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ISOMORPHISM CLASSES OF CAYLEY PERMUTATION GRAPHS

YUNSUN NAM

ABSTRACT. In this paper, we study the isomorphism problem of Cayley permutation graphs. We obtain a necessary and sufficient condition that two Cayley permutation graphs are isomorphic by a direction-preserving and color-preserving (positive/negative) natural isomorphism. The result says that if a graph G is the Cayley graph for a group Γ then the number of direction-preserving and color-preserving positive natural isomorphism classes of Cayley permutation graphs of G is the number of double cosets of Γ^ℓ in S_Γ, where S_Γ is the group of permutations on the elements of Γ and Γ^ℓ is the group of left translations by the elements of Γ. We obtain the number of the isomorphism classes by counting the double cosets.

1. Introduction

The permutation graphs were introduced as a generalization of the Petersen graph by Chartrand and Harary in [1]. For a labeled graph G with vertex set $V(G) = \{1, 2, 3, \cdots, n\}$ and a permutation α in S_n, the *α-permutation graph of* G, $P_\alpha(G)$, consists of two disjoint copies of G, G_v and G_w, with vertex sets $V(G_v) = \{v_1, v_2, \cdots, v_n\}$ and $V(G_w) = \{w_1, w_2, \cdots, w_n\}$, along with the edges $(v_i, w_{\alpha(i)})$ for $1 \leq i \leq n$.

The isomorphism problem of permutation graphs has been studied by many people. Holton and Stacey studied on path permutation graphs in [4] and many people studied on cycle permutation graphs in [5, 6, 8]. In this paper, we study the isomorphism problem of Cayley permutation graphs.

Let Γ be a finite group and X be a generating set for Γ. The *Cayley graph G for Γ and X* is the graph whose vertex set and edge set are

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defined as follows:

$$V(G) = \Gamma; \qquad E(G) = \{(g, gx) \mid g \in \Gamma,\, x \in X\}.$$

Let S_Γ denote the group of permutations on the elements of Γ. Let α be a permutation in S_Γ. Then the permutation graph $P_\alpha(G)$ has as its vertex set $\{v_g,\, w_g \mid g \in \Gamma\}$, and as its edge set $\{(v_g, v_{gx}),\, (w_g, w_{gx}),\, (v_g, w_{\alpha(g)}) \mid g \in \Gamma,\, x \in X\}$. Specially we call the permutation graph $P_\alpha(G)$ the *Cayley α-permutation graph of G* when G is a Cayley graph.

Throughout this paper, a graph G is the Cayley graph for a group Γ and a generating set X unless any remark is given.

Given a graph G and two permutations α and β in S_Γ. The graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a *positive natural isomorphism* ϕ if ϕ restricted to G_v is G_v and thus ϕ restricted to G_w is G_w. The graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a *negative natural isomorphism* ϕ if ϕ restricted to G_v is G_w and thus ϕ restricted to G_w is G_v. The graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a *natural isomorphism* ϕ if ϕ is either a positive natural isomorphism or a negative natural isomorphism.

A Cayley graph is called a *Cayley color graph* if a direction and a color are designated for each edge as follows: an edge (g, gx) has the direction from g to gx and the color x. An automorphism ϕ of a Cayley color graph Γ is said to be *direction-preserving and color-preserving* if an edge is mapped to an edge with the same color and the tail of an edge is mapped to the tail of the image edge, that is, $\phi(gx) = hx$ when $\phi(g) = h$. We extend the definition of direction-preserving and color-preserving to natural isomorphisms of Cayley permutation graphs. We say that *a natural isomorphism ϕ from $P_\alpha(G)$ to $P_\beta(G)$ is direction-preserving and color-preserving (DPCP)* if $\phi(v_{gx}) = v_{hx}$ and $\phi(w_{jx}) = w_{kx}$ when $\phi(v_g) = v_h$ and $\phi(w_j) = w_k$ (respectively, $\phi(v_{gx}) = w_{hx}$ and $\phi(w_{jx}) = v_{kx}$ when $\phi(v_g) = w_h$ and $\phi(w_j) = v_k$).

From the definition, we can easily tell that a DPCP positive natural isomorphic or a DPCP natural isomorphic relation is an equivalence relation on the set of all Cayley permutation graphs of G but a DPCP negative natural isomorphic relation is not.

In this paper, we obtain a necessary and sufficient condition that $P_\alpha(G)$ and $P_\beta(G)$ are isomorphic by a DPCP (positive/negative) natural isomorphism (Theorem 4). Using this condition, we show that the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of G is equal to the number of double cosets of Γ^ℓ in S_Γ, where Γ^ℓ is the group of left translations by the elements of Γ. Using group action we compute the number of double cosets of Γ^ℓ in S_Γ, and consequently obtain the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of G (Theorem 7).

