J. Korean Math. Soc. **34** (1997), No. 2, pp. 257–264

# ALMOST CAUSAL STRUCTURE IN SPACE-TIMES

JONG CHUL PARK

ABSTRACT. We shall introduce the concept of almost causality condition. By defining the almost causality condition we would like to examine the relationship between Woodhouse's causality principle and other known causality conditions. We show that a series of causality conditions can be characterized by using the almost causality condition.

## I. Introduction

The concept of almost causal precedence was proposed by Woodhouse[9]. There is an essential difference between the almost future and the Seifert future. The relation $J_s^+$ between events in a space-time is transitive one, i.e., $y \in J_s^+(x)$ and $z \in J_s^+(y)$ imply $z \in J_s^+(x)$. However, this is not case for the relation $A^+$.

In this paper, we shall introduce the concept of almost causality condition. By defining the almost causality condition we would like to examine the relationship between Woodhouse's causality principle and other known causality conditions. We show that a series of causality conditions can be characterized by using the almost causality condition.

## II. Preliminaries

By a space-time we mean a pair $(M, g)$ with $M$ an orientable, time orientable, connected paracompact and Hausdroff differentiable manifold without boundary and $g$ a Lorentzian metric defined globally on $M$. The chronological relation $\ll$ (causal relation $\leq$ ) between points of $M$ are defined by saying $x \ll y$ (causal relation $x \leq y$ ) if only there is a future-directed timelike (nonspacelike) curve from $x$ to $y$. The

Received November 29, 1993.

1991 Mathematics Subject Classification: 53C50, 83C75.

Key words and phrases: Seifert future set, almost causal, stable causal, Woodhouse causal.

chronological future $I^+(A)$ and causal future $J^+(A)$ of a subset $A$ of $M$ are defined as

$$I^+(A) = \{q \in M : \text{ there is } a\ p \in A \text{ with } p \ll q\},$$

$$J^+(A) = \{q \in M : \text{ there is } a\ p \in A \text{ with } p \leq q\}.$$

Dually, the chronological past $I^-(A)$ and the causal past $J^-(A)$ are defined. We note that $I^+(A)$ are open for all $A \subset M$. For a single point $p$, we abbreviate $I^+(\{p\})$ by $I^+(p)$ and similarly for $J^+$. The chronological common future $\uparrow U$ of an open subset $U$ of $M$ is defined by

$$\uparrow U = I^+(\{p \in M : u \ll p \text{ for all } u \in U\}).$$

The chronological common past $\downarrow U$ is defined dually.

We say that $p$ almost causally precedes $q$, denoted by $pAq$, if for all $x \in I^-(p)$, $I^+(q) \subset I^+(x)$; or equivalently, if for all $y \in I^+(q)$, $I^-(p) \subset I^-(y)$. We note that $pAq$ iff every neighborhood of $p$ would contain a point which will precede chronologically some points in any neighborhood of $q$. The almost future $A^+(p)$ and the almost past $A^-(p)$ of $p \in M$ are defined

$$A^+(p) = \{y \in M : pAy\}, \ A^-(p) = \{x \in M : xAp\}.$$

Distinguishing conditions were introduced to exclude situations in whi-ch there were nonspacelike curves which returned arbitrarily close to their point of origin. A space-time $M$ is future (resp., past) distinguishing, if $I^+(x) = I^+(y)$ (resp., $I^-(x) = I^-(y)$ implies $x = y$. An equivalent statement is that for any $p$ in $M$ and any neighborhood $Up$ of $p$, there is a neighborhood $Vp$ of $p$ contained in $Up$ such that no future- (resp., past-) directed nonspacelike curve through $p$ which leaves $Vp$ ever returns to it. A space-time $(M, g)$ is distinguishing if and only if it is future and past distinguishing. A space-time $M$ is future reflecting if $I^-(p) \subset I^-(q)$ implies $I^+(q) \subset I^+(p)$, and it is past reflecting if $I^+(p) \subset I^+(q)$ implies $I^-(q) \subset I^-(p)$ : whereas $M$ is reflecting if it is both past and future reflecting. The reflecting space-time was investigated by Hawking and Sachs[5].

