J. Korean Math. Soc. **34** (1997), No. 2, pp. 321–336

ON CURVATURE PINCHING FOR TOTALLY REAL SUBMANIFOLDS OF $HP^n(c)$

YOSHIO MATSUYAMA

ABSTRACT. Let S be the Ricci curvature of an n-dimensional compact minimal totally real submanifold M of a quaternion projective space $HP^n(c)$ of quaternion sectional curvature c. We proved that if $S \leq \frac{3(n-2)}{16}c$, then either $S \equiv \frac{n-1}{4}c$ (i.e. M is totally geodesic or $S \equiv \frac{3(n-2)}{16}c$. All compact minimal totally real submanifolds of $HP^n(c)$ satisfy in $S \equiv \frac{3(n-2)}{16}c$ are determined.

1. Introduction

Let $HP^n(c)$ be an $n-$dimensional quaternion projective space with constant quaternion sectional curvature $c\,(> 0)$ and let M be an $m-$dimensional totally real submanifold isometrically immersed in $HP^n(c)$. Let h be the second fundamental form of M in $HP^n(c)$.

In [4] Funabashi showed: Let M be an $n-$dimensional totally real minimal submanifold isometrically immersed in $HP^n(c)$. If

$$|h|^2 < \frac{n+1}{4(6n-2)}c$$

if and only if M is totally geodesic and of constant curvature $\frac{c}{4}$.

Recall the totally real imbeddings [4] and [11]:

$$\nu : RP^n(\frac{1}{4}) \to HP^n(1),$$

Received August 30, 1995.
1991 Mathematics Subject Classification: 53C40, 53C42.
Key words and phrases: quaternion projective space, totally real, minimal, Ricci curvature.

and the first standard imbeddings of projective spaces:

$$\overline{\psi_1} : RP^2(\frac{1}{12}) \to RP^4(\frac{1}{4})$$
$$\overline{\psi_2} : CP^2(\frac{1}{3}) \to RP^7(\frac{1}{4})$$
$$\overline{\psi_3} : HP^2(\frac{1}{3}) \to RP^{13}(\frac{1}{4})$$
$$\overline{\psi_4} : CayP^2(\frac{1}{3}) \to RP^{25}(\frac{1}{4})$$

Moreover, Houh [6] proved: (A) Let M be an $n-$dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. If the sectional curvature γ of M satisfies

$$\gamma \geq \frac{n-2}{4(2n-1)}c,$$

then either M is totally geodesic in $HP^n(c)$ or $n = 2$ and M is flat.
(B) Let M be an $m-$dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. If the sectional curvature γ of M satisfies

$$\gamma \geq \frac{m-1}{4(2m-1)}c$$

then either M is totally geodesic in $HP^n(c)$ or $m = 2, n \geq 4$ and M is the Veronese surface in $HP^n(c)$ with positive constant curvature $\frac{c}{12}$.

Using the method of Chen and Ogiue [1], we can prove that: (A1) Let M *be an* $n-$*dimensional compact totally real minimal submanifold isometrically immersed in* $HP^n(c)$. *If*

$$|h|^2 \leq \frac{n(n+1)}{4(2n-1)}c,$$

then either M *is totally geodesic in* $HP^n(2)$ *or* $n = 2$ *and* M *is flat.*
(B1) *Let* M *be an* m*-dimensional compact totally real minimal submanifold isometrically immersed in* $HP^n(c)$. *If*

$$|h|^2 \leq \frac{m^2}{4(2m-1)}c$$

then either M is totally geodesic in $HP^n(c)$ or $m = 2, n \geq 4$ and M is the Veronese surface in $HP^n(c)$ with positive constant curvatrue $\frac{c}{12}$.

