J. Korean Math. Soc. **34** (1997), No. 2, pp. 279–284

THE INDEX OF THE CORESTRICTION OF A VALUED DIVISION ALGEBRA

YOON SUNG HWANG

ABSTRACT. Let L/F be a finite separable extension of Henselian valued fields with same residue fields $\bar{L} = \bar{F}$. Let D be an inertially split division algebra over L, and let ${}^cD$ be the underlying division algebra of the corestriction $\mathrm{cor}_{L/F}(D)$ of D. We show that the index $\mathrm{ind}({}^cD)$ of ${}^cD$ divides $[Z(\bar{D}) : Z(\overline{{}^cD})] \cdot \mathrm{ind}(D)$, where $Z(\bar{D})$ is the center of the residue division ring $\bar{D}$.

For any finite separable extension L/F of fields and any central simple algebra A over L, the corestriction of A is a central simple F-algebra obtained as the fixed point algebra under a Galois group action (cf.[**Ri**]). This induces the map from the Brauer group $Br(L)$ to $Br(F)$ corresponding to the homological corestriction. Though this algebraic corestriction is an important tool in the theory of division algebras, it is actually very hard to work with. To gain a better insight into the behavior of the corestriction, we analyze here the corestriction for valued division algebras over Henselian valued fields, for which there is a well-developed structure theory.

For any ring R we write $Z(R)$ and R^* for the center of R and the group of units of R, respectively. We will consider only central simple algebras A *finite-dimensional* over a field F. By Wedderburn's theorem, $A \cong M_n(D)$, a matrix ring over a division algebra D, which is called the *underlying division algebra* of A.

A valued field (F, v) is called *Henselian* if v extends uniquely to each field algebraic over F. For a nice account for several other characterizations of Henselian valuations, see Ribenboim's paper[**Rb**]. Recall (e.g.

Received July 20, 1994.
1991 Mathematics Subject Classification: 16K20.
Key words and phrases: Corestriction, Division Algebras, Henselian valuation.
This author was partially supported by NON DIRECTED RESEARCH FUND, Korea.

from[**W**]) that if D is a central division algebra over a Henselian valued field (F, v), there exists one and only one valuation on D extending v on F.

For a central L-division algebra D, ${}^cD$ denote the underlying division algebra of the corestriction $cor_{L/F}(D)$ of D. The index $ind({}^cD)$ of ${}^cD$ divides $ind(D)^{[L:F]}$. ([**D**, Lemma 7, p. 54]) We will show that when D is inertially split over L and L/F is a finite separable extension of Henselian valued fields with same residue fields $\overline{L}=\overline{F}$, the index $ind({}^cD)$ of ${}^cD$ divides $[Z(\overline{D}): Z(\overline{{}^cD})] \cdot ind(D)$, where $Z(\overline{D})$ is the center of the residue division ring $\overline{D}$. (See below for terminology.)

We now fix most of the basic terminology and notation that we will employ throughout this paper.

Let $(L, v) \supseteq (F, v)$ be a finite separable extension of Henselian fields. We say that L is inertial (or unramified) over F if $[\overline{L}: \overline{F}] = [L: F]$ and $\overline{L}$ is separable over $\overline{F}$.

Let (F, v) be a Henselian valued field. Let D be a central division F-algebra (with a unique valuation extending v on F). We say D is *tame and totally ramified* over F if char $(\overline{F}) \nmid [D: F]$ and $|\Gamma_D : \Gamma_F| = [D: F]$. D is said to be *inertially split* over F if D is split by F_{nr} where F_{nr} is the maximal unramified extension in some algebraic closure of F. Also, D is said to be *tame* if char $(\overline{F}) = 0$ or char $(\overline{F}) = q \neq 0$ and the q-primary component of D is split by F_{nr}. (See [**JW**, Lemma 5.1] and [**JW**, Lemma 6.1] for other characterizations of inertially split and tame division algebras.) Recall also that D is said to be inertial over F if $[\overline{D}: \overline{F}] = [D: F]$ and $Z(\overline{D}) = \overline{F}$. D is said to be nicely semiramified over F if D has a maximal subfield L which is inertial over F, and another maximal subfield K which is totally ramified of radical type over F. (Then, $\overline{D} = \overline{L}, \Gamma_D = \Gamma_K$ and $[\overline{D}: \overline{F}] = |\Gamma_D : \Gamma_F| = ind(D)$.) (See [**JW**, Sec. 4].) Let

$$\mathcal{D}(F) = \{D \mid D \text{ is a central division F-algebra with } [D: F] < \infty\}$$
$$\mathcal{D}_{ttr} = \{D \in \mathcal{D}(F) \mid D \text{ is tame and totally ramified over } F\}$$
$$\mathcal{D}_i(F) = \{D \in \mathcal{D}(F) \mid D \text{ is inertial over } F\}$$
$$\mathcal{D}_{is}(F) = \{D \in \mathcal{D}(F) \mid D \text{ is inertially split over } F\} \quad \text{and}$$
$$\mathcal{D}_t(F) = \{D \in \mathcal{D}(F) \mid D \text{ is tame over } F\}.$$

