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THE FIXED POINT INDEX FOR ACCRETIVE MAPPINGS WITH k-SET CONTRACTION PERTURBATIONS IN CONES

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1. Introduction

The fixed point index plays an important role in solving the positive solutions of nonlinear equations in ordered Banach spaces ([7], [10], [11], [14], [15]). Many authors have studied the existence problems of positive solutions of nonlinear equations for nonlinear mappings ([1]-[5], [7], [9], [10], [14], [15]).

Let E be a Banach space and P be a closed cone in E, i.e., P is a closed convex set in E and

$$\lambda P \subset P \quad \text{for all} \quad \lambda \geq 0 \quad \text{and} \quad P \cap (-P) = \{0\}.$$

Let Ω be a nonempty open bounded subset of E. Let $A : D(A) \subset P \to 2^P$ be a multivalued accretive mapping, i.e., for all $x, y \in D(A)$, $a_1 \in Ax$ and $a_2 \in Ay$,

$$\|x - y\| \leq \|x - y + \lambda(a_1 - a_2)\|,$$

and let $K : \Omega \cap P \to P$ be a strict k-set contraction for $0 \leq k < 1$ ([12], [14]).

In this paper, we prove that if $(I+A)(D(A)) = P$ and $x \notin -Ax+Kx$ for all $x \in \partial\Omega \cap D(A)$, then the fixed point index is defined for a mapping $-A + K$ and $-A + K$ has a fixed point in $\Omega \cap D(A)$.

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2. Main Results

Let E be a Banach space, P be a closed cone in E and " $\leq$ " be the order induced by P in E, i.e., $x \leq y$ if and only if $y - x \in P$.

PROPOSITION 1. *If $A : D(A) = P \to P$ is a continuous accretive mapping and for each $x \in P$, there exists $\beta(x) > 0$ such that $Ax \leq \beta(x) \cdot x$, then $(\lambda I + A)(P) = P$ for all $\lambda > 0$.*

Proof. For each $z \in P$, consider the following differential equation

$$\text{(2.1)} \qquad \begin{cases} x'(t) &= -(\lambda I + A)x(t) + z, \quad t \in [0, +\infty), \\ x(0) &= u \in P. \end{cases}$$

For each $x \in P$, since $Ax \leq \beta(x) \cdot x$, there exists $W(x) \in P$ such that $\beta(x) \cdot x = Ax + W(x)$.
Thus we have

$$x + \epsilon(-\lambda x - Ax + z) = (1 - \epsilon\lambda - \epsilon\beta(x))x + \epsilon W(x) + \epsilon z$$

and so for sufficiently small $\epsilon > 0$ such that $1 - \epsilon\lambda - \epsilon\beta(x) > 0$, we have

$$(1 - \epsilon\lambda - \epsilon\beta(x))x + \epsilon W(x) + \epsilon z \in P.$$

Hence it follows that for all $x \in P$

$$\lim_{\epsilon \to 0_+} \frac{1}{\epsilon}\rho(x + \epsilon(-\lambda x - Ax + z), P) = 0$$

and, by Deimling [6], we know that (2.1) has only one solution in P. Let $x(t, u)$ be the unique solution of (2.1) with $x(0) = u$.

Now, we define a mapping $B_T : P \to P$ by

$$B_T u = x(T, u) \quad \text{for all} \quad u \in P \quad \text{and a constant} \quad T > 0.$$

For all $u,\ v \in P$, let $\phi(t) = \|x(t, u) - x(t, v)\|$. Then we have

$$\phi(t)D^-\phi(t) \leq (x'(t, u) - x'(t, v), x(t, u) - x(t, v))_-,$$

where $D^-\phi(t) = \limsup_{h\to 0^+} \frac{\phi(t)-\phi(t-h)}{h}$ and $(\cdot,\ )_-$ is the semi-inner product ([6]) and so

$$\begin{aligned}&\phi(t)D^-\phi(t)\\&\le(-\lambda x(t,u)-Ax(t,u)+\lambda x(t,v)+Ax(t,v),x(t,u)-x(t,v))_-.\end{aligned}$$

Since A is accretive, we have

$$\begin{aligned}&(-Ax(t,u)+Ax(t,v),x(t,v)-x(t,v))_-\\&=-(Ax(t,u)-Ax(t,v),x(t,u)-x(t,v))_+\\&\le 0.\end{aligned}$$