Recently, Coulton and Gauchman [3] proved the following: Let M be an m−dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. Then

$$|h(v,v)|^2 \leq \frac{1}{12}c$$

for all unit tangent vector $v \in T_xM$ if and only if one of the following conditions is satisfied: a) $|h(v,v)|^2 \equiv 0$ and M is totally geodesic, b) Max $\{|h(v,v)|^2\} = \frac{1}{12}c$ and M is either congruent to one of the imbeddings $\psi_i = \nu \circ \overline{\psi_i}$ or to the immersion $\psi_5 = \psi_1 \circ \pi$, where $\pi : S^2(\frac{1}{12}c) \to RP^2(\frac{1}{12}c)$ is the covering map.

Moreover, using the methods of Gauchman [5] and Xia [12], we can prove that: (A2) *Let M be an n−dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. If n is odd and*

$$|h(v,v)|^2 \leq \frac{n+1}{12n-8}c,$$

then M is totally geodesic. (B2) *Let M be an m−dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. If m is odd and*

$$|h(v,v)|^2 \leq \frac{m}{12m-8}c,$$

the M is totally geodesic. (A3) *Let M be an n−dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. If*

$$|h|^2 \leq \frac{n+1}{6}c,$$

then either M is totally geodesic or $n = 2$ and M is flat. (B3) *Let M be an m−dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. If*

$$|h|^2 \leq \frac{m}{6}c,$$

then either M is totally geodesic in $HP^n(c)$ or $m = 2, n \geq 4$ and M is the Veronese surface in $HP^n(c)$ with positive constant curvature $\frac{c}{12}$.

The purpose of this paper is to prove the following:

THEOREM 1. *Let M be an $n-$dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. Then the Ricci curvature S of M satisfies*

$$S \geq \frac{3(n-2)}{16}c$$

if and only if the following condition is satisfied: M is totally geodesic and M is an imbedded submanifold congruent to one of the standard immbedding $i : RP^n(\frac{c}{4}) \to HP^n(c)$ or to the standard immersion $i \circ \pi : S^n(\frac{c}{4}) \to HP^n(c)$, where $\pi : S^n(\frac{c}{4}) \to RP^n(c)$ or $n = 2$ and M is flat.

THEOREM 2. *Let M be an $m-$dimensional compact totally real minimal submanifold isometrically immersed in $HP^n(c)$. Then S of M satisfies*

$$S \geq (\frac{m-1}{4} - \frac{3(m+2)}{8(2m+5)})c$$

if and only if one of the following conditions is satisfied:

A) $S = \frac{m-1}{4}c$ and M is totally geodesic,

B) $S = (\frac{m-1}{4} - \frac{3(m+2)}{8(2m+5)})c$ and M is an imbedded submanifold congruent to one of the imbeddings $\psi_i = \nu \circ \overline{\psi_i}$ or to the immersion $\psi_5 = \psi_1 \circ \pi$, where $\pi : S^2(\frac{1}{12}c) \to RP^2(\frac{1}{12}c)$ is the covering map.

2. Preliminaries

Let $\overline{M}$ be a differentiable manifold of dimension $4n$, and assume that there is a 3-dimensional vector bundle V, [7], consisting of tensors of type (1,1) over $\overline{M}$ satisfying the following condition: in any coordinate neighborhood U of $\overline{M}$ there is a local base $\{I, J, K\}$ of V called a *canonical local base* of V such that

$$\begin{aligned} &I^2 = J^2 = K^2 = -Id, \\ &IJ = -JI = K; JK = -KJ = I; KI = -IK = J, \end{aligned} \tag{2.1}$$

where Id denotes the identity tensor field of type (1,1). If $\overline{M}$ is a manifold and V is a bundle over $\overline{M}$ satisfying the above condition then $(\overline{M}, V)$ is called an *almost quaternion* manifold. If $\overline{g}$ is a Riemannian metric for $(\overline{M}, V)$ such that $\overline{g}(\varphi\overline{X}, \overline{Y}) + \overline{g}(\overline{X}, \varphi\overline{Y}) = 0$, holds for any cross section φ of V, with $\overline{X}, \overline{Y} \in T\overline{M}$, then $(\overline{M}, V, \overline{g})$ is called an *almost quaternion metric* manifold.

