J. Korean Math. Soc. **34** (1997), No. 2, pp. 265–277

ZEEMAN'S THEOREM IN NONDECOMPOSABLE SPACES

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ABSTRACT. Let E be a real, non-degenerate, indefinite inner product space with dim $E \geq 3$. It is shown that any bijection of E which preserves the light cones is an affine map.

0. Introduction

The aim of this paper is to obtain the famous E. C. Zeeman's theorem: "Causality implies the Lorentz group" in the most general frame, namely, that of a real, non-degenerate, indefinite inner product space with dimension greater than three (but not necessarily decomposable!). We shall give two proofs of this result. Before proceeding to the statement of our general theorem, we shall give an outline of some previous generalizations of the Zeeman's theorem.

0.1. Let M_4 denote Minkowskian space-time, well-known as the real 4-dimensional "continuum" of special relativity. The indefinite quadratic form

$$(*) \qquad t^2 - x^2 - y^2 - z^2$$

on M_4, naturally gives rise to a causal relation, i.e. a relation of precedence, which in the mathematical sense simply is a partial order between events.

We note by K the causal group and by L the group generated by Lorentz transformations (isometries), translations and homotheties. In 1964, E. C. Zeeman published the surprising result (see [7]) that $K = L$. Actually, according to the paper of W. F. Pfeffer [6], this result

Received December 30, 1993.
1991 Mathematics Subject Classification: 46C20, 47B50, 83A05.
Key words and phrases: Minkowski space-time, Lorentz group, causality, indefinite inner product spaces, light cones, neutral automorphisms.

was also obtained by V. Knichal (in 1962, unpublished). The physical interpretation of Zeeman's theorem is the following : if the light cones of the quadratic form (*) are preserved, then the general causality principle can be established by checking the causality only once, either for a photon or for a heavy particle.

0.2. Another proof of Zeeman's theorem in its initial form was given by A. J. Briginshaw [4], who showed that the nonaffine conformal transformations of compactified M_4 are necessarily global causality violators. In 1972 Zeeman's theorem was extended to $R \times R^n$ with $n \geq 2$ by H. J. Borchers and G. C. Hegerfeldt (see [3]), using geometrical arguments.

If $n = 1$, then the general causal automorphism would map the space and time axes into curved lines, as is shown by the following

EXAMPLE. (E. C. Zeeman [7]) Let M_2 denote two-dimensional Minkowski space with characteristic quadratic form

$$Q(e) = t^2 - x^2 \quad , \qquad e = (t, x) \in M_2.$$

Choose new coordinates

$$s = t - x, \qquad y = t + x.$$

Let $f, g : R \longrightarrow R$ be two arbitrary nonlinear orientation-preserving homeomorphisms of the real line onto itself. Define $h : M_2 \longrightarrow M_2$ by

$$h(s, y) = (f(s), g(y)).$$

Then h is a nonlinear causal automorphism.

0.3. In 1980 Zeeman's theorem was obtained by W. F. Pfeffer [6] in the very general frame of a real Hilbert space, as follows:

THEOREM. *Let H be a real Hilbert space with dim $H \geq 3$ and let $(.,.) : H \times H \longrightarrow R$ be a symmetric, continuous, nonsingular, bilinear form in H, for which the associated Gram operator is surjective. Then each bijection $f : H \longrightarrow H$, which preserves the null vectors relative to $(.,.)$, is an affine map.*

REMARK 0.4. Actually, W. F. Pfeffer [6] omitted in the formulation of the theorem the assumption that the Gram operator is surjective. The following example, which disproves a claim of W. F. Pfeffer [6, Proposition 1.2.], was given by T. Bălan [1].