Since the quantum uncertainty principle implies impreciseness in the measurements, it will not be possible to measure that exact values of the metric at any point and hence the metric perturbations are to be taken into account. To achieve this, Seifert[7] defined the $J_s^+(x)$ as follows:

$$J_s^+(x) = \bigcap_{\overline{g}>g} J^+(x,\overline{g})$$

Here $\overline{g} > g$ means that the null corn of $\overline{g}$ are everywhere wider than $g$, i.e., every nonspacelike vector with respect to $g$ becomes timelike with respect to $\overline{g}$. We call the set $J_s^+(x)$ as the Seifert future set of $x$.

We shall that a space-time $(M, g)$ is stably causal if there exists a Lorentzian metric $\overline{g}$ such such that $(M, \overline{g})$ is causal and for each $p \in M$ and $v \in T_p(M)$ with $v \neq 0$, $g(v,v) \leq 0$ implies $\overline{g}(v,v) < 0$. We say that $M$ is causally continuous if it is distinguishing and reflecting. There are many equivalent descriptions of this important causality conditions[5]. For these space-times the causal structure may be extended to the causal boundary[3]. Furthermore, a metrizable topology may be defined on the causal completion of a causally continuous space-times[2]. A distinguishing space-time $M$ is causally simple if $J^+(p)$ and $J^-(p)$ are closed for all $p \in M$.

All of the other notations and terminologies of this paper will be refered to Beem and Ehrlich[1] and Hawking and Ellis[3].

## III. Almost causal structure in space-times

The following lemmas that are proved in [1] and [5] will be used to show our theorems in this paper.

LEMMA 1. *Let $M$ be a space-time. Then for all $x \in M$,*

$$Int(A^+(x)) = \uparrow I^-(x).$$

LEMMA 2. *Let $M$ be a reflecting space-time. Then, for all $x \in M$,*

$$\uparrow I^-(x) = I^+(x), \ \downarrow I^+(x) = I^-(x).$$

It is known that for a general space-time, the causal future $J^+(x)$ of some $x \in M$ need not be closed in the manifold topology. However, we know that the almost furure of an event is always closed [1].

LEMMA 3. *Let $M$ be a space-time. Then the almost future $A^+(x)$ is closed in the manifold topology for all events $x \in M$.*

LEMMA 4. *Let $M$ be a distinguishing space-time. Then $M$ is reflecting if and only if $J_s^{\pm}(x) = \overline{J^{\pm}(x)}$ for all events $x \in M$.*

The relationship between the almost future and Seifert future of an event in a general space-time is obtained in the following result.

PROPOSITION 5. *Let $M$ be a space-time. Then for all $x \in M$,*

$$A^+(x) \subset J_s^+(x).$$

*Proof.* From the definition of $J_s^+(x)$, we have $I^+(x) \subset J_s^+(x)$. Note that $J_s^+(x)$ is closed for all $x \in M$. Hence $\overline{I^+(x)} \subset \overline{J_s^+(x)} = J_s^+(x)$. Let now $p \in A^+(x)$. Then $I^+(p) \subset I^+(z)$ for all $z \in I^-(x)$. Consider now a sequence $p_n$ in $I^-(x)$ converging to $x$. Then we have $\bigcap J_s^+(p_n) \subset J_s^+(x)$. Now $I^+(p) \subset I^+(z)$ for all $z \in I^-(x)$ implies that $p \in \overline{I^+(z)}$ for all $z \in I^-(x)$. So $p \in J_s^+(z)$ for all $z \in I^-(x)$ and so $p \in \bigcap J_s^+(p_n)$ since $p \in I^-(x)$ for all $n$. Hence $p \in J_s^+(x)$. □

However, for a general space-time they are not equal (see Figure 1).

With this motivation we shall define the following definition:

DEFINITION 6. A space-time $M$ is said to be almost causal if, for all $x \in M$,

$$A^+(x) = J_s^+(x).$$

The following proposition shows that for a reflecting space-time, the almost future of a point turns out to be a simple and well-known set.

