambda, \mu, \kappa) \in G$, $f \in \mathcal{H}_{S,\mathcal{M}}$ and $\xi \in \mathbb{R}^{(h,g)}$. In order to emphasize $\mathcal{M}$, sometimes we call $U^{S,\mathcal{M}}$ the Schrödinger representation of G of *index* $\mathcal{M}$. The inner product $(\ ,\ )_{S,\mathcal{M}}$ on $\mathcal{H}_{S,\mathcal{M}}$ is given by

$$(f_1, f_2)_{S,\mathcal{M}} := \int_{\mathbb{R}^{(h,g)}} f_1(U)\, \overline{f_2(U)}\, dU, \quad f_1, f_2 \in \mathcal{H}_{S,\mathcal{M}}.$$

And we define the norm $\|\ \|_{S,\mathcal{M}}$ on $\mathcal{H}_{S,\mathcal{M}}$ by

$$\|f\|_{S,\mathcal{M}}^2 := \int_{\mathbb{R}^{(h,g)}} |f(U)|^2\, dU, \quad f \in \mathcal{H}_{S,\mathcal{M}}.$$

THEOREM 3.10. *The Fock representation $(U^{F,\mathcal{M}}, \mathcal{H}_{F,\mathcal{M}})$ of G is untarily equivalent to the Schrödinger representation $(U^{S,\mathcal{M}}, \mathcal{H}_{S,\mathcal{M}})$ of G of index $\mathcal{M}$. Therefore the Fock representation $U_{F,\mathcal{M}}$ is irreducible. The intertwining unitary isometry $I_{\mathcal{M}} : \mathcal{H}_{S,\mathcal{M}} \longrightarrow \mathcal{H}^{F,\mathcal{M}}$ is given by*

$$(I_{\mathcal{M}} f)(W) := \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(\xi, W)\, f(\xi)\, d\xi, \tag{3.46}$$

where $f \in \mathcal{H}_{S,\mathcal{M}} = L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$, $W \in \mathbb{C}^{(h,g)}$ and $k_{\mathcal{M}}(\xi, W)$ is a function on $\mathbb{R}^{(h,g)} \times \mathbb{C}^{(h,g)}$ defined by (3.41).

Proof. For any $f \in \mathcal{H}_{S,\mathcal{M}} = L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$, we define

$$(I_{\mathcal{M}} f)(W) := \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(\xi, W)\, f(\xi)\, d\xi, \quad W \in \mathbb{C}^{(h,g)}.$$

Now we will show the following (I1), (I2) and (I3):

(I1) The image of $\mathcal{H}_{S,\mathcal{M}}$ under $I_{\mathcal{M}}$ is contained in $\mathcal{H}_{F,\mathcal{M}}$.

(I2) $I_{\mathcal{M}}$ preserves the norms, i.e., $\|f\|_{S,\mathcal{M}} = \|I_{\mathcal{M}} f\|_{\mathcal{M}}$.

(I3) $I_{\mathcal{M}}$ is a bijective operator of $\mathcal{H}_{S,\mathcal{M}}$ onto $\mathcal{H}_{F,\mathcal{M}}$.

Before we prove (I1), (I2) and (I3), we prove the following lemma. □

LEMMA 3.11. *For a fixed $U \in \mathbb{R}^{(h,g)}$, we consider the Taylor expansion*

$$k_{\mathcal{M}}(U, W) = \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} h_J(U)\, \Phi_{\mathcal{M},J}(W), \quad W \in \mathbb{C}^{(h,g)} \tag{3.47}$$

of the holomorphic function $k_{\mathcal{M}}(U, \cdot\,)$ on $\mathbb{C}^{(h,g)}$. Then the set $\left\{h_J \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)}\right\}$ forms a complete orthonormal system in $L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$. Moreover, for a fixed $W \in \mathbb{C}^{(h,g)}$, (3.47) is the Fourier expansion of $k_{\mathcal{M}}(\,\cdot\,, W)$ with respect to this orthonormal system $\left\{ h_J \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)} \right\}$.

Proof. Following Igusa [2], pp. 33-34, we can prove it. The detail will be left to the reader.