EXAMPLE. In the real Hilbert space $L^2([-1,1])$, using the function $a : [-1,1] \longrightarrow R$ expressed by

$$a(x) = \begin{cases} -\exp(x + 1/x) & \text{if } x \in [-1,0) \\ 0 & \text{if } x = 0 \\ \exp(x - 1/x) & \text{if } x \in (0,1], \end{cases}$$

we may construct the symmetric, continuous, nonsingular, bilinear form $(.,.) : L^2([-1,1]) \times L^2([-1,1]) \longrightarrow R$ by the formula

$$(f,g) = \int_{-1}^{+1} a(x)f(x)g(x)dx$$

Then we may see that the corresponding Gram operator is not surjective.

1. Generalizing Zeeman's theorem along W. F. Pfeffer's lines

1.1. It is our object in the present paper to sharpen and extend the above theorem for a significantly wider class of inner product spaces. In this section our discussion has been very much influenced by W. F. Pfeffer's paper [6].

First, let us recall that an inner product on a real vector space E is a symmetric bilinear mapping $(.,.) : E \times E \longrightarrow R$. The pair $(E, (.,.))$ is called a real inner product space (see J. Bognár [2]). If for every nonzero element $x \in E$ there exists $y \in E$ such that $(x,y) \neq 0$, we say that $(E, (.,.))$ is non-degenerate. If E contains positive as well as negative elements, we say that $(E, (.,.))$ is an indefinite inner product space.

DEFINITION 1.2. Let $(E, (.,.))$ be an arbitrary inner product space. We say that the bijection $f : E \longrightarrow E$ is a neutral automorphism if we have $(x-y,\ x-y)=0$ iff $(f(x)-f(y), f(x)-f(y))=0$. The set of all neutral automorphisms is denoted by K. If in addition $f(0)=0$ we say that f is a centered neutral automorphism (briefly, CNA) and we note the set of all such automorphisms by K_0.

DEFINITION 1.3. We say that $N \subset E$ is a neutral set if a) $0 \in N$ and b) $(x-y,\ x-y)=0$ for all $x, y \in N$. The family of all maximal neutral sets will be noted by M. If the neutral set V is a linear subspace of E, we say that V is a neutral subspace.

1.4. The formulation of our result is as follows:

THEOREM. *Let $(E, (.,.))$ be a real non-degenerate indefinite inner product space with dim $E \geq 3$. If $f \in K$, then f is an affine transformation of E, for which there exists $c \in R^*$ such that*

$$(f(x)-f(y),\ f(x)-f(y)) = c(x-y,\ x-y)$$

for all $x, y \in E$.

1.5. a) One feature of our discussion which distinguishes it from that given by W. F. Pfeffer [6] is the avoidance of assumptions in Theorem 1.4. on the decomposability of the space E (according to J. Bognár [2, Theorem IV. 5.2.], if the inner product space E admits a Hilbert majorant, then E is decomposable).

b) In order to prove Theorem 1.4., we need the following result of J. Bognár [2, Theorem II.6.3.], concerning the mutual behaviour of two inner products $(.,.)$ and $(.,.)_1$ defined on the same vector space E.

THEOREM. *If $(x,x)=0$ implies $(x,x)_1=0$, then for some real number a we have*

$$(x,y)_1 = a(x,y) \qquad (x,y \in E)$$

c) Let us remark that, if $f \in K$ and $y \in E$, then the mapping $g: E \longrightarrow E$, expressed by

$$g(x) = f(x+y) - f(y) \qquad (x \in E)$$

is a CNA.

THEOREM 1.6. *Let $(E, (.,.))$ be a real, non-degenerate inner product space and let F be a neutral subspace of E. Suppose that $F = F^{\perp\perp}$. Then we have*

$$F = \cap\{N \in M \; ; \; F \subseteq N\}.$$

Proof of Theorem 1.6. It suffices to show that, given $N \in M$ with $N \supset F$ and $e \in N \backslash F$, there exists $N_1 \in M$ such that $F \subseteq N_1$ and $e \notin N_1$.