The group Γ can be identified with the subgroup Γ^ℓ of S_Γ by the natural monomorphism $g \mapsto \ell_g$ where ℓ_g is the left translation by g. In this sense, the number of double cosets of Γ in S_Γ is equal to the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of G.

2. Necessary and sufficient condition

In this section, we obtain a necessary and sufficient condition that two Cayley permutation graphs are isomorphic by a DPCP (positive/negative) natural isomorphism.

The following three lemmas will be used to prove Theorem 4, which gives a necessary and sufficient condition for two permutation graphs $P_\alpha(G)$ and $P_\beta(G)$ to be isomorphic by a DPCP (positive/negative) natural isomorphism. The first lemma is a partial result of Frucht's theorem and we state it here without proof (see [2, p. 70]).

LEMMA 1. [2] *Let G be the Cayley graph for a group Γ and a generating set X. Then the set of automorphisms that are direction-preserving and color-preserving forms a group with function composition, which is the group $\Gamma^\ell = \{\ell_g \mid g \in \Gamma\}$.*

We have used ℓ_g to denote the left translation by an element g in Γ. We will let r_g denote the right translation by g. If an edge e is in G_v, then the ends of e are v_g and v_{gx} for some $g \in \Gamma$ and $x \in X$. Since gx is $r_x(g)$, the ends of e are v_g and $v_{r_x(g)}$. It is similar for an edge in G_w.

LEMMA 2. *Let G be the Cayley graph for a group Γ and a generating set X, and let α and β be two permutations in S_Γ. Then*

(i) *the graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a DPCP positive natural isomorphism if and only if there exists $a \in \Gamma$ such that $\alpha^{-1}r_x\alpha = \ell_a^{-1}\beta^{-1}r_x\beta\ell_a$ for all $x \in X$;*

(ii) *the graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a DPCP negative natural isomorphism if and only if there exists $a \in \Gamma$ such that $\alpha^{-1}r_x\alpha = \ell_a^{-1}\beta r_x\beta^{-1}\ell_a$ for all $x \in X$; and*

(iii) *the graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a DPCP natural isomorphism if and only if there exists $a \in \Gamma$ such that $\alpha^{-1}r_x\alpha = \ell_a^{-1}\beta^{-1}r_x\beta\ell_a$ for all $x \in X$ or $\alpha^{-1}r_x\alpha = \ell_a^{-1}\beta r_x\beta^{-1}\ell_a$ for all $x \in X$.*

Proof. (i) Suppose that $P_\alpha(G)$ and $P_\beta(G)$ are isomorphic by a DPCP positive natural isomorphism ϕ. Then the restriction of ϕ to the G_v of $P_\alpha(G)$ is a DPCP isomorphism to the G_v of $P_\beta(G)$, which induces a left translation ℓ_a in Γ^ℓ (by Lemma 1). The path $v_g w_{\alpha(g)} w_{r_x\alpha(g)} v_{\alpha^{-1}r_x\alpha(g)}$ in $P_\alpha(G)$ is mapped to the path $v_{\ell_a(g)} w_{\beta\ell_a(g)} w_{r_x\beta\ell_a(g)} v_{\beta^{-1}r_x\beta\ell_a(g)}$ in $P_\beta(G)$. Thus $\alpha^{-1}r_x\alpha = \ell_a^{-1}\beta^{-1}r_x\beta\ell_a$.

Now prove the other direction. Suppose that $\alpha^{-1}r_x\alpha = \ell_a^{-1}\beta^{-1}r_x\beta\ell_a$ for some $a \in \Gamma$. Define the map $\phi : P_\alpha(G) \to P_\beta(G)$ by $\phi(v_g) = v_{\ell_a(g)}$ and $\phi(w_g) = w_{\beta\ell_a\alpha^{-1}(g)}$. Then the map ϕ is a DPCP positive natural isomorphism.

We prove only (i) here, because the proof of (ii) is similar to that of (i) and (iii) follows from (i) and (ii). □

LEMMA 3. *Let Γ be a group and X be a generating set for Γ. Let $X^r = \{r_x \mid x \in X\}$. Then the center of X^r in S_Γ, $C(X^r)$, is the set Γ^ℓ.*

Proof. Clearly, a left translation ℓ_a is in $C(X^r)$. If σ is in $C(X^r)$, then $\sigma(gx) = \sigma(g)x$ for all $x \in X$ and all $g \in \Gamma$. If $g = x_1x_2\cdots x_k$, then

$$\begin{aligned}\sigma(g) &= \sigma(x_1x_2\cdots x_k) = \sigma(x_1x_2\cdots x_{k-1})x_k = \cdots \\ &= \sigma(x_1)x_2\cdots x_k = \sigma(e)x_1x_2\cdots x_k = \sigma(e)g,\end{aligned}$$

where e is the identity of the group Γ. Thus the permutation σ is the left translation by $\sigma(e)$. □

Now, we give one of our main theorems.

