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ON 2-CARDINALLY PERMUTABLE GROUPS

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I. Introduction

In recent years there has been much interest in the study of groups satisfying various permutability conditions (see, for instance, [1], [2] and [3]). More recently, the following condition has been studied: for some m, if S is any subset of m elements of a group G, then $|S^2| < m^2$ (where, for subsets A, B of G, AB stands for $\{ab; a \in A, b \in B\}$). It was shown that groups with this property are finite-by-abelian-by-finite. In [5], more generalized condition, collapsing condition, was introduced by Semple and Shalev. A group G is called n-collapsing if for every subset of n-element in G, $|S^n| < n^n$ and G is collapsing if it is n-collapsing for some n. They proved that for a finitely generated residually finite group G, it is collapsing if and only if it is nilpotent-by-finite. Now we consider a similar notion of permutable subsets.

DEFINITIONS. (a) A subset S is said to be *special* if there exists a subset T of a cyclic subgroup of G such that $S = xTy$ or $xTy \bigcup \{t\}$ where $x, y \in G$, $t \in T$. In other word, a special subset of G is of the form $\{xt^{n_1}y, xt^{n_2}y, \dots, xt^{n_r}y\}$ or $\{t^{n_i}, xt^{n_1}y, xt^{n_2}y, \dots, xt^{n_r}y\}$ where $1 \leq i \leq r$ and $x, y, t \in G$.

(b) For integers $1 < m \leq n$, a group G is said to be 2-*cardinally permutable to* (m, n) if (i) G has at least one m-element special subset and one n-element special subset, and (ii) for every m-element special subset A and n-element special subset B, AB and BA have same cardinalities.

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We call a group G 2-*cardinally permutable* if it is 2-cardinally permutable to (m,n) for some $1 < m \leq n$.

For every integer pair $1 < m \leq n$, it seems hard to completely characterize groups which are 2-cardinally permutable to (m,n). However in this note we will completely characterize groups which are 2-cardinally permutable to (m,n), where $1 < m \leq n \leq 3$ and then show that 2-cardinally permutable groups are center-by-(finite exponent). Moreover nonperiodic 2-cardinally permutable groups are nothing but abelian. As an immediate corollary, we note that 2-cardinally permutable groups are collapsing.

II. Results

LEMMA 1. *Let G be 2-cardinally permutable to (m,n) with $1 < m \leq n \leq 3$ and $x, y \in G$. Then*
a) *if $x^2 = 1$, then x lies in the center of G;*
b) *if $[x,y] \neq 1$, then $y^x = y^{-1}$.*

Proof. (a) Suppose that x has order 2 and $[x,y] \neq 1$ for some $y \in G$. If $y^2 = 1$, then we take $A = \{1, x, xy\}$ and $B = \{1, y, xy\}$. Then $|AB| < |BA|$, a contradiction. If $y^2 \neq 1$ then we may take special subsets $A = \{1, y, yx\}$ and $B = \{1, x, xy\}$. Then $AB = \{1, x, y, xy, yx, yxy, y^2\}$ and $BA = \{1, y, x, yx, xy, xyx, xy^2, xy^2x\}$. Since $|AB| < |BA|$, there must be one collapsing in BA. The only possible cases are (i) $y = xy^2x$ or (ii) $yx = xy^2x$. These two relations are same and so there must be a collapsing in AB. However $x = yxy$ is the only possible case in AB. Hence we have $y^3 = 1$ and $x = yxy$. Now we take $A = \{1, x, xy\}$ and $B = \{1, y, xy\}$. Then we get $|AB| < |BA|$. For the cases of $m = n = 2$ and $m = 2, n = 3$, we may take $A = \{1, x\}$, $B = \{y, xy\}$ and $A = \{1, x\}$, $B = \{1, y, xy\}$ respectively.

(b) Let $[x,y] \neq 1$. Take $A = \{1, x, xy\}$ and $B = \{1, y, x^{-1}y\}$. Then $AB = \{1, y, x^{-1}y, x, xy, xy^2, xyx^{-1}y\}$ and $BA = \{1, x, xy, y, yx, yxy, x^{-1}y, x^{-1}yx, x^{-1}yxy\}$. Note $|AB| < |BA| = |AB| + 2$. Hence there must be collapsing in BA and we have seven possible cases, namely, $1 = x^{-1}yxy$, $x = yxy$, $x = x^{-1}yxy$, $xy = x^{-1}yx$, $yx = x^{-1}y$, $yx = x^{-1}yxy$, and $yxy = x^{-1}yx$. Now we have to find compatible 2 cases. By simple check, we have at least one of the following three relations,