F being weakly closed, there is a weak neighborhood (see J. Bognár [2, p. 60]) U of e such that $U \cap F = \emptyset$. Then there exist $\varepsilon > 0$ and a finite set $G \subset E$ such that

$$\{x \in E \; ; \; |(g, x - e)| < \varepsilon \;\text{ for all }\; g \in G\} \subseteq U.$$

Let us introduce the seminorm p, expressed by

$$p(y) = \max_{g \in G} |(g, y)| \qquad (y \in E).$$

We also introduce the seminorm q, defined by

$$q(y) = \inf_{x \in F} p(y - x) \qquad (y \in E).$$

Then we have $q(e) \geq \varepsilon$ and, using the Hahn-Banach theorem, we obtain a linear functional φ such that

$$\varphi(e) = \varepsilon$$

$$|\varphi(x)| \leq q(x) \qquad (x \in E).$$

Since $q(y) \leq p(y) \quad (y \in E)$, it follows that φ is weakly continuous, hence there is a $z \in E$ such that

$$\varphi(y) = (y, z) \qquad (y \in E).$$

Now, $x \in F$ implies that $q(x) = 0$, thus $\varphi(x) = 0$ and therefore $z \in F^{\perp}$. On the other hand, we have $(e, z) = \varepsilon > 0$. Setting

$$z^* = z + \delta_{(z,z),o} e,$$

where $\delta_{i,j}$ is the Kronecker's delta, we obtain

$$(z^*, z^*) \neq 0 ;$$
$$z^* \in F^{\perp} ;$$
$$(z^*, e) > 0 .$$

Then we may define the isometric bijection $f : E \longrightarrow E$ by

$$f(x) = x - 2(z^*, z^*)^{-1}(x, z^*)z^* \qquad (x \in E).$$

Let us remark that $f^2 = id_E$.

We define $N_1 = f(N)$. Since $N \in M$, it follows that $N_1 \in M$. Since e is neutral, we obtain $(e, f(e)) \neq 0$. It follows that $f(e) \notin N$, whence $e = f(f(e)) \notin N_1$. Now, if $x \in F$, then we have $(x, e) = 0, (x, z) = 0$ and therefore $(x, z^*) = 0$. It follows that $f(x) = x (x \in F)$, thus $f(F) = F$. We obtain that $F \subseteq N_1$. □

1.7. In what follows we note by Lin A the linear span of the subset A of a linear space.

PROPOSITION. *If $(E, (.,.))$ is a real non-degenerate inner product space, then for each finite neutral set N and $f \in K_0$ we have*

$$f(Lin\ N) = Lin\ f(N)$$

(see W. F. Pfeffer [6, Proposition 3.6.]).

COROLLARY 1.8. *Let $(E, (.,.))$ be a real non-degenerate inner product space, $e \in E$ with $(e, e) = 0$ and $f \in K_0$. Then f preserves the colinearity with e, i.e. for each $a \in R$ there exists $b \in R$ such that $f(ae) = bf(e)$.*

(see W. F. Pfeffer [6, Corollary 3.7.]).

LEMMA 1.9. *Let $E, (.,.))$ be a real non-degenerate inner product space, $e \in E$ with $(e, e) = 0$ and $f \in K_0$. Then $(e, x) = 0$ iff $(f(e), f(x)) = 0$.*

(see W. F. Pfeffer [6, Lemma 4.1.]).

DEFINITION 1.10. The inner product space $(E, (.\,,.))$ is called an event space (the term "scalar event world" is also used) if $E = R \times H$, where $(H, < .\,,. >)$ is a real pre-Hilbert space with $\dim H \geq 2$ and

$$((t;x),(s;y)) = ts - < x, y >$$

for all $(t;x)$ and $(s;y)$ in E.