Assume that the Riemannian connection $\overline{\nabla}$ of $(\overline{M}, V, \overline{g})$ satisfies the following condition: if φ is a local cross section of the bundle V, then $\overline{\nabla}_{\overline{X}}\varphi$ is also a local cross section of V, where $\overline{X}$ is an arbitrary vector field. In this case $\overline{M} = (\overline{M}, V, \overline{g})$ is called a *Kaehler quaternion* manifold.

Let $\overline{x} \in \overline{M}$ and $\overline{X} \in T_{\overline{x}}\overline{M}$. Consider the 4-dimensional subspace $Q(\overline{X})$ in $T_{\overline{x}}\overline{M}$ defined by

$$Q(\overline{X}) = \mathrm{Span}_R\{\overline{X}, I\overline{X}, J\overline{X}, K\overline{X}\}$$

We call this the $Q-$section generated by $\overline{X}$. If for all $\overline{x} \in \overline{M}, \overline{X} \in T_{\overline{x}}\overline{M}$ and $\overline{Y}, \overline{Z} \in Q(\overline{X})$ the sectional curvature $\sigma(\overline{Y}, \overline{Z}) = c$ (a constant), then we say that $\overline{M}$ is a Kaehler quaternion manifold of constant $Q-$sectional curvature c. In addition, such a manifold is called a *quaternion space form.*

The curvature operator $\overline{R}$ of a quaternionic space form $\overline{M} = (\overline{M}, V, \overline{g})$ has the form:

$$\begin{aligned}
\overline{R}(\overline{X}, \overline{Y})\overline{Z} = &\frac{c}{4}\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y} + \overline{g}(I\overline{Y}, \overline{Z})I\overline{X} \\
&- \overline{g}(I\overline{X}, \overline{Z})I\overline{Y} + 2\overline{g}(\overline{X}, I\overline{Y})I\overline{Z} + \overline{g}(J\overline{Y}, \overline{Z})J\overline{X} \\
&- \overline{g}(J\overline{X}, \overline{Z})J\overline{Y} + 2\overline{g}(\overline{X}, J\overline{Y})J\overline{Z} + \overline{g}(K\overline{Y}, \overline{Z})K\overline{X} \\
&- \overline{g}(K\overline{X}, \overline{Z})K\overline{Y} + 2\overline{g}(\overline{X}, K\overline{Y})K\overline{Z}\},
\end{aligned} \tag{2.2}$$

where c is the $Q-$sectional curvature. It is well known that the quaternion projective space $HP^n(c)$ is a compact $4n-$dimensional quaternion space form.

Let $(\overline{M}, V, \overline{g})$ be a Kaehler quaternion manifold and let M be a Riemannian submanifold isometrically immersed in $\overline{M}$. We say that M is a totally real submanifold of $\overline{M}$, [4], if

$$\theta(T_xM) \perp T_xM$$

for any $x \in M$, and any $\theta \in V_x$, where V_x is the fibre of V over x. Recall that h is the second fundamental form.

LEMMA 1. *Assume that M is a totally real submanifold of a Kaehler quaternion manifold. Then*
$\overline{g}(h(X,Y),IZ),\overline{g}(h(X,Y),JZ)$ *and* $\overline{g}(h(X,Y),KZ)$ *are symmetric with respect to* X,Y *and* $Z \in T_xM, x \in M$.

Let M be a compact Riemannian manifold, UM its unit tangent bundle, and UM_x the fibre of UM over a point x of M. We denote by dx, dv and dv_x denote the canonical measures on M, UM and UM_x respectively.

For any continuous function $f : UM \to R$, we have

$$\int_{UM} f dv = \int_M (\int_{UM_x} f dv_x) dx.$$

If T is a $k-$covariant tensor on M and ∇T is its covariant derivative, then we have:

$$\int_{UM} \{\sum_{i=1}^{m} (\nabla T)(e_i, e_i, v, \cdots, v)\} dv = 0,$$

where $e_1, \cdots, e_m$ is an orthonormal basis of $T_xM,\ x \in M$.