(i) $x = yxy$, (ii) $x^3 = y^2$(with $y = xyx$) and (iii) $x^3 = 1$(with $y = xyx$). If (i) happens, we are done. If (ii) or (iii) happens, then we take replace x, y by xy, y respectively. Then we have at least one of three relations, namely, (i′) $xy = yxyy$, (ii′) $xyxyxy = y^2$ and (iii′) $xyxyxy = 1$. Since $y = xyx$, (ii′) or (iii′) can not happen. If (i′) happens, $y = xyx = yxyyx$ and so $y^2 = x^{-2}$. This is final contradiction. For the case of $m = n = 2$ and $m = 2, n = 3$, we may take $A = \{1, y\}$, $B = \{x, yx\}$ and $A = \{1, y\}$, $B = \{1, x, yx\}$ respectively. □

THEOREM A. *G is 2-cardinally permutable to* $(2,2)$ *or* $(2,3)$ *if and only if either G is abelian or the direct product of a quaternion group of order 8 and an elementary abelian 2-group.*

Proof. Let G be 2-cardinally permutable to $(2,2)$ or $(2,3)$. Then by Lemma 1(b), $x^y = x^{\pm 1}$, any x, y in G. So G is a Dedekind group and every element of odd order is in the centre of G. If G is not abelian, then G has no elements of odd order, otherwise, with x, y, z in G, $[x, y] \neq 1$, z of odd order, we get $(xz)^y = x^{-1}z \neq (xz)^{\pm 1}$. Now the result follows from the structure of Dedekind groups (see [4], p. 139).

For the converse, let $G = Q \times D$ where D is an elementary abelian 2-group and Q a quaternion group of order 8. First we show that G is 2-cardinally permutable to $(2,3)$. Actually we do not have to take special subsets. Let $A = \{g_1, g_1ax\}$, $B = \{g_2, byg_2, czg_2\}$ be given two subsets of G, where $a, b \in Q$, $x, y \in D$ and $g_1, g_2 \in G$. Write $A' = \{\ 1,\ ax\ \}$, $A'' = \{\ 1,\ a^{\epsilon}x\ \}$ and $B' = \{\ 1,\ by,\ cz\ \}$ where $\epsilon = 1$ if g_2g_1 lies in the centeralizer of a, and $\epsilon = -1$ if not. Then $|AB| = |A'B'|$ and $A'B' = \{1, by, cz, ax, abxy, acxz\}$. And $|BA| = |B'A''|$ and $B'A'' = B' \cup B' \cdot a^{\epsilon}x = \{1, by, cz, a^{\epsilon}x, ba^{\epsilon}xy, ca^{\epsilon}xz\}$. In every case, if there is one collapsing in $A'B'$, there is a corresponding one in $B'A''$. For example, if $cz = abxy$, then $z = xy$ and $c = ab$. Hence $c = ba$ or $c = ba^{-1}$. Thus $cz = baxy$ or $by = ca$ if $\epsilon = 1$, and $by = ca^{-1}$ or $cz = ba^{-1}xy$ if not.

For the case of $m = n = 2$, we may apply a similar argument. □

THEOREM A′. *G is 2-cardinally permutable to* $(3,3)$ *if and only if G is abelian.*

Proof. Note that a quaternion group $Q = \langle x, y | x^4 = 1, y^2 = x^2, yx = x^3y\rangle$ is not 2-cardinally permutable to $(3,3)$. For $A = \{1, x, xy\}$ and

$B = \{1, x, xyx\}$, we have $|AB| < |BA|$. So the result follows by the same argument in Theorem A. $\square$

LEMMA 2. *A 2-cardinally permutable group G is center-by-(finite exponent).*

Proof. Let G be 2-cardinally permutable to (m, n). We claim that there exists an integer k such that $[y^k, x] = 1$ for all $x, y \in G$. Let $x, y \in G$. We consider two n-element subsets A and B where $A = \{1,\ y,\ y^2, \cdots,\ y^{m-1}\}$ and $B = \{x,\ yx,\ y^2x, \cdots,\ y^{n-1}x\}$. Then $AB = \{x,\ yx,\ y^2x, \cdots,\ y^{m+n-2}x\}$ and so $|AB| = m + n - 1$. Now $BA = B \bigcup By \bigcup By^2 \cdots \bigcup By^{m-1}$.

If $|B \bigcup By| - |B| > 1$, then there is some integer $\ell < m$ such that $T = B \bigcup By \bigcup \cdots \bigcup By^{\ell} \supset By^{\ell+1}$. Then $Ty^h \subset T$ for all integer h. Hence y has bounded order.