If $f \in \mathcal{H}_{S,\mathcal{M}}$, then by the Schwarz inequality and lemma 3.8, (3.43), we have

$$\begin{aligned} |\,(I_{\mathcal{M}} f\,)(W)| &\leq \left(\int_{\mathbb{R}^{(h,g)}} |k_{\mathcal{M}}(U, W)|^2\, dU \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^{(h,g)}} |f(U)|^2\, dU \right)^{\frac{1}{2}} \\ &= (det\, \mathcal{M})^{-\frac{g}{4}} \cdot e^{\pi\sigma(\mathcal{M} W\, {}^t\overline{W})}\, \|f\|_{S,\mathcal{M}}. \end{aligned}$$

Thus the above integral $(I_{\mathcal{M}} f)(W)$ converges uniformly on any compact subset of $\mathbb{C}^{(h,g)}$ and hence $(I_{\mathcal{M}} f)(W)$ is holomorphic in $\mathbb{C}^{(h,g)}$. And according to lemma 6.11, we get

$$\begin{aligned} (\,I_{\mathcal{M}} f\,)(W) &= \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} \int_{\mathbb{R}^{(h,g)}} h_J(U)\, f(U)\, \Phi_{\mathcal{M},J}(W)\, dU \\ &= \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} (h_J, \bar{f})_{S,\mathcal{M}}\, \Phi_{\mathcal{M},J}(W). \end{aligned}$$

Therefore we get

$$\begin{aligned}\|I_{\mathcal{M}} f\|^2_{F,\mathcal{M}} &= \int_{\mathbb{C}^{(h,g)}} |I_{\mathcal{M}} f(W)|^2 \, d\mu_{\mathcal{M}}(W) \\ &= \sum_{J,\,K\in\mathbb{Z}^{(h,g)}_{\geq 0}} (h_J,\, \bar{f})_{S,\mathcal{M}} \cdot \overline{(h_K,\, \bar{f})} \\ &\qquad \int_{\mathbb{C}^{(h,g)}} \Phi_{\mathcal{M},J}(W)\, \overline{\Phi_{\mathcal{M},K}(W)}\, d\mu_{\mathcal{M}}(W) \\ &= \sum_{J\in\mathbb{Z}^{(h,g)}_{\geq 0}} |(h_j,\, \bar{f})_{S,\mathcal{M}}|^2 \qquad \text{(by (3.37))} \\ &= \|f\|^2_{S,\mathcal{M}} < \infty.\end{aligned}$$

This proves (I1) and (I2). It is easy to see that $I_{\mathcal{M}} \overline{h_J} = \Phi_{\mathcal{M},J}$ for all $J \in \mathbb{Z}^{(h,g)}_{\geq 0}$. Since the set $\left\{ \Phi_{\mathcal{M},J} \mid J \in \mathbb{Z}^{(h,g)}_{\geq 0} \right\}$ forms a complete orthonormal system of $\mathcal{H}_{F,\mathcal{M}}$, $I_{\mathcal{M}}$ is surjective. Obviously the injectivity of $I_{\mathcal{M}}$ follows immediately from the fact that $I_{\mathcal{M}} \overline{h_J} = \Phi_{\mathcal{M},J}$ for all $J \in \mathbb{Z}^{(h,g)}_{\geq 0}$. This proves (I3).

On the other hand, we let $f \in \mathcal{H}_{S,\mathcal{M}}$ and $g = (\lambda, \mu, \kappa) \in G$. We put $\zeta = \lambda + i\mu$. Then we get

$$\begin{aligned}&\left(U^{F,\mathcal{M}}(g)(I_{\mathcal{M}} f) \right)(W) \\ &= e^{2\pi i \sigma(\mathcal{M}\kappa)} \cdot e^{-\pi\sigma\{\mathcal{M}(\zeta\, {}^t\bar{\zeta} + 2W\, {}^t\bar{\zeta})\}}\, (I_{\mathcal{M}} f)(W+\zeta) \quad (\text{by (3.44)}) \\ &= e^{2\pi i \sigma(\mathcal{M}\kappa)} \cdot e^{-\pi\sigma\{\mathcal{M}(\zeta\, {}^t\bar{\zeta} + 2W\, {}^t\bar{\zeta})\}} \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U, W+\zeta)\, f(U)\, dU.\end{aligned}$$

We define the function $A : \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)} \longrightarrow \mathbb{C}$ by

$$A_{\mathcal{M}}(U, W) := \sigma \left\{ \mathcal{M} \left(-U\, {}^tU - \frac{W\, {}^tW}{2} + 2U\, {}^tW \right) \right\}. \tag{3.48}$$

Obviously $\kappa_{\mathcal{M}}(U, W) = e^{2\pi A_{\mathcal{M}}(U,W)}$ for $U \in \mathbb{R}^{(h,g)}$ and $W \in \mathbb{C}^{(h,g)}$.