Therefore, we have

$$\phi(t)D^-\phi(t)\le-\lambda\phi^2(t)\quad\text{and}\quad\phi(t)\le e^{-\lambda T}\phi(0)$$

and so $\|B_Tu-B_Tv\|\le e^{-\lambda T}\|u-v\|$. Hence, B_T has a unique fixed point $u_0\in P$, i.e., $B_Tu_0=u_0$. This implies that $x'(t,u_0)=0$ for all $t>0$. Therefore, we have $0=-\lambda u_0-Au_0+z$ and so $z\in(A+\lambda I)(P)$. This completes the proof. □

Next, we assume that $A: D(A)\subset P\to 2^P$ is a multivalued accretive mapping and $(A+I)(D(A))=P$. Then it is well-known that $(I+A)^{-1}$ is nonexpansive ([4]).

Let Ω be a open bounded subset of E and $K:\overline{\Omega}\cap P\to P$ be a strict k-set contraction for $k\in[0,1)$. Suppose that $D(A)\cap\overline{\Omega}\neq\emptyset$ and $x\notin -Ax+Kx$ for all $x\in\partial\Omega\cap D(A)$. Then $x\neq(I+A)^{-1}Kx$ for all $x\in\partial\Omega\cap P$ and $(I+A)^{-1}K$ is also a strict k-set contraction. Therefore, the fixed point index $i((I+A)^{-1}K,\ \Omega\cap P)$ is well defined ([1], [10]). Now, we define

$$i(-A+K,\ \Omega\cap D(A))=i((I+A)^{-1}K,\ \Omega\cap P).$$

Then we have the following:

THEOREM 2. (1) *If* $\Omega=B(0,r)$ *and* $Kx=x_0\in B(0,r)\cap P$ *for all* $x\in\overline{B(0,r)}\cap P$, *then*

$$i(-A+K,B(0,r)\cap D(A))=1.$$

(2) *If* $\Omega = \Omega_1 \cup \Omega_2$ *and* $\Omega_1 \cap \Omega_2 = \emptyset$, *then*

$$i(-A+K, \Omega \cap D(A)) = i(-A+K, \Omega_1 \cap D(A)) + i(-A+K, \Omega_2 \cap D(A)).$$

(3) *Let* $H(t,x) : [0,1] \times (\overline{\Omega} \cap P) \to P$ *be uniformly continuous in* x *for each* $t \in [0,1]$ *and* $H(t,\cdot)$: $\Omega \cap P \to P$ *be a strick* k*-set contraction, where* k *does not depend on* t. *If, for all* $t \in [0,1]$,

$$x \notin -Ax + H(t,x) \quad \text{for all} \quad x \in \partial\Omega \cap D(A),$$

then $i(-A + H(t,x), \Omega \cap D(A))$ *does not depend on* t.
(4) *If* $i(-A+K, \Omega \cap D(A)) \neq 0$, *then* $x \in -Ax + Kx$ *has a solution in* $\Omega \cap D(A)$, *i.e.,* $-A + K$ *has a fixed point in* $\Omega \cap D(A)$.

Proof. (2), (3) and (4) are obvious ([1], [10]).
Now, we prove (1). First, we have

$$0 \in D(A) \quad \text{and} \quad 0 \in A0 \tag{2.2}$$

In fact, since $(A+I)(D(A)) = P$, there exist $x \in D(A)$ and $a \in Ax$ such that $x + a = 0$. Since $x \geq 0$ and $a \geq 0$, we have $x = 0$ and $a = 0 \in A0$. Hence we have

$$(A+I)^{-1}0 = 0. \tag{2.3}$$

Next, we prove that

$$i((I+A)^{-1}K, \Omega \cap P) = 1, \ \Omega = B(0,r). \tag{2.4}$$

Since $(I+A)^{-1}Kx = (I+A)^{-1}x_0$ for all $x \in \overline{\Omega} \cap P$ and

$$\|(I+A)^{-1}x_0 - (I+A)^{-1}0\| \leq \|x_0\| < r,$$

it follows that $(I+A)^{-1}x_0 \in \Omega \cap P = B(0,r) \cap P$ and, by Amman [1] and Li [10],