PROPOSITION 7. *Let $M$ be a reflecting space-time. Then, for all $x \in M$,*

$$A^+(x) = \overline{I^+(x)}.$$

*Proof.* By lemma 1, $Int(A^+(x)) = \uparrow I^-(x)$. Now by lemma 2, $\uparrow I^-(x) = I^+(x)$. Since $A^+(x)$ is closed in the manifold topology, $A^+(x) = \overline{A^+(x)} = \overline{IntA^+(x)} = \overline{\uparrow I^-(x)} = \overline{I^+(x)}$. □

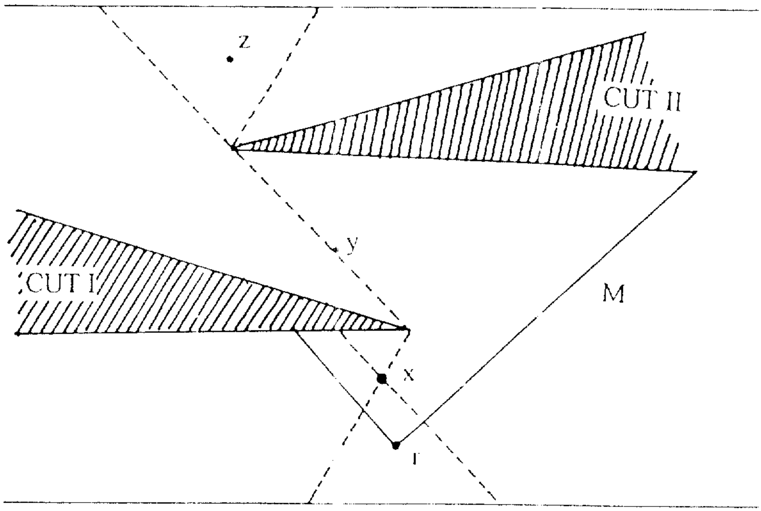


Figure 1. In the space-time here, $z \notin A^+(x)$ because the future of any event $r \in A^-(x)$ is fully obstructed by cuts in $M$ so it cannot contain the future of the event $z$. But clearly for all metrics $\overline{g} > g$, $z \in J^+(x, \overline{g})$ because a slightest perturbation in the metric would open up the null cones to give way to nonspacelike curve to reach from $x$ to $z$. Hence $z \in J_s^+(x)$ and the two futures $A^+(x)$, $J_s^+(x)$, are not equal.

THEOREM 8. *Let $M$ be a causally continuous space-time. Then $M$ satisfies the almost causal condition.*

*Proof.*

$$
\begin{aligned}
J_s^+(x) &= \bigcap_{\overline{g}>g} J^+(x, \overline{g}) \\
&= \overline{J^+(x)} \\
&= \overline{I^+(x)} \\
&= A^+(x)
\end{aligned}
$$

The following proposition was proved by Seifert[7]. □

PROPOSITION 9. *A space-time $M$ is stably causal if $J_s^+$ is a partial ordering on $M$ (i.e., $p \in J_s^+(q)$ and $q \in J_s^+(p)$ implies $p = q$).*

THEOREM 10. *Let $M$ be a almost causal space-time. Then $M$ is stably causal if and only if $M$ satisfies the Woodhouse causality principle.*

*Proof.* As we have shown in Proposition 9, Seifert has shown that the stable causality of $M$ implies that $x \in J_s^+(y)$ and $y \in J_s^+(x)$ implies $x = y$. Now suppose $xAy$ and $yAx$ hold. Then $y \in A^+(x)$ and $x \in A^+(y)$. Since $A^+(x) \subset J_s^+(x)$ for all $x \in M$, this implies $x = y$, that is, $M$ is Woodhouse causal. Conversely, suppose that $x \in J_s^+(y)$ and $y \in J_s^+(x)$. Since $M$ is almost causal, $x \in J_s^+(y) = A^+(x)$ and $y \in J_s^+(x) = A^+(x)$.
Hence $x = y$. So $M$ is a stable causal space-time. □

The following corollary is straightforward from Theorem 10.