LEMMA 1.11. *Let $(E, (.\,,.))$ be a real, indefinite, non-degenerate inner product space with $\dim E \geq 3$, $e \in E$ with $(e,e) < 0$ and $F = e^{\perp}$. Then*

a) $E = F \oplus$ *Lin* e
b) *F is non-degenerate*
c) *If $(E, -(.\,,.))$ is not an event space, then*
  c_1) *F is indefinite and*
  c_2) *if $f \in K_0, x \in E$ and $(f(x), f(x)) < 0$, then f preserves the colinearity with x*

(see W. F. Pfeffer [6, Lemma 4.2. and Proposition 4.3.]).

PROPOSITION 1.12. *Suppose that $(E, (.\,,.))$ and $(E, -(.\,,.))$ are not event spaces and that dim $E \geq 3$. Then each CNA of E preserves the colinearity with every $x \in E$.*
(see W. F. Pfeffer [6, Corollary 4.4.]).

1.13. Using Lemma 1.11 c) and Lemma 1.9 we obtain the following

PROPOSITION. *Let $(E, (.\,,.))$ be an event space, $e \in E$ and $f \in K_0$. Then $(e,e) < 0$ implies that $(f(e), f(e)) < 0$.*

COROLLARY 1.14. *The CNAs of the event spaces preserve the colinearity with any negative vector.*

LEMMA 1.15. *Let f be a CNA of the event space $(E, (.\,,.))$. Suppose that $x, y \in E$ are such that*

$$(x,x) < 0, \ (y,y) < 0 \ \text{ and } \ (x-y, \ x-y) < 0.$$

Then for all a, b in R there exist $\alpha, \beta \in R$ such that

$$f(ax + by) = \alpha f(x) + \beta f(y)$$

(this follows from the above Corollary).

LEMMA 1.16. *Let F be a two-dimensional subspace of the event space $(E, (.,.))$. Then F has a basis $\{x, y\}$ such that $(x, x) < 0, (y, y) < 0$ and $(x - y, x - y) < 0$.*
(see W. F. Pfeffer [6, Lemma 4.9]).

LEMMA 1.7. *Let f be a CNA of the event space $(E, (.,.))$. If $x, y \in E$ are such that $(x - y, x - y) < 0$, then $(f(x) - f(y), f(x) - f(y)) < 0$. (this follows from the Proposition 1.13 and Remark 1.5 c)).*

THEOREM 1.18. *Let f be a CNA of the event space $(E, (.,.))$ and let F be a 1-dimensional subspace of E. Then $f(F)$ is a 1-dimensional subspace, too.*
(see W. F. Pfeffer [6, Theorem 4.13]).

1.19. Now, Proposition 1.12 and Theorem 1.18 imply that, if $(E, (.,.))$ is a real, non-degenerate, indefinite inner product space with dim $E \geq 3$ and $f \in K_0$, then f preserves the colinearity with all the vectors of E.

Using the fact that in the conditions of the above theorem the image of every line is also a line, and the fundamental theorem of the projective geometry, we obtain:

THEOREM. *If $(E, (.,.))$ is a real, non-degenerate, indefinite inner product space with dim $E \geq 3$ and $f \in K_0$, then f is a linear transformation of E (and, consequently, each $f \in K$ is affine).*

Our main result (Theorem 1.4.) is an immediate consequence of Theorems 1.5 b) and 1.19.

2. Short proof of Zeeman's theorem

In this section we shall give an alternative proof of Theorem 1.4. In this new proof, instead of using neutral sets as above (as was done in W. F. Pfeffer [6]), we shall use somewhat more natural properties of indefinite inner product spaces.