Now, we suppose that M is an $m-$dimensional compact Riemannian manifold isometrically immersed in a Riemannian manifold $\overline{M}$. To simplify notation, we henceforth write $\overline{g}(\ ,\ ) = <\ ,\ >$. We denote by $<\ ,\ >$ the metric of $\overline{M}$ as well as that induced on M. Let h be the second fundamental form of the immersion.

Let X, Y, Z and W denote the tangent vector fields on M. Then if ∇h and $\nabla^2 h$ denote the first and second covariant derivatives of h, respectively, one has that ∇h is symmetric and $\nabla^2 h$ satisfies the following relation:

$$\begin{aligned}(\nabla^2 h)(X,Y,Z,W) =& (\nabla^2 h)(Y,X,Z,W) + R^{\perp}(X,Y)h(Z,W) \\ & - h(R(X,Y)Z,W) - h(Z,R(X,Y)W),\end{aligned} \tag{2.3}$$

where $R^{\perp}$ and R are the curvature operators of the normal and tangent bundles over M, respectively.

If S is the Ricci curvature of M and M is minimally immersed in $\overline{M}$, from Gauss equation we have :

$$S(v,w) = \sum_{i=1}^{m} \overline{R}(v, e_i, e_i, w) - \sum_{i=1}^{m} < A_{h(v,e_i)} e_i, w >, \tag{2.4}$$

where $\overline{R}$ is the curvature operator of $\overline{M}$.

Now let $v \in UM_x, x \in M$. If $e_2, \cdots, e_m$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2, \cdots, e_m\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1, e_2, \cdots, e_m\}$ is an orthonormal basis of T_xM. If we denote the Laplacian of $UM_x \cong S^{m-1}$ by Δ, then $\Delta f = e_2e_2f + \cdots + e_me_mf$, where f is a differentiable function on UM_x.

Define a function f_1 on $UM_x, x \in M$, by

$$\begin{aligned} f_1(v) &= |A_{h(v,v)}v|^2 \\ &= \sum_{i=1}^{m} < h(v,v), h(v,e_i) >^2 . \end{aligned}$$

Using the minimality of M we can prove that

$$\begin{aligned} (\Delta f_1)(v) =& -6(m+4)f_1(v) \\ &+ 8\sum_{i=1}^{m} < A_{h(v,v)}v, A_{h(v,e_i)}e_i > \\ &+ 8\sum_{i=1}^{m} < A_{h(v,v)}e_i, A_{h(v,e_i)}v > \\ &+ 8\sum_{i=1}^{m} < A_{h(v,e_i)}v, A_{h(v,e_i)}v > \\ &+ 2\sum_{i=1}^{m} < A_{h(v,v)}e_i, A_{h(v,v)}e_i > . \end{aligned} \tag{2.5}$$

Similarly, define $f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9$ and f_{10} by

$$\begin{aligned} f_2(v) &= \sum_{i=1}^{m} < A_{h(v,e_i)}v, A_{h(v,e_i)}v >, \\ f_3(v) &= \sum_{i=1}^{m} < A_{h(v,e_i)}v, A_{h(v,v)}e_i >, \\ f_4(v) &= \sum_{i,j=1}^{m} < A_{h(e_j,e_i)}e_j, A_{h(v,v)}e_i >, \end{aligned}$$

$$\begin{aligned}
f_5(v) &= \sum_{i=1}^{m} < A_{h(v,v)}v, A_{h(v,e_i)}e_i >,\\
f_6(v) &= \sum_{i,j=1}^{m} < A_{h(e_j,e_i)}e_j, A_{h(v,e_i)}v >,\\
f_7(v) &= \sum_{i,j=1}^{m} < A_{h(e_i,v)}e_i, A_{h(v,e_j)}e_j >,\\
f_8(v) &= \sum_{i=1}^{m} < A_{h(v,v)}e_i, A_{h(v,v)}e_i >,\\
f_9(v) &= |h(v,v)|^2,\\
f_{10}(v) &= \sum_{i=1}^{m} < A_{h(v,e_i)}e_i, v >,
\end{aligned}$$