It is clear that $\mathcal{D}_i(F) \subseteq \mathcal{D}_{is}(F) \subseteq \mathcal{D}_t(F)$ and $\mathcal{D}_{ttr}(F) \subseteq \mathcal{D}_t(F)$.

$(K/F,\sigma,a)_n$ is the cyclic F-algebra generated over K by a single element x with defining relations $xcx^{-1} = \sigma(c)$ for all $c \in K$ and $x^n = a \in F^*$, where K is a Galois extension of F with cyclic Galois group generated by σ and $n = [K : F]$.

Now, we give a lemma to compute the corestriction $cor_{L/F}(D)$ of D of a NSR cyclic division algebra D over L when L/F is a finite separable extension of Henselian valued fields with $\overline{L} = \overline{F}$.

LEMMA 1. *Let L/F be a finite separable extension of Henselian valued fields with same residue fields $\overline{L}=\overline{F}$. Let $D = (M'/L,\sigma,\alpha)_{t'}$ be a NSR cyclic division algebra over L. (So, M'/L is inertial with cyclic Galois group generated by σ.) Let M be the inertial lift of $\overline{M'}$ over F. Then,*

$$cor_{L/F}(D) \sim (M/F,\sigma,N_{L/F}(\alpha))_{t'},$$

where $N_{L/F}$ is the norm map from L to F.

Proof. Let $L_{sep} = F_{sep}$ be the separable closure of L and F. Let $G = Gal(F_{sep}\,/\,F)$ and $H = Gal(L_{sep}\,/\,L)$

Since M/F is Galois and $L \cap M = F$, L and M are linearly disjoint over F and $L \otimes_F M$ is the field $L \cdot M = M'$. Let $N = Gal(F_{sep}\,/\,M)$. Then since M/F is Galois and $L \cap M = F$, N is normal in G and $G = HN$. Also, $Gal(M\,/\,F) \cong G/N \cong <\sigma>$ and $Gal(M'\,/\,L) \cong H/(H \cap N) \cong <\sigma>$. Since $L \otimes_F M$ is the field M', by [**D**, p. 56, Ex. 1] $cor_{L/F}(D) \otimes_F M \sim cor_{M'\,/\,M}(D \otimes_L M') \sim cor_{M'\,/\,M}(M') \sim M$ in $Br(M)$.

Since $D = (M'/L,\sigma,\alpha)_{t'} \in Br(M'\,/\,L) \cong H^2(H/(H \cap N), M'^*)$, D is represented by $\inf^H_{H/(H\cap N)}(f)$ where $f \in H^2(H/(H \cap N), M'^*)$ is given by $(\sigma^i,\sigma^j) \mapsto 1$ if $0 \leq i+j \leq n-1$ and $(\sigma^i,\sigma^j) \mapsto \alpha$ if $i+j \geq n$. Since the algebraic corestriction corresponds to the homological corestriction, in $Br(F)$, $[cor_{L/F}(D)]$ is represented by $cor^G_H(\inf^H_{H/(H\cap N)}(f))$. But, by [**H**, Th. 5] $cor^G_H(\inf^H_{H/(H\cap N)}(f)) = \inf^G_{G/N}(\mathcal{N}^*_{G/N}(f))$, where $\mathcal{N}^*_{G/N}\colon H^2(H/(H\cap N), M'^*) \to H^2(G/N, M^*)$ is induced by the norm map from M^* to M. Hence, $cor_{L/F}(D) \sim (M/F,\sigma,N_{L/F}(\alpha))_{t'}$ in $Br(F)$. □

We now can prove our theorem.

THEOREM 2. *Let L/F be a finite separable extension of Henselian valued fields with same residue fields $\overline{L}=\overline{F}$. If D is inertially split over L, then the index $ind({}^cD)$ of ${}^cD$ divides $[Z(\overline{D}):Z(\overline{{}^cD})]\cdot ind(D) = |\Gamma_D : \Gamma_{{}^cD}|\cdot ind(D)$, where $Z(\overline{D})$ is the center of the residue division ring $\overline{D}$, and Γ_D is the value group of D.*

Proof. Since D is inertially split over L, by [**JW**, Lemma 5. 14] there exist $I', N' \in \mathcal{D}(L)$ with I' inertial over L and N' NSR over L, such that $D \sim I' \otimes_L N'$ in $Br(L)$. Then by [**JW**, Th. 4. 4], $N' = \bigotimes_{i=1}^{k} (M_i'/L, \sigma_i, \alpha_i)_{t_i'}$ where M_i'/L is inertial cyclic Galois with $Gal(M_i' / L) =< \sigma_i >$.