Suppose that $|B \bigcup By| - |B| = 1$. We then have a relation $x = yxy$. Take $m - 2$ distinct integers, $a_1, a_2, ..., a_{m-2}$ with $3 < a_i < p$ for some big positive integer p, and $n - 2$ distinct integers, $b_1, b_2, ..., b_{n-2}$ with $p + 1 < b_1$ and $b_{i+1} = b_i + p$. Here we consider another two special subsets of G, $A = \{1, x\} \bigcup A_1$ and $B = \{y, xy\} \bigcup B_1$ where $A_1 = \{xy^{a_1}, xy^{a_2}, ..., xy^{a_{m-2}}\}$ and $B_1 = \{xy^{b_1}, xy^{b_2}, ..., xy^{b_{n-2}}\}$. Then

$$AB = \{y, xy, xxy\} \bigcup A_1xy \bigcup A_1y \bigcup B_1 \bigcup xB_1 \bigcup A_1B_1 \ ,$$
$$BA = \{\ y,\ xy,\ xyx,\ yx\} \bigcup xyA_1 \bigcup yA_1 \bigcup B_1 \bigcup B_1x \bigcup B_1A_1.$$

By the choice of a_i, b_j, $|AB| < |BA|$. $\square$

THEOREM B. *Nonperiodic 2-cardinally permutable groups are abelian.*

Proof. Let G be 2-cardinally permutable to (m, n). Then G has nonperiodic centre Z containing a torsion-free element z. Suppose $x, y \in G$ and $[x, y] \neq 1$. Let xZ be of order k in G/Z. Take $m - 2$ distinct integers, $a_1, a_2, ..., a_{m-2}$ with $3 < a_i < p$ for some big positive integer p, and $n - 2$ distinct integers, $b_1, b_2, ..., b_{n-2}$ with $p + 1 < b_1$ and $b_{i+1} = b_i + p$, where a_i and b_j are 1 under modulo k. Now we consider two special subsets of G, $A = \{1, (xz)\} \bigcup A_1$ and $B =$

$\{y,(xz)y\}\bigcup B_1$ where $A_1 = \{(xz)^{a_1},(xz)^{a_2},...,(xz)^{a_{m-2}}\}$ and $B_1 = \{(xz)^{b_1}y,(xz)^{b_2}y,...,(xz)^{b_{n-2}}y\}$. Then

$$AB = \{y, xyz, xxyz^2\}\bigcup A_1xy\bigcup A_1y\bigcup B_1\bigcup xB_1\bigcup A_1B_1 \ ,$$
$$BA = \{\ y,\ xyz,\ xyxz^2,\ yxz\}\bigcup xyA_1\bigcup yA_1\bigcup B_1\bigcup B_1x\bigcup B_1A_1.$$

By the choice of a_i, and b_j, all elements in AB are distinct. Hence there should be at least one collapsing in BA. Since x is not in centre Z, we have finitely many types of possible relations.

Note that the above argument is independent of the power of z. That is, we can replace z in A_1 and B_1 by z^ℓ for all integers ℓ and get the same possible relations. Thus at least one type of relation should hold for infinitely many integers. This is clearly impossible. □

Note. (i) If a finite group G has a cyclic subgroup K of index 2, then G is 2-cardinally permutable to (m,n), where $1 < m \leq n = |K|+1$. For suppose that G is not abelian. G has $|K|+1$ element special subset and it is of the form, $\{h\}\bigcup Ha$ where $a \notin H$, $h \in H$ and H is a cyclic subgroup of order $|K|$. Let A be an n element special subset of G and B an $|K|+1$ element special subset. If $A\cap H \neq \emptyset$ and $A \cap H\alpha \neq \emptyset$ for some $\alpha \notin H$, then $AH\alpha = H\alpha A = G$ and so $AB = BA = G$. If not, $|AB| = |BA| = |A| + |H|$. In particular, a symmetric group of degree 3 is 2-cardinally permutable to $(2,4)$, $(3,4)$ and $(4,4)$ and a dihedral group of order $2n$ is 2-cardinally permutable to $(m, 2n+1)$, where $m = 2,...,2n+1$. From these examples and Theorem A, A', we know that there are no systematic set-inclusion relations depending on m and n in the class of 2-cardinally permutable groups to (m,n).