By an easy computation, we get

$$A_{\mathcal{M}}(U, W+\zeta) - A(U-\lambda, W) = \sigma \left\{ \mathcal{M} \left(\frac{\zeta\, {}^t\bar{\zeta}}{2} + W\, {}^t\bar{\zeta} - i\lambda\, {}^t\mu + 2iU\, {}^t\mu \right) \right\}.$$

Therefore we get

$$\begin{aligned} & k_{\mathcal{M}}(U, W+\zeta) \\ & = e^{2\pi A_{\mathcal{M}}(U-\lambda, W)} \cdot e^{2\pi\sigma\left\{\mathcal{M}\left(\frac{1}{2}\zeta\, {}^t\zeta + W\, {}^t\bar{\zeta} - i\lambda\, {}^t\mu + 2iU\, {}^t\mu\right)\right\}} \\ & = k_{\mathcal{M}}(U-\lambda, W) \cdot e^{2\pi\sigma\left\{\mathcal{M}\left(\frac{1}{2}\zeta\, {}^t\bar{\zeta} + W\, {}^t\bar{\zeta} - i\lambda\, {}^t\mu + 2iU\, {}^t\mu\right)\right\}}. \end{aligned}$$

Hence we have

$$\begin{aligned} & \left(U^{F,\mathcal{M}}(g)(I_{\mathcal{M}} f)\right)(W) \\ & = \int_{\mathbb{R}^{(h,g)}} e^{2\pi i\sigma\left\{\mathcal{M}(\kappa + 2U\, {}^t\mu - \lambda\, {}^t\mu)\right\}}\, k_{\mathcal{M}}(U-\lambda, W)\, f(U)\, dU \\ & = \int_{\mathbb{R}^{(h,g)}} e^{2\pi i\sigma\left\{\mathcal{M}(\kappa + 2\lambda\, {}^t\mu + 2U\, {}^t\mu - \lambda\, {}^t\mu)\right\}}\, k_{\mathcal{M}}(U, W)\, f(U+\lambda)\, dU \\ & = \int_{\mathbb{R}^{(h,g)}} e^{2\pi i\sigma\left\{\mathcal{M}(\kappa + 2U\, {}^t\mu + \lambda\, {}^t\mu)\right\}}\, k_{\mathcal{M}}(U, W)\, f(U+\lambda)\, dU \\ & = \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U, W)\, \left(U^{S,\mathcal{M}}(g) f\right)(U)\, dU \qquad \text{(by (3.45))} \\ & = \left(I_{\mathcal{M}}\left(U^{S,\mathcal{M}}(g) f\right)\right)(W). \end{aligned}$$

So far we proved that $U^{F,\mathcal{M}} \circ I_{\mathcal{M}} = I_{\mathcal{M}} \circ U^{S,\mathcal{M}}(g)$ for all $g \in G$. That is, the unitary isometry $I_{\mathcal{M}}$ of $\mathcal{H}_{S,\mathcal{M}}$ onto $\mathcal{H}_{F,\mathcal{M}}$ is the intertwining operator. This completes the proof. □

The infinitesimal representation $dU^{F,\mathcal{M}}$ associated to the Fock representation $U^{F,\mathcal{M}}$ is given as follows.

PROPOSITION 3.12. *Let $\mathcal{M}$ be as before. We put*

$$\mathcal{M} = (\mathcal{M}_{kl}), \quad (2\pi\mathcal{M})^{\frac{1}{2}} = (\tau_{kl}),$$

where $\tau_{kl} \in \mathbb{R}$ and $1 \leq k, l \leq h$. For each $J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)}$ and $W = (W_{ka}) \in \mathbb{C}^{(h,g)}$, we have

$$\text{(3.49)} \quad dU^{F,\mathcal{M}}(D_{kl}^0)\, \Phi_{\mathcal{M},J}(W) = 2\pi i\, \mathcal{M}_{kl}\, \Phi_{\mathcal{M},J}(W), \quad 1 \leq k \leq l \leq h.$$

(3.50)

$$dU^{F,\mathcal{M}}(D_{ka})\,\Phi_{\mathcal{M},J}(W) = -2\pi\left(\sum_{m=1}^{h}\mathcal{M}_{mk}W_{ma}\right)\Phi_{\mathcal{M},J}(W) + \sum_{m=1}^{h}\tau_{mk}J_{ma}^{\frac{1}{2}}\,\Phi_{\mathcal{M},J-\epsilon_{ma}}(W).$$