respectively. Then we know that

$$\begin{aligned}
(2.6)\qquad (\Delta f_2)(v) = &- 4(m+2)f_2(v) + 4f_6(v)\\
&+ 4\sum_{i,j=1}^{m} < A_{h(e_j,e_i)}v, A_{h(v,e_i)}e_j >\\
&+ 2\sum_{i,j=1}^{m} < A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v >\\
&+ 2\sum_{i,j=1}^{m} < A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j >,
\end{aligned}$$

$$\begin{aligned}
(2.7)\qquad (\Delta f_3)(v) = &- 4(m+2)f_3(v) + 2f_4(v)\\
&+ 4\sum_{i,j=1}^{m} < A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i >\\
&+ 4\sum_{i,j=1}^{m} < A_{h(v,e_i)}e_j, A_{h(e_j,v)}e_i >,
\end{aligned}$$

$$(2.8)\qquad (\Delta f_4)(v) = -2mf_4(v),$$

$$(2.9)\qquad \begin{aligned}(\Delta f_5)(v) = &-4(m+2)f_5(v) + 4f_6(v)\\ &+4f_7(v) + 2f_4(v),\end{aligned}$$

$$(2.10)\qquad (\Delta f_6)(v) = -2mf_6(v) + 2\sum_{i,j,k=1}^{m} < A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k >,$$

$$(2.11)\qquad (\Delta f_7)(v) = -2mf_7(v) + 2\sum_{i,j,k=1}^{m} < A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k >,$$

$$(2.12)\qquad \begin{aligned}(\Delta f_8)(v) = &-4(m+2)f_8(v)\\ &+8\sum_{i,j=1}^{m} < A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i >,\end{aligned}$$

$$(2.13)\qquad (\Delta f_9)(v) = -4(m+2)f_9(v) + 8\sum_{i=1}^{m} < A_{h(v,e_i)}e_i, v > .$$

$$(2.14)\qquad (\Delta f_{10})(v) = -2mf_{10}(v) + 2|h|^2.$$

Then we have the following (See [8] and [9]):

LEMMA 2. *Let M be an $m-$dimensional compact minimal submanifold isometrically immersed in $\overline{M}$. Then for all $x \in M$, we have*

$$(2.15)\qquad \begin{aligned}&\int_{UM_x} |A_{h(v,v)}v|^2 dv_x\\ &\geq \frac{2}{m+2}\int_{UM_x}\sum_{i=1}^{m} < A_{h(v,e_i)}e_i, A_{h(v,v)}v > dv_x,\end{aligned}$$

where $\{e_i\}_{i=1}^m$ is an orthonormal basis of the tangent space T_xM to M at x.

The following is the well known Chern-Do Carmo-Kobayashi inequality (Lemma 1 in [2]):

LEMMA 3. *Under the same assumption of Lemma 2 we have*

$$(2.16)\qquad \begin{aligned}&(m+4)\int_{UM_x} f_1(v)dv_x - 4\int_{UM_x} f_5(v)dv_x\\ &-3\int_{UM_x} f_8(v)dv_x \geq \frac{-4}{m(m+2)}\int_{UM_x}|h|^4 dv_x\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{m}(\nabla^2 f_9)(e_i,e_i,v) =& \sum_{i=1}^{m} < (\nabla^2 h)(e_i,e_i,v,v), h(v,v) >\\ &+\sum_{i=1}^{m} < (\nabla h)(e_i,v,v), (\nabla h)(e_i,v,v) >,\end{aligned}$$

we have

LEMMA 4. *Let M be an $m-$dimensional totally real minimal submanifold isometrically immersed in $HP^n(c)$. Then for $v \in UM_x$ we have*