(ii) For prime numbers $p < q$, if G is a group of order pq, then G is 2-cardinally permutable to $(q+1, q+1)$. Note that the Sylow q-subgroup Q is normal. Thus the result follows easily from the fact that every $q+1$ element special subset is of the form $\{q\}\cup Qg$ for some $g \in G$. Actually by using the normality of Q, we can show that G is 2-cardinally permutable to $(m, q+1)$ where $2 < m \leq q+1$.

(iii) Not every finite group is 2-cardinally permutable.
Let $G = \langle x,y,z | x^3 = y^3 = z^3 = 1, xy = yxz, xz = zx, yz = zy\rangle$. Suppose that G is 2-cardinally permutable. Then it is 2-cardinally

permutable to (m,n) for some $1 < m \leq n \leq 4$. By Theorem A and A', we can assume that (m,n) is $(2,4)$, $(3,4)$, or $(4,4)$. These possibilities can be easily removed by taking special subsets $\{1, x, ..., xy^{m-2}\}$ and $\{1, y, xy, x^2y\}$ for the case of $(m,4)$.

(iv) Clearly 2-cardinally permutable groups are collapsing by Lemma 2. We consider two 2-cardinally permutable groups, the direct product of a quaternion group of order 8 and an elementary abelian 2-group, and an infinite cyclic group. Then the direct product of above two groups is not 2-cardinally permutable by Theorem B for it is a non-periodic nonabelian group. Hence the class of 2-cardinally permutable groups is not closed under a direct product.

Question. Are 2-cardinally permutable groups subgroup-closed ?

An infinite 2-cardinally permutable group is abelian if it is nonperiodic by Theorem B. It however can be much more complicated if it is periodic. For example, the direct product of a symmetric group S_3 of degree 3 and an infinite elementary abelian 2-group is 2-cardinally permutable. But $S_3 \times S_3 \times \cdots$ can not be 2-cardinally permutable. Hence it seems hard to characterize periodic infinite 2-cardinally permutable groups. Here we have some information on 2-cardinally permutable groups. Suppose that G is 2-cardinally permutable to (m,n) and has an element x of prime order $p = n-1$. Then the subgroup $\langle x \rangle$ is of interest. In Lemma 1(a) we have that such a subgroup of 2-cardinally permutable group G to $(m,3)$ lies in the centre of G. We can not expect such property for $n \geq 4$ (for example, the symmetric group of degree 3 in note (i)). However we will see that if $\langle x \rangle$ is normal in G in the following theorem. This shows that 'Tarski Monster' groups can not be 2-cardinally permutable.

THEOREM C. *Let G be 2-cardinally permutable to (m,n) and p, q primes. Then for an element x of order $p^s q^t \geq n-1$, $\langle x \rangle$ contains a proper normal subgroup of G.*

Proof. Case I. Let $m \neq n$ or $p^s q^t > n-1$.

For $y \in G$, we may take $A = \{1, x, \ldots, x^{m-1}\}$ and $B = \{1\} \bigcup B_1$ where $B_1 = \{y, xy, \ldots, x^{n-2}y\}$. Then $AB = A \cup \bigcup_{i=0}^{n-2} Ax^i y$ and $|AB| \leq |A| + n + m - 2$. Now $BA = A \cup \bigcup_{i=0}^{m-1} B_1 x^i$. If $A \cap B_1 x^i \neq \emptyset$ for some i, then $x^j = x^\ell y x^i$ and so $y \in N(\langle x \rangle)$, the normalizer of $\langle x \rangle$ in G.

If not, note that $\sum_{i=0}^{m-1}|Bx^i| = (n-2)(m-1)+n+m-2$. Thus $B_1x^i \cap B_1x^j \neq \emptyset$ for some i, j and so $y \in N(\langle x^i \rangle)$ for some i.

Case II. Let $m = n$ and $p^sq^t = n-1$.

(i) $p^sq^t > 3$.

For the case of $|y| = \ell > 2$, we take $A = \{1\} \bigcup A_1$ and $B = \{1\} \bigcup B_1$ where $A_1 = \{y^{\ell-1}, y^{\ell-1}x, \ldots, y^{\ell-1}x^{n-2}\}$ and $B_1 = \{y, xy, \ldots, x^{n-2}y\}$. Since $|AB| \leq 3p^sq^t + 1 < |B_1||A_1| = (p^sq^t)^2$, there should be at least one collapsing in B_1A_1. Thus $x^iy = yx^j$ for some i, j. Hence we get $y \in N(\langle x^i \rangle)$ for some i. For the case of $y^2 = 1$, we may follow the same argument as above for $A = \{x\} \bigcup A_1$ and $B = \{1\} \bigcup B_1$ where $A_1 = \{1, yxy, \ldots, yx^{n-2}y\}$ and $B_1 = \{y, xy, \ldots, x^{n-2}y\}$.