By Lemma 1 above, $cor_{L/F}(N') \sim \bigotimes_{i=1}^{k} (M_i/F, \sigma_i, N_{L/F}(\alpha_i))_{t_i'}$ where M_i is the inertial lift of $\overline{M_i'}$ over F.

Let $a_i = N_{L/F}(\alpha_i)$ and let $v(a_i)$ map to an element of $\Gamma_F/t_i'\Gamma_F$ of order t_i. So, $t_i v(a_i) = t_i' v(p_i)$ for some $p_i \in F^*$, and $a_i = u_i p_i^{s_i}$ where $s_i = t_i'/t_i$ and u_i is a v-unit of F. Let K_i be an extension of F of degree t_i with $F \subseteq K_i \subseteq M_i$ and $Gal(K_i / F) =< \overline{\sigma_i} >$ where $\overline{\sigma_i}$ is the restriction of σ_i to K_i. Then by [**R**, Th. 30. 10, p. 262] $cor_{L/F}(N') \sim \bigotimes_{i=1}^{k} (M_i/F, \sigma_i, u_i)_{t_i'} \otimes_F (\bigotimes_{i=1}^{k} (K_i/F, \overline{\sigma_i}, p_i)_{t_i})$. Also, by [H, Lemma 4] ${}^cI' \in \mathcal{D}_i(F)$ and $\overline{{}^cI'} \sim \overline{{}^cI'}^{\otimes[L:F]}$ in $Br(F)$.

Let I be the underlying division algebra of ${}^cI' \otimes_F (\bigotimes_{i=1}^{k} (M_i/F, \sigma_i, u_i)_{t_i'})$ and let $N = \bigotimes_{i=1}^{k} (K_i/F, \overline{\sigma_i}, p_i)_{t_i}$. Then ${}^cD \sim I \otimes_F N$ in $Br(F)$ with I inertial over F and N NSR over F. So, by [**JW**, Th. 5. 15 (a)] $ind(D) = ind(\overline{I'}_{\overline{N'}}) \cdot |\Gamma_{N'} : \Gamma_L| = ind(\overline{I'}_{\overline{N'}}) \cdot \prod_{i=1}^{k} t_i'$, and $ind({}^cD) = ind(\overline{I}_{\overline{N}}) \cdot |\Gamma_N : \Gamma_F| = ind(\overline{I}_{\overline{N}}) \cdot \prod_{i=1}^{k} t_i$. But $ind(\overline{I}_{\overline{N}})$ divides $ind(\overline{{}^cI'}_{\overline{N}}) \cdot \prod_{i=1}^{k} ind((\overline{M_i}/\overline{F}, \sigma_i, \overline{u_i})_{t_i'} \otimes_{\overline{F}} \overline{N})$. Since $\overline{{}^cI'} \sim \overline{I'}^{\otimes[L:F]}$, by [**P**, Prop. 13. 4] and [**D**, Th. 12,p. 67] $ind(\overline{{}^cI'}_{\overline{N}}) \mid ind(\overline{I'}_{\overline{N}})$ and $ind(\overline{I'}_{\overline{N}}) \mid ind(\overline{I'}_{\overline{N}}) \cdot [\overline{N'} : \overline{N}]$.

Note that $Gal(\overline{M_i N} / \overline{N}) \cong Gal(\overline{M_i} / \overline{K_i}) =< \sigma_i^{t_i} >$ as $\overline{M_i} \cap \overline{N} =$