(2.17)

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{m}(\nabla^2 f_9)(e_i,e_i,v) =& \sum_{i=1}^{m}|(\nabla h)(e_i,v,v)|^2 + \frac{m}{4}c|h(v,v)|^2\\ &+2\sum_{i=1}^{m} < A_{h(v,v)}e_i, A_{h(e_i,v)}v >\\ &-2\sum_{i=1}^{m} < A_{h(v,e_i)}e_i, A_{h(v,v)}v >\\ &-\sum_{i=1}^{m} < A_{h(v,v)}e_i, A_{h(v,v)}e_i >\\ &+\frac{c}{4}\sum_{i=1}^{m}(< h(v,v), Ie_i >^2 + < h(v,v), Je_i >^2\\ &+ < h(v,v), Ke_i >^2)\end{aligned}$$

3. Proof of theorem 1

Since by (2.4) it holds

$$S(v,w) = \frac{n-1}{4}c - \sum_{i=1}^{n} < A_{h(v,e_i)}e_i, w >,$$

we have only to prove Theorem 1 under the assumption of

$$\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > \leq \frac{n+2}{16}c. \tag{3.1}$$

Let $v \in UM_x, x \in M$. Since M is totally real, the following equations hold:

$$\begin{aligned} &\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i > \\ &= \sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(v,e_j)}e_i > \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v > \\ &= \sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j > . \end{aligned} \tag{3.3}$$

In terms of (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.17), (3.2) and (3.3) we obtain

(3.4)

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_9)(e_i,e_i,v) - \frac{1}{6}(\Delta f_1)(v) - \frac{1}{3(n+2)}(\Delta f_2)(v) \\ &\quad + \frac{1}{6(n+2)}(\Delta f_3)(v) + \frac{1}{3n(n+2)}(\Delta f_4)(v) + \frac{1}{6(n+2)}(\Delta f_5)(v) \\ &- \frac{1}{3n(n+2)}(\Delta f_6)(v) + \frac{1}{3n(n+2)}(\Delta f_7)(v) + \frac{1}{6(n+2)}(\Delta f_8)(v) \\ &= \sum_{i=1}^{n} |(\nabla h)(e_i,v,v)|^2 + \frac{n+1}{4}cf_9(v) \\ &\qquad + (n+4)f_1(v) - 4f_5(v) - 2f_8(v). \end{aligned}$$

From (2.15) we get

$$
\begin{aligned}
(3.5) \qquad 0 \geq & \int_{UM_x} \sum_{i=1}^{n} |(\nabla h)(e_i, v, v)|^2 dv_x \\
& + \int_{UM_x} (\frac{n+1}{4} c f_9(v) - \frac{2n}{n+2} f_5(v) - 2 f_8(v)) dv_x.
\end{aligned}
$$

Define a self-adjoint operator $L : T_x M \to T_x M$ by $Lv = \sum_{i=1}^{n} A_{h(v,e_i)} e_i$. If $e_1, \cdots, e_n$ is an orthonormal basis of $T_x M, x \in M$, such that $Le_i = \alpha_i e_i$ from (3.1) we have

$$
\begin{aligned}
(3.6) \qquad f_5(v) &= < L_v, A_{h(v,v)} v > \\
&\leq \frac{n+2}{16} c f_9(v)
\end{aligned}
$$

Since

$$
f_8(v) = |h(v,v)|^2 \sum_{i=1}^{n} < A_{\frac{h(v,v)}{|h(v,v)|}} e_i, A_{\frac{h(v,v)}{|h(v,v)|}} e_i >,
$$

we have

$$
(3.7) \qquad f_8(v) \leq \frac{n+2}{16} c f_9(v)
$$

Combining (3.5) and (3.6) with (3.7), we obtain

$$
\begin{aligned}
0 \geq & \int_{UM_x} \sum_{i=1}^{n} |(\nabla h)(e_i, v, v)|^2 dv_x \\
& + \int_{UM_x} (\frac{n+1}{4} c - \frac{n}{8} c - \frac{n+2}{8} c) f_9(v) dv_x.
\end{aligned}
$$

Thus we see that M is a submanifold of $HP^n(c)$ with parallel second fundamental form.