In what follows we assume in addition that E is non-decomposable and that f is a CNA of E. We are going to prove that f is linear and there is a $c \in R^*$ such that

$$(f(x), f(y)) = c(x, y) \qquad (x, y \in E).$$

Our proof is divided into four steps :

2.1. f preserves the colinearity with any neutral vector

Suppose that there is a neutral $e \in E$ and $a \in R$ such that $f(ae) \notin \mathrm{Lin}\, f(e)$. Clearly, $e \neq 0$. Put $u = f(ae)$ and $v = f(e)$. Then u and v are neutral orthogonal vectors and $v \neq 0$. E being non-degenerate, there exists $y \in E$ such that $(v, y) \neq 0$. Then we have $u \neq cv$, where $c = (v,y)^{-1}(u,y)$. Since E is non-degenerate, there is $z \in E$ such that $(u - cv, z) \neq 0$. Let us define

$$w = z - (v,y)^{-1}(v,z)y.$$

Then we have $w \perp v$ and $(w,u) \neq 0$. Let $F = \mathrm{Lin}\,\{u,v,w,y\}$. We claim that F is non-degenerate. Indeed, let $h \in F \cap F^{\perp}$. Then there exist $\alpha, \beta, \gamma, \delta, \in R$ such that

$$h = \alpha u + \beta v + \gamma w + \delta y.$$

Since $h \perp u$, it follows that $\gamma(w,u) + \delta(y,u) = 0$.

Since $h \perp v$, it follows that $\delta(y, v) = 0$, thus $\delta = 0$ and, consequently, $\gamma = 0$, hence $h = \alpha u + \beta v$. Since $h \perp w$, it follows that $\alpha(u, w) = 0$, thus $\alpha = 0$. Therefore $h \beta v$. Since $h \perp y$, we obtain $\beta(v, y) = 0$, thus $\beta = 0$. This proves our claim.

Being also finite-dimensional, F is ortho-complemented. Let $P : E \longrightarrow F$ be the ortho-projector onto F. Suppose that $F^{\perp}$ is definite, say positive. Since E is nondecomposable, it cannot be quasi-positive, therefore we may choose a 5-dimensional negative definite subspace $H \subset E$. Then $P|_H$ is one-to-one, whence $\dim F \geq 5$, contradiction. The case when $F^{\perp}$ is negative definite is similar. It follows that $F^{\perp}$ is indefinite. Let us choose $r \in F^{\perp}$ with $(r,r) = -(w,w)$ and let us note $t = w + r$. Then t is neutral, $t \perp v$ and $(t,u) \neq 0$. Since f^{-1} is CNA, we obtain $f^{-1}(t) \perp e$ and $(f^{-1}(t), ae) \neq 0$, contradiction. It follows that there is $b \in R$ such that $f(ae) = bf(e)$.

2.2. f preserves the colinearity with any non-neutral vector

Let $a \in R^*$ and $e \in E$ with $(e,e) \neq 0$. Let $u = f(ae), v = f(e)$ and

$$w = u - (v,v)^{-1}(u,v)v$$

(since f is CNA, we have $(v,v) \neq 0$). Clearly, $w \perp v$. Let us take a neutral q in $v^{\perp}$.

Suppose that $(f^{-1}(q), e) \neq 0$. Let us define

$$a^* = \frac{1}{2}(f^{-1}(q), e)^{-1}(e,e).$$

Using the first stage of our proof, we obtain a $b^* \in R$ such that

$$f(a^* f^{-1}(q)) = b^* q.$$

Then $e - a^* f^{-1}(q)$ is neutral and consequently

$$(f(e) - b^* q,\ f(e) - b^* q) = 0.$$

Therefore we have $(f(e), f(e)) = 0$, hence $(e,e) = 0$, contradiction.

Then $f^{-1}(q) \perp e$, thus $f^{-1}(q) \perp ae$. Suppose that $(q,u) \neq 0$. Let $a^{**} = \frac{1}{2}(u,q)^{-1}(u,u)$. Using the first step for f^{-1} we obtain a b^{**} in R such that

$$f^{-1}(a^{**}q) = b^{**} f^{-1}(q).$$

Then $u - a^{**}q$ is neutral, hence

$$(ae - b^{**} f^{-1}(q), ae - b^{**} f^{-1}(q)) = 0,$$

therefore $a^2(e,e) = 0$, thus $(e,e) = 0$, contradiction.