COROLLARY 11. *Let $M$ satisfies the Woodhouse causality principle. If $M$ is almost causal, then $M$ is stable causal.*

*Proof.* Assume that $x \in J_s^+(y)$ and $y \in J_s^+(x)$. Then, from the theorem above, $x \in J_s^+(y) = A^+(y)$ and $y \in J_s^+(x) = A^+(y)$. So $x = y$. □

In particular, $\overline{J^+(x)} \subset J_s^+(x)$ for all events $x \in M$. Figure 2 shows that $\overline{J^+(x)} \neq J_s^+(x)$.

THEOREM 12. *Let $M$ be a reflecting space-time. If $M$ is almost causal, then $\overline{J^+(x)} = J_s^+(x)$ for all $x \in M$.*

*Proof.* $A^+(x) = \overline{I^+(x)} = \overline{J^+(x)} = J_s^+(x) \subset A^+(x)$. □

Recall that $M$ is reflecting space-time if and only if $[I^+(p) \supset I^+(q)$ iff $I^-(p) \subset I^-(q)]$, if and only if $[q \in \overline{I^+(p)}$ iff $p \in \overline{I^-(q)}]$.

THEOREM 13. *Let a space-time $M$ satisfies the following condition: $J^+(p)$ and $J^-(p)$ are closed in $M$ for every $p \in M$. Then $M$ is reflecting.*

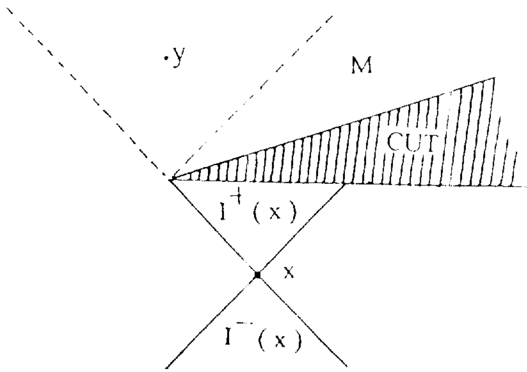


Figure 2. The space-time $M$ is conformal to a submanifold of 2-dimen-sional Minkowski space. Here, $\overline{J^+(x)} \neq J_s^+(x)$ corresponding to a lack of smoothness under small perturbation of the metric.

*Proof.* Since $q \in \overline{I^+(p)} = \overline{J^+(p)} = J^+(p)$, $p \in J^-(q) = \overline{J^-(q)} = \overline{I^-(q)}$. Similarly, if $p \in \overline{I^-(q)}$, then $q \in \overline{I^+(p)}$. □

We get the following the result of Theorem 13.

THEOREM 14. *Let $M$ be a causally simple space-time. Then, for all event $x \in M$,*

$$A^+(x) = J_s^+(x) = J^+(x).$$

*Proof.*

$$A^+(x) = J_s^+(x)(\leftarrow \text{ almost causal})$$

$$= \overline{J^+(x)}(\leftarrow \text{ causally continuous})$$

$$= J^+(x)(\leftarrow \text{causally simple}).$$

□

## References

[1] G. M. Akolia, P. S. Joshi and U. D. Vyas, *On almost causality*, J. Math. Phys. **22** (1981), 1234-1247.

[2] J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry,* **67**, *Pure and Applied Mathematics*, Marcel Dekker New York and Basel, 1981.

[3] R. Budic and R. K. Sachs, *Causal boundaries for general relativistic space-times*, J. Math. Phys. **15** (1974), 1302-1309.

[4] S. W. Hawking and G. F. R. Ellis, *The large scale Structure of space-time*, Cambridge Monographs on mathematical physics, Cambridge University Press, Cambridge, 1973.

[5] S. W. Hawking and R. K. Sachs, *Causally continuous space-times*, Comm. Math. Phys. **35** (1974), 287-296.

[6] Y. Matori, *Totally vicious space-times and reflectivity*, J. Math. Phys. **29** (1988), 823-824.

[7] H. J. Seifert, *The causal boundary of space-times*, Gen. Rel. Grav. **3** (1971), 247-259.

[8] U. D. Vyas, and G. M. Akolia, *Causally continuous space-times*, Gen. Rel. Grav. **18** (1986), 309 - 314.

[9] N. M. J. Woodhouse, *The differentiable and causal structure of space-time*, J. Math. Phys. **14** (1973), 495-501.

Department of Mathematics
Yeungnam University
Kyongsan 712-749, Korea