$\overline{K_i}$. So, by [**R**, Th. 30. 8., p. 261] $(\overline{M_i}/\overline{F}, \sigma_i, \overline{u_i})_{t_i'} \otimes_{\overline{F}} \overline{N} \sim (\overline{M_i N} / \overline{N}, \sigma_i^{t_i}, \overline{u_i})_{s_i}$, whence $ind((\overline{M_i}/\overline{F}, \sigma_i, \overline{u_i})_{t_i'} \otimes_{\overline{F}} \overline{N})$ divides s_i. Therefore, $ind({}^cD)$ divides $[\overline{N'} : \overline{N}] \cdot ind(\overline{I'}_{\overline{N'}}) \cdot \prod_{i=1}^{k} s_i t_i = [\overline{N'} : \overline{N}] \cdot ind(D)$. Since $Z(\overline{D}) = \overline{N'}$ and $Z(\overline{{}^cD}) = \overline{N}$ by [**JW**, Th. 5.15 (a)], $ind({}^cD)$ divides $[Z(\overline{D}) : Z(\overline{{}^cD})] \cdot ind(D)$. Note that $[Z(\overline{D}) : Z(\overline{{}^cD})] = |\Gamma_D : \Gamma_{{}^cD}|$ since $\Gamma_D = \Gamma_{N'}$ and $\Gamma_{{}^cD} = \Gamma_N$ by [**JW**, Th. 5. 15 (a)] and $[\overline{N'} : \overline{N}] = |\Gamma_{N'} : \Gamma_N|$ as N' is NSR over L and N is NSR over F. □

This theorem gives us a best relation between $ind(D)$ and $ind({}^cD)$ when D is inertially split over L and L/F is a finite separable extension of Henselian valued fields with same residue fields $\overline{L}=\overline{F}$, as the following examples illustrare.

EXAMPLE 3. Let L/F be as above and let D be inertial over L. Then by [**H**, Lemma 4] ${}^cD$ is inertial over F and $\overline{{}^cD} \sim \overline{D}^{\otimes[L:F]}$ in $Br(\overline{F})$. So $Z(\overline{D}) = \overline{L} = \overline{F} = Z(\overline{{}^cD})$, and $ind({}^cD) = ind(\overline{{}^cD}) = ind(\overline{D}^{\otimes[L:F]}) = ind(D^{\otimes[L:F]})$ by [**JW**, Th. 2. 8 (b)]. So, by [**P**, Prop. 13. 4] $ind({}^cD) \mid ind(D)$. Suppose gcd $(ind(D), [L : F]) = 1$, $D \sim D^{\otimes[L:F]\cdot r}$ in $Br(L)$ for some integer r, as gcd $(\exp(D), [L : F]) = 1$. So by [**P**, Prop. 13. 4] again, $ind(D) \mid ind(D^{\otimes[L:F]}) = ind({}^cD)$. Hence $ind({}^cD) = ind(D) = [Z(\overline{D} : Z(\overline{{}^cD})] \cdot ind(D)$.

EXAMPLE 4. Let (F, v) be a Henselian field with $\Gamma_F = \mathbb{Z}$ and $\pi \in F$ with $v(\pi) = 1$. Let $L = F(\sqrt[n]{\pi})$. (So $\overline{L} = \overline{F}$ and $\Gamma_F = \frac{1}{n}\mathbb{Z}$). Let $t \geq 1$ with gcd $(n, t) = 1$ and $D = (M'/L, \sigma, \pi)_t$ be a NSR division algebra over L, where M'/L isinertial with $Gal(M / L) =< \sigma >$ and $[M' : L] = t$. Then by Lemma 1, ${}^cD \sim cor_{L/F}(D) \sim (M/F, \sigma, \pi^n)_t$ in $Br(F)$, where M is the inertial lift of $\overline{M'}$ over F. But since $(M/F, \sigma, \pi^n)_t$ is a NSR division algebra over F as shown in [**JW**, Ex. 4. 3], $ind({}^cD) = t = ind(D) = [Z(\overline{D} : Z(\overline{{}^cD})] \cdot ind(D)$.

References

[1] P. Draxl, *Skew Fields*, London Math. Soc. Lecture Note Series, **81**, Cambridge Univ. Press, Cambridge, 1983.

[2] Y. Hwang, *The corestriction of valued division algebras over Henselin Fields II*, Pacific J. Math. **170** (1995), 83-103.
[3] B. Jacob and A. Wadsworth, *Division algebras over Henselian fields*, J. Algebra **128** (1990), 126-179.
[4] R. S. Pierce, *Associative Algebras*, Springer-Verlag, New York, 1982.
[5] I. Reiner, *Maximal Orders*, Academic Press, London, 1975.
[6] P. Ribenboim, *Equivalent forms of Hensel's lemma*, Expositiones Math. **3** (1985), 3-24.
[7] C. Riehm, *The corestriction of algebraic structures*, Inventiones Math. **11** (1970), 73-98.
[8] A. Wadsworth, *Extending valuations to finite dimensional division algebras*, Proc. Amer. Math. Soc. **98** (1986), 20-22.

Department of Mathematics
Korea University
Anam-Dong, Sungbuk-ku
Seoul 136-701, Korea
E-mail: yhwang@semi.korea.ac.kr