THEOREM 4. *Let G be the Cayley graph for a group Γ and a generating set X, and let α and β be two permutations in S_Γ. Then*

(i) *the graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a DPCP positive natural isomorphism if and only if $\beta \in \Gamma^\ell\alpha\Gamma^\ell$;*
(ii) *the graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a DPCP negative natural isomorphism if and only if $\beta \in \Gamma^\ell\alpha^{-1}\Gamma^\ell$; and*
(iii) *the graph $P_\alpha(G)$ is isomorphic to $P_\beta(G)$ by a DPCP natural isomorphism if and only if $\beta \in \Gamma^\ell\alpha\Gamma^\ell \cup \Gamma^\ell\alpha^{-1}\Gamma^\ell$.*

Proof. (i) Suppose that the graph $P_\alpha(G)$ and $P_\beta(G)$ are isomorphic by a DPCP positive natural isomorphism. By Lemma 2, there exists

$a \in \Gamma$ such that $\alpha^{-1} r_x \alpha = \ell_a^{-1} \beta^{-1} r_x \beta \ell_a$ for all $x \in X$. Thus $\beta \ell_a \alpha^{-1}$ is in the center $C(X^r)$ $(= \Gamma^\ell)$ and so β belongs to $\Gamma^\ell \alpha \Gamma^\ell$.

Now suppose that β is in $\Gamma^\ell \alpha \Gamma^\ell$. Then $\beta = \ell_a \alpha \ell_b$ for some a and b in Γ. Since ℓ_a is in the center $C(X^r)$, $\beta^{-1} r_x \beta = \ell_b^{-1} \alpha^{-1} r_x \alpha \ell_b$ for all $x \in X$. By Lemma 2, the graph $P_\alpha(G)$ and $P_\beta(G)$ are isomorphic by a DPCP positive natural isomorphism.

We prove only (i) here, because the proof of (ii) is similar to that of (i) and (iii) follows from (i) and (ii). □

3. Counting

We have already noted in Section 1 that a DPCP positive natural isomorphic or a DPCP natural isomorphic relation is an equivalence relation on the set of all Cayley permutation graphs of G, but a DPCP negative isomorphic relation is not. In this section, we compute the numbers of the equivalence classes, i.e., the DPCP positive natural isomorphism classes and the DPCP natural isomorphism classes. Theorem 4 says that the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of G is equal to the number of double cosets of Γ^ℓ in S_Γ. Using group action we compute the number of the double cosets, and consequently obtain the number of the DPCP positive natural isomorphism classes.

In Section 1, we already noted that the group Γ can be identified with the subgroup Γ^ℓ of S_Γ and in this sense the number of double cosets of Γ in S_Γ is equal to the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of G.

Now we consider the following group action to compute the number of double cosets of Γ^ℓ in S_Γ. Define an action $[\Gamma \times \Gamma, S_\Gamma] \to S_\Gamma$ by $(a, b)\alpha \mapsto \ell_a \alpha \ell_b$. Let $\mathrm{Fix}_{(a,b)}$ denote the set of fixed points of (a, b), i.e.,

$$\mathrm{Fix}_{(a,b)} = \{\alpha \in S_\Gamma \mid (a, b)\alpha = \alpha\}.$$

We let $o(g)$ denote the order of g in Γ.

LEMMA 5. *Let a and b be any elements in a group Γ.*

(i) *If $o(a)$ and $o(b)$ are different, then $\mathrm{Fix}_{(a,b)} = \emptyset$.*

(ii) *If $o(a)$ and $o(b)$ are the same, then*

$$|\mathrm{Fix}_{(a,b)}| = i! \cdot d^i$$

where $d = o(a) = o(b)$ and $i = \frac{|\Gamma|}{d}$.

Proof. (i) Assume that there exists a permutation α in $\mathrm{Fix}_{(a,b)}$. Then $\alpha(g) = \ell_a \alpha \ell_b(g)$ for all $g \in \Gamma$, i.e., $\alpha(bg) = a^{-1}\alpha(g)$ for all $g \in \Gamma$. Since $\alpha(b^{o(a)}) = a^{-o(a)}\alpha(e) = \alpha(e)$ and α is one-to-one, $b^{o(a)} = e$. So $o(b)$ divides $o(a)$. Similarly, we conclude that $o(a)$ divides $o(b)$. This contradicts that $o(a)$ and $o(b)$ are different.