4. Proof of theorem 2

As in the proof of Theorem 1 by (2.4) since it holds

$$
S(v,w) = \frac{m-1}{4} c - \sum_{i=1}^{m} < A_{h(v,e_i)} e_i, w >,
$$

we have only to prove Theorem 2 under the assumption of

$$\sum_{i=1}^{m} < A_{h(v,e_i)}e_i, v > \leq \frac{3(m+2)}{8(2m+5)}c. \tag{4.1}$$

Let $v \in UM_x, x \in M$. Since M is totally real, in this case the following equations also hold:

$$\begin{aligned} &\sum_{i,j=1}^{m} < A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i > \\ &= \sum_{i,j=1}^{m} < A_{h(v,e_i)}e_j, A_{h(v,e_j)}e_i >, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\sum_{i,j=1}^{m} < A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v > \\ &= \sum_{i,j=1}^{m} < A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j >, \end{aligned} \tag{4.3}$$

(4.4)

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{m}(\nabla^2 f_9)(e_i, e_i, v) - \frac{1}{6}(\Delta f_1)(v) - \frac{1}{3(m+2)}(\Delta f_2)(v) \\ &+ \frac{1}{6(m+2)}(\Delta f_3)(v) + \frac{1}{3m(m+2)}(\Delta f_4)(v) + \frac{1}{6(m+2)}(\Delta f_5)(v) \\ &- \frac{1}{3m(m+2)}(\Delta f_6)(v) + \frac{1}{3m(m+2)}(\Delta f_7)(v) + \frac{1}{6(m+2)}(\Delta f_8)(v) \\ &= \sum_{i=1}^{m} |(\nabla h)(e_i, v, v)|^2 + \frac{m}{4}cf_9(v) + (m+4)f_1(v) - 4f_5(v) - 2f_8(v) \\ &+ \frac{c}{4}\sum_{i=1}^{m}(< h(v,v), Ie_i >^2 + < h(v,v), Je_i >^2 + < h(v,v), Ke_i >^2). \end{aligned}$$

Integrating (4.4) and multiplying it by $\frac{3}{2}$, we have

$$\begin{aligned}0 =&\frac{3}{2}\int_{UM_x}\sum_{i=1}^{m}|(\nabla h)(e_i,v,v)|^2 dv_x+\frac{3m}{8}c\int_{UM_x}f_9(v)dv_x\\&+\frac{3}{2}(m+4)\int_{UM_x}f_1(v)dv_x-6\int_{UM_x}f_5(v)dv_x-3\int_{UM_x}f_8(v)dv_x\\&+\frac{3c}{8}\int_{UM_x}\sum_{i=1}^{m}(<h(v,v),Ie_i>^2+<h(v,v),Je_i>^2\\&+<h(v,v),Ke_i>^2)dv_x.\end{aligned}$$

Using (2.16), we get

(4.5)

$$\begin{aligned}0 \geq&\frac{3}{2}\int_{UM_x}\sum_{i=1}^{m}|(\nabla h)(e_i,v,v)|^2 dv_x+\frac{3m}{8}c\int_{UM_x}f_9(v)dv_x\\&+\frac{1}{2}(m+4)\int_{UM_x}f_1(v)dv_x-2\int_{UM_x}f_5(v)dv_x-\frac{4}{m(m+2)}\int_{UM_x}|h|^4 dv_x\\&+\frac{3c}{8}\int_{UM_x}\sum_{i=1}^{m}(<h(v,v),Ie_i>^2+<h(v,v),Je_i>^2\\&+<h(v,v),Ke_i>^2)dv_x.\end{aligned}$$