(3.51)

$$dU^{F,\mathcal{M}}(\hat{D}_{lb}\,\Phi_{\mathcal{M},J}(W) = 2\pi i\left(\sum_{m=1}^{h}\mathcal{M}_{ml}W_{mb}\right)\Phi_{\mathcal{M},J}(W) + i\sum_{m=1}^{h}\tau_{ml}J_{mb}^{\frac{1}{2}}\,\Phi_{\mathcal{M},J-\epsilon_{lb}}(W).$$

Proof. We put $E_{kl}^0 = \frac{1}{2}(E_{kl}+E_{lk})$, where $1 \le k \le l \le h$.

$$\begin{aligned}
dU^{F,\mathcal{M}}(D_{kl}^0)\,\Phi_{\mathcal{M},J}(W) &= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}(\exp tX_{kl}^0)\,\Phi_{\mathcal{M},J}(W)\\
&= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}\left((0,0,tE_{kl}^0)\right)\,\Phi_{\mathcal{M},J}(W)\\
&= \lim_{t\to 0}\frac{e^{2\pi i\sigma(t\mathcal{M}E_{kl}^0)} - I}{t}\,\Phi_{\mathcal{M},J}(W)\\
&= \lim_{t\to 0}\frac{e^{2\pi it\mathcal{M}_{kl}} - I}{t}\,\Phi_{\mathcal{M},J}(W)\\
&= 2\pi i\,\mathcal{M}_{kl}\,\Phi_{\mathcal{M},J}(W).
\end{aligned}$$

And we have

$$\begin{aligned}
&dU^{F,\mathcal{M}}(D_{ka})\,\Phi_{\mathcal{M},J}(W)\\
&\quad= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}(\exp tX_{ka})\,\Phi_{\mathcal{M},J}(W)\\
&\quad= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}\left((tE_{ka},0,0)\right)\Phi_{\mathcal{M},J}(W)\\
&\quad= \frac{d}{dt}\Big|_{t=0} e^{-\pi t^2\sigma(\mathcal{M}E_{ka}\,{}^tE_{ka})-2\pi t\sigma(\mathcal{M}W\,{}^tE_{ka})}\,\Phi_{\mathcal{M},J}(W+tE_{ka})
\end{aligned}$$

$$= -2\pi \left(\sum_{m=1}^{h} \mathcal{M}_{mk} W_{ma} \right) \Phi_{\mathcal{M},J}(W) + \left.\frac{d}{dt}\right|_{t=0} \Phi_{\mathcal{M},J}(W + tE_{ka})$$

$$= -2\pi \left(\sum_{m=1}^{h} \mathcal{M}_{mk} W_{ma} \right) \Phi_{\mathcal{M},J}(W) + \sum_{m=1}^{h} \tau_{mk}\, J_{ma}^{\frac{1}{2}}\, \Phi_{\mathcal{M},J-\epsilon_{ma}}(W)$$

Finally,

$$dU^{F,\mathcal{M}}(\hat{D}_{lb})\, \Phi_{\mathcal{M},J}(W)$$

$$= \left.\frac{d}{dt}\right|_{t=0} U^{F,\mathcal{M}}(\exp t\hat{X}_{lb})\Phi_{\mathcal{M},J}(W)$$

$$= \left.\frac{d}{dt}\right|_{t=0} U^{F,\mathcal{M}}\left((0, tE_{lb}, 0)\right) \Phi_{\mathcal{M},J}(W)$$

$$= \left.\frac{d}{dt}\right|_{t=0} e^{-\pi t^2 \sigma(\mathcal{M}E_{lb}\, {}^tE_{lb}) + 2\pi i t\, \sigma(\mathcal{M}W\, {}^tE_{lb})}\, \Phi_{\mathcal{M},J}(W + itE_{lb})$$

$$= 2\pi i \left(\sum_{m=1}^{h} \mathcal{M}_{ml} W_{mb} \right) \Phi_{\mathcal{M},J}(W) + \left.\frac{d}{dt}\right|_{t=0} \Phi_{\mathcal{M},J}(W + itE_{lb})$$

$$= 2\pi i \left(\sum_{m=1}^{h} \mathcal{M}_{ml} W_{mb} \right) \Phi_{\mathcal{M},J}(W) + i \sum_{m=1}^{h} \tau_{ml}\, J_{mb}^{\frac{1}{2}}\, \Phi_{\mathcal{M},J-\epsilon_{mb}}(W).$$

□

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