(ii) Let α be any element in $\mathrm{Fix}_{(a,b)}$. Let $\{<a>a_1, <a>a_2, \cdots, <a>a_\imath\}$ and $\{<b>b_1, <b>b_2, \cdots, <b>b_\imath\}$ be the sets of all right cosets of $<a>$ and of $<b>$, respectively. (Here, $<a>$ denotes the cyclic subgroup of Γ generated by a.) If $\alpha(b_k) = a^l a_j$, then $\alpha(b^m b_k) = a^{l-m}a_j$ for all m. Since all the elements in the same right coset of $<b>$ are mapped by α to elements in the same right coset of $<a>$ and the order of a coset of $<a>$ is the same as that of $<b>$, there is one-to-one correspondence between the sets of right cosets of $<a>$ and of $<b>$. And the images of all the elements in $<b>b_k$ are uniquely determined by the image $\alpha(b_k)$. Thus $|\mathrm{Fix}_{(a,b)}|$ is $\imath! \cdot d^\imath$. □

LEMMA 6. *Let Γ be a group. Then the number of double cosets of Γ^ℓ in S_Γ is*

$$\frac{1}{|\Gamma|^2} \sum_{d||\Gamma|} n_d^2 \cdot \imath! \cdot d^\imath,$$

where n_d is the number of the elements of order d in Γ and $\imath = \frac{|\Gamma|}{d}$.

Proof. The orbit of α is the double coset $\Gamma^\ell \alpha \Gamma^\ell$. Thus it follows from Burnside's lemma [7, p. 163] and Lemma 5. □

Now we are ready to compute the number of DPCP (positive) isomorphism classes of Cayley permutation graphs of G.

THEOREM 7. *Let G be the Cayley graph for a group Γ and a generating set X. Then*

(i) *the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of G is*

$$\frac{1}{|\Gamma|^2} \sum_{d||\Gamma|} n_d^2 \cdot \imath! \cdot d^\imath,$$

where n_d is the number of the elements of order d in Γ and $\imath = \frac{|\Gamma|}{d}$; and

(ii) *the number of DPCP natural isomorphism classes of Cayley permutation graphs of G is*

$$\frac{1}{2}\left\{ \frac{1}{|\Gamma|^2}\sum_{d||\Gamma|} n_d^2 \cdot \imath! \cdot d^\imath + |\{\overline{\alpha} \mid \alpha\Gamma^\ell\alpha \cap \Gamma^\ell \neq \emptyset, \alpha \in S_\Gamma\}| \right\},$$

where n_d and $\imath$ are the same as in (i) *and $\overline{\alpha}$ is the orbit of α under the group action defined at the beginning of this section.*

Proof. (i) It follows from Theorem 4 (i) and Lemma 6.

(ii) By Theorem 4 (iii), the DPCP natural isomorphism class of $P_\alpha(G)$ is $\overline{\alpha} \cup \overline{\alpha^{-1}}$. The orbits $\overline{\alpha}$ and $\overline{\alpha^{-1}}$ are the same if and only if $\alpha^{-1} \in \Gamma^\ell\alpha\Gamma^\ell$. And $\alpha^{-1} \in \Gamma^\ell\alpha\Gamma^\ell$ if and only if $\alpha\Gamma^\ell\alpha \cap \Gamma^\ell \neq \emptyset$. □

Example:

1. Let G be the Cayley graph for the cyclic group Z_p (p is a prime) and the generating set $\{1\}$. Then G is the cycle C_p of length p. Since every element except the identity has order p, the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of C_p is:

$$\frac{1}{p}[(p-1)! + (p-1)].$$

2. Let G be the Cayley graph for the group $Z_p \times Z_q$ (p and q are distinct primes) and the generating set $\{(1,0),(0,1)\}$. Then G is the discrete torus $C_p \times C_q$. Since the group has one element of order one, $p-1$ elements of order p, $q-1$ elements of order q and $(p-1)(q-1)$ elements of order pq, the number of DPCP positive natural isomorphism classes of Cayley permutation graphs of $C_p \times C_q$ is:

$$\frac{1}{pq}[(pq-1)!+(p-1)^2(q-1)!p^{(q-1)}+(q-1)^2(p-1)!q^{(p-1)}+(p-1)^2(q-1)^2].$$

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Global Analysis Research Center
Department of Mathematics
Seoul National University
Seoul 151-742, Korea