$$i((I+A)^{-1}K, B(0,r) \cap P) = 1.$$

Therefore, $i(-A+K, B(0,r) \cap D(A)) = 1$. This completes the proof.□

LEMMA 3. *If* $Kx \not\geq x$ *for all* $x \in \partial\Omega \cap P$ *and* $0 \in \Omega$, *then*

$$i(-A+K, \Omega \cap D(A)) = 1.$$

Proof. Let $H(t,x) = tKx$ for all $t \in [0,1]$ and $x \in \overline{\Omega} \cap P$. If $x \in -Ax + tKx$ for some $x \in \partial\Omega \cap D(A)$ and $t \in [0,1]$, then $t \neq 0$ (otherwise, we have $x = 0 \in \partial\Omega$, which is a contradiction). Thus we have $Kx \geq x/t \geq x$, which contradicts $Kx \not\geq x$. Hence, $H(t,x)$ satisfy all the conditions of (3) in Theorem 2 and so

$$i(-A+K, \Omega \cap D(A)) = i(-A+0, \Omega \cap D(A)).$$

By (2.3), we have $(I+A)^{-1}0 = 0 \in \Omega \cap P$ and so $i((I+A)^{-1}0, \Omega \cap P) = 1$ and

$$i(-A+0, \Omega \cap D(A)) = 1. \tag{2.5}$$

Therefore, we have

$$i(-A+K, \Omega \cap D(A)) = 1.$$

This completes the proof. □

COROLLARY 4. *If* $0 \in \Omega$ *and* $Kx < x$ *for all* $x \in \partial\Omega \cap P$, *then* $-A+K$ *has a fixed point in* $\Omega \cap D(A)$.

Proof. By hypothesis, it is obvious to prove that $Kx \not\geq x$ for all $x \in \partial\Omega \cap P$. By Lemma 3,

$$i(-A+K, \Omega \cap D(A)) = 1.$$

Therefore, by (4) of Theorem 2, $-A+K$ has a fixed point in $\Omega \cap D(A)$. □

LEMMA 5. *Suppose that* $u_0 \neq 0$ *for* $u_0 \in P$ *and* $x - tu_0 \notin -A(x - tu_0) + Kx$. *If* $x \in \partial\Omega \cap P$ *and* $x - tu_0 \in D(A)$ *for* $t \geq 0$, *then*

$$i(-A+K, \Omega \cap D(A)) = 0.$$

Proof. Suppose that $i((I+A)^{-1}K, \Omega \cap D(A)) \neq 0$ and for each $\tau > 0$, let $H(t,x) = (I+A)^{-1}K + t\tau u_0$ for all $x \in \Omega \cap P$ and $t \in [0,1]$.

It is obvious that $H(t,x)$ is uniformly continuous in x for each t and $H(t,\cdot)$ is a strict k-set contradiction for each t. By Amman [1] and Li [8], we have

$$i((I+A)^{-1}K + \tau u_0, \Omega \cap P) = i((I+A)^{-1}K, \Omega \cap P) \neq 0.$$

Thus there exists $x_\tau \in \Omega \cap P$ such that

$$x_\tau - (I+A)^{-1}Kx_\tau = \tau u_0. \tag{2.6}$$

Letting $\tau \to \infty$, the left side of (2.6) is bounded, but the right side of (2.6) is unbounded, which is a contraction. Therefore, we have

$$i(-A+K, \Omega \cap D(A)) = 0.$$

This completes the proof. □

THEOREM 6. *Let $A : D(A) \subset P \to 2^P$ be a multivalued accretive mapping such that $(I+A)(D(A)) = P$ and let Ω_1, Ω_2 be two open bounded subsets of E with $0 \in \Omega_1 \subset \Omega_2$. Suppose that if $K : \overline{\Omega}_2 \cap P \to$ is a strict k-set contraction mapping and $0 \neq u_0 \in P$, then*