From (2.15) we know that

$$\begin{aligned}&\frac{1}{2}(m+4)\int_{UM_x}f_1(v)dv_x-2\int_{UM_x}f_5(v)dv_x\\&\geq\frac{-m}{m+2}\int_{UM_x}f_5(v)dv_x.\end{aligned}\tag{4.6}$$

By means of (4.1) we obtain

$$f_5(v)\leq\frac{3(m+2)}{8(2m+5)}c|h(v,v)|^2,\tag{4.7}$$

$$|h|^2\leq\frac{3m(m+2)}{8(2m+5)}c.\tag{4.8}$$

Combining (4.5), (4.6) and (4.7) with (4.8), we have

$$\begin{aligned}0 \geq & \frac{3}{2}\int_{UM_x}\sum_{i=1}^{m}|(\nabla h)(e_i,v,v)|^2 dv_x + \frac{3m}{8}c\int_{UM_x} f_9(v)dv_x \\ & + \frac{3m}{8(2m+5)}\int_{UM_x} f_9(v)dv_x - \frac{12}{8(2m+5)}\int_{UM_x}|h|^2 dv_x \\ & + \frac{3c}{8}\int_{UM_x}\sum_{i=1}^{m}(< h(v,v), Ie_i >^2 + < h(v,v), Je_i >^2 \\ & + < h(v,v), Ke_i >^2)dv_x.\end{aligned}$$

Noting (2.13) and (2.14), we get

$$\int_{UM_x}|h|^2 dv_x = \frac{m(m+2)}{2}\int_{UM_x}|h(v,v)|^2 dv_x.$$

Hence,

$$\begin{aligned}0 \geq & \frac{3}{2}\int_{UM_x}\sum_{i=1}^{m}|(\nabla h)(e_i,v,v)|^2 dv_x \\ & + \int_{UM_x}(\frac{3m}{8}c - \frac{3m}{8}c)f_9(v)dv_x \\ & + \frac{3c}{8}\int_{UM_x}\sum_{i=1}^{m}(< h(v,v), Ie_i >^2 + < h(v,v), Je_i >^2 \\ & + < h(v,v), Ke_i >^2)dv_x.\end{aligned}$$

This proves Theorem 2.

References

[1] B. -Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1974), 257-266.

[2] S. -S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of sphere with second fundamental form of a constant length*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, III 1968), Springer, New York (1970), 59-75.

[3] P. Coulton and H. Gauchman, *Submanifolds of quaternion projective space with bounded second fundamental form*, Kodai Math. J. **12** (1989), 296-307.
[4] S. Funabashi, *Totally real submanifolds of a quaternionic Kaehlerian manifold*, Kodai Math. J. **29** (1978), 261-270.
[5] H. Gauchman, *Pinching theorems for totally real minimal submanifolds of $CP^n(c)$*, Tohoku Math. J. **41** (1989), 249-257.
[6] C. -S. Houh, *Some totally real submanifolds in a quaternion projective space*, Rocky Mountain J. Math. **10** (1980), 509-515.
[7] S. Ishihara, *Quaternion Kaehlerian manifolds*, J. Differential Geometry **9** (1974), 483-500.
[8] Y. Matsuyama, *On some pinchings of minimal submanifolds*, Proceedings of the Workshop on Geometry and its applications in Honor of Morio Obata, November 1991, edited by Tadashi Nagano et al., World Scientific, Singapore (1993), 121-134.
[9] S. Montiel, A. Ros and F. Urbano, *Curvature pinching and eigenvalue rigidity for minimal submanifolds*, Math. Z. **191** (1986), 537-548.
[10] H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex projective space*, J. Math. Soc. Japan **28** (1976), 638-667.
[11] K. Tsukada, *Parallel submanifolds in quaternion projective space*, Osaka J. Math. **22** (1985), 187-241.
[12] C. Xia, *Minimal submanifolds with bounded second fundamental form*, Math. Z. **208** (1991), 537-543.

Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112, Japan