(ii) $p^sq^t = 3$.

If $y^2 = 1$, then simply we may take $A = \{x, 1, yxy, yx^2y\}$ and $B = \{1, y, xy, x^2y\}$ and get a result. If $y^2 \neq 1$, then we take $A = \{1\} \cup y\langle x \rangle$ and $B = \{1\} \cup \langle x \rangle y$. Note that $|AB| \leq 9$. Since 1 is distinct from all elements in $\langle x \rangle y \cdot y \langle x \rangle$, there should be at least one relation of the form $y^2 = x^iy^2x^j$ in $\langle x \rangle y \cdot y\langle x \rangle$, a subset of BA. Hence y^2 lies in $N(\langle x \rangle)$. Thus every element of order 3, 5 or 7 lies in $N(\langle x \rangle)$ and so does an element of order 6. For other cases we may take $A = \{1, x, xy, xy^2\}$ and $B = \{1, y, y^2, y^3\}$. Suppose that $7 < |y|$, the order of y and $y \notin N(\langle x \rangle)$. Then $|AB| = 10$. Now $BA = B \cup Bx \cup Bxy \cup Bxy^2$. Clearly $B \cap By = B \cap Bxy = \emptyset$. Thus $Bx \cap Bxy \neq \emptyset$. Note that the once we have a relation for xy, then the other elements are fixed automatically, for example, if $xy = y^2x$, then $yxy = y^3x$ and so on. Since $y \notin N(\langle x \rangle)$, we have the only non-trivial possible relation $xy = y^{-1}x$. The other cases are easily removed. For example, suppose $xy = y^2x$. Then by simple calculations we have $BA = B \cup \{x, yx, \ldots, y^7x\}$ and $|BA| > 10$, a contradiction. Let $xy = y^{-1}x$, that is, $x = yxy$. However this relation and the above relation $y^2 = x^iy^2x^j$ can not be compatible. Finally we assume that $|y| = 4$. Then $|AB| = 8$. Clearly $B \cap By = B \cap Bxy = \emptyset$ and so Bx should be Bxy. Hence $xy = y^3x$, i.e., $x = yxy$. If $|xy| \neq 4$, then by the above argument $xy \in N$ and so does y. Thus $|xy| = 4$, that is, $xyxyxyxy = x^4 = x = 1$, a final contradiction.

(iii) $p^sq^t = 2$.

This case is already treated in Lemma 1.

Hence for every $y \in G$, y lies in $N(\langle x^i \rangle)$ for some i. Note that there are only two minimal subgroups, $\langle x^{p^sq^{t-1}} \rangle$ and $\langle x^{p^{s-1}q^t} \rangle$ of $\langle x \rangle$. That is $y \in N(\langle x^{p^s q^{t-1}} \rangle)$ or $N(\langle x^{p^{s-1} q^t} \rangle)$. Since no group is a union of two proper subgroups, $G = N(\langle x^{p^s q^{t-1}} \rangle)$ or $N(\langle x^{p^{s-1} q^t} \rangle)$. □

COROLLARY 6. *Let G be 2-cardinally permutable to (m, n). If M is an abelian maximal subgroup containing an element of order $> n$, then M contains a nontrivial normal subgroup of G.*

Proof. Let M contain an element x of order $> n$. Then for $y \in G \backslash M$, $y \in N(\langle x^i \rangle)$ for some i by the proof of the above theorem. Since $M \subset N(\langle x^i \rangle)$ and $y \in N(\langle x^i \rangle)$, $\langle x^i \rangle$ is normal in G. □

As an easy consequence of above corollary, we have that A_4, the alternating group of degree 4, is the smallest non-trivial finite group which is not 2-cardinally permutable. For, if A_4 is 2-cardinally permutable, every subgroup of order 3 is normal, a contradiction.

Finally we consider a finiteness condition on 2-cardinally permutable groups. Recall that a group is called *locally graded* if every finitely generated non-trivial subgroup has a finite non-trivial quotient.

THEOREM D. *If G is a locally graded 2-cardinally permutable group, then it is abelian or locally finite.*

Proof. Let G be finitely generated and periodic, and let N the finite residual of G. Then G is center-by-(finite exponent). Hence G/N is a residually finite center-by-(finite exponent) and collapsing. Now G/N is nilpotent-by-finite and so finite. Suppose $N \neq 1$. Since G is locally graded, N has a non-trivial finite factor group N/K. But then $G/core_G(K)$ is finite, contrary to the choice of N. Hence G is finite. □

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