(i) *for each $x \in \partial\Omega_2$, $x \not\leq Kx$ and for each $x \in \partial\Omega_1 \cap P$ and $t \geq 0$,*

$$x - tu_0 \in D(A) \quad \text{and} \quad x - tu_0 \notin -A(x - tu_0) + Kx$$

or

(ii) *for each $x \in \partial\Omega_1$, $x \not\leq Kx$ and for each $x \in \partial\Omega_2 \cap P$ and $t \geq 0$,*

$$x - tu_0 \in D(A) \quad \text{and} \quad x - tu_0 \notin -A(x - tu_0) + Kx.$$

If either (i) *or* (ii) *is satisfied, then $-A+K$ has a fixed point in $(\Omega_2 - \overline{\Omega}_1) \cap D(A)$.*

Proof. Suppose that the condition (i) is satisfied. By Lemma 3 and Lemma 5, we have

$$i(-A+K, \Omega_2 \cap D(A)) = 1 \tag{2.7}$$

and

$$i(-A+K, \Omega_1 \cap D(A)) = 0, \tag{2.8}$$

respectively. Thus, from (2) of Theorem 2, (2.7) and (2.8), it follows that

$$i(-A+K, (\Omega_2 - \overline{\Omega}_1) \cap D(A)) = 1.$$

Therefore, by (4) of Theorem 2, $-A+K$ has a fixed point in $(\Omega_2 - \overline{\Omega}_1) \cap D(A)$.
If (ii) is satisfied, the proof is similar to the case (i). This completes the proof. □

THEOREM 7. *If, for each $x \in \partial\Omega \cap D(A)$, $\|Kx\| \leq \|x\|$ and $0 \in \Omega$, then $-A+K$ has a fixed point in $\overline{\Omega} \cap D(A)$.*

Proof. we may assume that

$$x \notin -Ax + Kx \quad \text{for all} \quad x \in \partial\Omega \cap D(A). \tag{2.9}$$

Let $H(t,x) = tKx$ for all $x \in \partial\Omega \cap P$ and $t \in [0,1]$. It is obvious that $H(t,x)$ is uniformly continuous in x for each $t \in [0,1]$ and $H(t,\cdot)$ is a strict k-set contraction for each t. We show that

$$x \notin -Ax + H(t,x) \quad \text{for all} \quad x \in \partial\Omega \cap D(A) \quad \text{and} \quad t \in [0,1]. \tag{2.10}$$

If $x \in -Ax + H(t,x)$ for some $x \in \partial\Omega \cap D(A)$ and $t \in [0,1]$, then $x = (I+A)^{-1}H(t,x)$. Since $(I+A)^{-1}$ is nonexpansive and $(I+A)^{-1}0 = 0$, we have

$$\|x\| \leq \|H(t,x)\| = \|tKx\| \leq t\|x\|.$$

Therefore, we have $t = 1$, which contradicts (2.8). Thus, by (3) of Theorem 2,

$$i(-A+K, \overline{\Omega} \cap D(A)) = i(-A+0, \overline{\Omega} \cap D(A))$$

and so (2.5) implies $i(-A+K, \overline{\Omega} \cap D(A)) = 1$. Therefore, by (4) of Theorem 2, $-A+K$ has a fixed point in $\overline{\Omega} \cap D(A)$. This completes the proof. □

THEOREM 8. *If $0 \in \Omega$ and $\|Kx\| \leq \|x + a\|$ for all $x \in \partial\Omega \cap D(A)$ and $a \in Ax$, then $-A + K$ has a fixed point in $\overline{\Omega} \cap D(A)$.*

Proof. We may assume that $x \notin -Ax + Kx$ for all $x \in \partial\Omega \cap D(A)$. Let $H(t, x) = tKx$ for all $t \in [0, 1]$ and $x \in \overline{\Omega} \cap P$. If $x \in -Ax + tKx$ for some $x \in \partial\Omega \cap D(A)$ and $t \in [0, 1]$, then $tKx \in x + Ax$. Thus there exists $a \in Ax$ such that $tKx = x + a$ and so we have $\|Kx\| \leq t\|Kx\|$. By the assumption, since $t \neq 1$, we have $Kx = 0$ and so $x + a = 0$. Therefore, by (2.3), we have $x = 0 \in \partial\Omega$, which contradicts $0 \in \Omega$. Thus we have $x \notin -Ax + H(t, x)$ for all $x \in \partial\Omega \cap D(A)$ and $t \in [0, 1]$. The rest of the proof is similar to that of Theorem 7. This completes the proof. □

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