Then we have $q \perp u$, whence $q \perp w$. Since E is non-decomposable, it follows that $v^{\perp}$ is indefinite.

Since we have $w \perp p$ for all neutral p in $v^{\perp}$, it follows that $w \perp v^{\perp}$. Since $w \perp v$ and $E = \text{Lin } v \oplus v^{\perp}$, we obtain $w \perp E$, hence $w = 0$. It follows that $f(ae) = bf(e)$ for some b in R.

2.3. f maps lines onto lines

Let $D \subset E$ be a line and $x, y \in D$, $x \neq y$. Let D' be the line generated by $f(x)$ and $f(y)$ and let $z \in D$. Then there is $a \in R$ such that $z = (1-a)x + ay$, hence $z - x = a(y - x)$. We define $f_x : E \longrightarrow E$ by $f_x(u) = f(x+u) - f(x)$. Then f_x is a CNA. Then there is a $b \in R$ such that

$$f_x(z - x) = b f_x(y - x),$$

thus

$$f(z) - f(x) = b[f(y) - f(x)],$$

whence

$$f(z) = (1-b)f(x) + bf(y) \in D'.$$

It follows that $f(D) \subseteq D'$. By a similar argument we get $f^{-1}(D') \subseteq D$, whence $f(D) \supseteq D'$. It follows that $f(D) = D'$. Using now the fundamental theorem of the projective geometry, we obtain that f is linear.

2.4. There exists $c \in R^*$ such that $(f(x), f(y)) = c(x,y)$ $(x, y \in E)$

Let us remark that the formula

$$(x,y)_1 = (f(x), f(y))$$

defines a new inner product on E. Since $(x,x) = 0$ implies $(x,x)_1 = 0$, by Theorem 1.5 b) we obtain a $c \in R$ such that

$$(x,y)_1 = c(x,y) \qquad \forall x, y \in E.$$

Suppose that $c = 0$. Then we have $(f(x), f(x)) = 0$ $\forall x \in E$. Since f is CNA, it follows that $(x,x) = 0$ $\forall x \in E$, thus E is degenerate, contradiction. In conclusion $c \in R^*$.

Appendix

Let E and F be two real, non-degenerate, indefinite inner product spaces. Trying more to extend the Zeeman's Theorem, it is natural to raise the following related

PROBLEM. Suppose that there exists a one-to-one mapping $f : E \longrightarrow F$ such that $(x-y, x-y) = 0$ iff $(f(x) - f(y), f(x) - f(y)) = 0$. Is there a linear isometric operator from E to F ?

Unfortunately, the following example shows that the answer is negative.

EXAMPLE. Let E be the real G W. Mackey's space (see J. Bognár [2, Example I. 11.3]) and let F be a real linear space with

$$\mathrm{Dim}\ F = \mathrm{card}\ (M \times \{0,1\})$$

(here "Dim" means algebraic dimension). Let

$$\{b_{x,i}\ ;\ x \in E,\quad i \in \{0,1\}\}$$

be a Hamel basis for F. Let us define the inner product

$$(.,.) : F \times F \longrightarrow R,$$

expressed by

$$(b_{x,i}, b_{y,j}) = ij\ sgn(x - y, x - y) + |i - j|\delta_{x,y}$$

$(x, y \in E\ ;\ i, j \in \{0,1\})$.

Then it is easy to see that $(F, (.,.))$ is indefinite and non-degenerate. Clearly, $(F, (.,.))$ admits a normed majorant, defined by the norm $\|u\| = \sum |u(x,i)|$ and the mapping

$$x \longrightarrow e_{x,1} \qquad (x \in E)$$

satisfies the Zeeman's condition. But, according to J. Bognár [2, Example III. 3.2.], E cannot have a normed majorant, so that E cannot be (isometrically) isomorphic to any subspace of F.

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