J. Korean Math. Soc. 34(1997), No. 1, pp. 181–193

LATTICE PATH COUNTING IN A BOUNDED PLANE

H. G. PARK, D. S. YOON[1] AND S. H. CHOI[2]

1. Introduction

The enumeration of various classes of paths in the real plane has an important implications in the area of combinatorics with statistical applications. In 1887, D. André [3, pp. 21] first solved the famous ballot problem, formulated by Bertrand [2], by using the well-known reflection principle which contributes tremendously to resolve the problems of enumeration of various classes of lattice paths in the plane. First, it is necessary to state the definition of NSEW-paths in the plane which will be employed throughout the paper. From [3, 10, 11], we can find results concerning many of the basics discussed in section 1 and 2.

We mean by an NSEW-path a lattice path in the plane consisting of unit steps, each in a direction North, South, East, or West. In [4], DeTemple and Robertson found the formula for the number of NSEW-paths of fixed length joining the origin to a point in the plane. And also, they discussed the number of all such paths which do not cross the line of the form $y = ax + b$ for some integers a, b with $a \neq 0$ (see also [5]). An analoguous situation with certain restriction on the line was considered by Guy, Krattenthaler and Sagang [9], Flajolet [6], and Arques [1]. While Gouyou-Beauchamps [7] showed that formula for the number of sub-diagonal paths of length n from the origin to a point on the x-axis which do not cross the line $y = x$, can be written as a product of Catalan numbers. Furthermore, he proved in [8] that the above formula is identical with the number of standard Young tableaux of n cells for height 4 or 5.

Received May 16, 1996.

1991 Mathematics Subject Classification: Primary 05A15; Secondary 03D40.

Key words and phrases: ballot number, Catalan number, dyck language, dyck path, lattice path.

[1] Supported by Faculty Research Grant of Hanyang University, 1996.

[2] Supported by Faculty Research Fund of Jeonju University, 1996.

To explain our main results of this article, we first propose the following sections in the plane; for a positive integer c,

$$\Pi = \{(x, y) \in R^2 | x \geq 0, \ -x \leq y \leq x\}$$

Π_c : the set of points below the line $y = -x - c$ in Π, except the points on the line

Π_{-c} : the set of points above the line $y = x - c$ in Π, except the points on the line

$\Pi_{\pm c} = \Pi_c \cap \Pi_{-c}$

If p and q are integers such that $|q| \leq p$, it will be shown that the number of NSEW-paths of length n in Π, which join the origin with (p, q), can be written explicitly as

$$\frac{(p-q+1)(p+q+1)(n!)^2}{(\frac{n+p-q}{2}+1)!(\frac{n-p+q}{2})!(\frac{n+p+q}{2}+1)!(\frac{n-p-q}{2})!},$$

where n and $p+q$ have the same parity. By using the above fact and the reflection principle, the number of all such NSEW-paths of fixed length in Π_c, which join the origin with $(p, q) \in \Pi_c$, will be obtained in section 3.1. And also, we give the formula for the same question regarding Π_{-c} and $\Pi_{\pm c}$.

2. Preliminaries

Consider an alphabet X having a finite number of elements, called letters. A word is a finite sequence of letters in X and the empty word will be denoted by 1 as a unit. We denote by X^* the set of all words generated by letters in the alphabet, together with 1. And, we define a binary operation on X^* by concatenation of two words. Then, X^* becomes a free monoid on X. The length $|f|$ of $f \in X^*$ is the number of letters in f. For a letter x in f, $|f|_x$ means the number of x in f. A word f' is called a left factor of a word $f \in X^*$ if $f = f'f''$ for some $f'' \in X^*$.

Let $P = \{x, \bar{x}, y, \bar{y}\}$ and $A = \{a, \bar{a}\}$ be alphabets. For each fixed $\nu \in \{x, y\}$ and all $f \in P^*$, we define a mapping $\delta_\nu : P^* \to N$ by

$$\delta_\nu(f) = |f|_\nu - |f|_{\bar{\nu}},$$

where N is the set of all integers. And we define a mapping $\delta : A^* \longrightarrow N$ by

$$\delta(f) = |f|_a - |f|_{\bar{a}}$$

for all $f \in A^*$.

An alphabet D is called a Dyck language if every word f of A^* satisfies the following conditions:

(i) $\delta(f) = 0$, and

(ii) $\delta(f') \geq 0$ for any left factor f' of f.

Consider the set $D_{n,r}$ of all left factors f' of every word f in the Dyck language D such that $|f'| = n$ and $\delta(f') = r$, where n and r have the same parity. If we code an East step by a and a North step by $\bar{a}$, then it is clear that Dyck words represent the minimal sub-diagonal paths joining the origin with a point on the line of the form $y = x$. And, if we code a North-East step by a and a South-East step by $\bar{a}$, then the Dyck words represent the Dyck paths. Since each word of $D_{n,r}$ represents the minimal sub-diagonal path of length n and height r, it can be shown that

$$\begin{aligned} |D \cap A^{2n}| &= C_n = \frac{1}{n+1}\binom{2n}{n}, \text{ and} \\ |D_{n,r}| &= \lambda_{n,r} = \frac{(r+1)(n!)}{(\frac{n-r}{2})!(\frac{n+r}{2}+1)!}, \end{aligned} \tag{1}$$

where C_n is called the n^{th} Catalan number and $\lambda_{n,r}$ is called the ballot number. If n and $p+q$ have the same pairity for integers p, q with $|q| \leq p$, then we let $P^*_{n,p,q}$ be the language composed of words f of P^* satisfying the following properties:

(i) $|f| = n$.

(ii) $\delta_x(f) = p$, and $\delta_y(f) = q$.

(iii) $\delta_x(f') \geq |\delta_y(f')| \geq 0$ for every left factor f' of f.

Consider the quarter plane Π as earlier mentioned in section 1. If we code a East (or West) step by x (or $\bar{x}$), and a North (or South) step by y (or $\bar{y}$), then the language $P^*_{n,p,q}$ can be identical with the set of NSEW-paths of length n, lying on the plane Π and joining the origin with a point (p, q). Each path of $P^*_{n,p,q}$ do not cross the lines $y = x$ and $y = -x$.

Next, we set a language $D_{n,p-q} \times D_{n,p+q} \subseteq A^* \times A^*$ which is the Cartesian product of $D_{n,p-q}$ and $D_{n,p+q}$. Then, we want to verify that a bijection between $D_{n,p-q} \times D_{n,p+q}$ and $P^*_{n,p,q}$. It is noted that $A^* \times A^*$ is a free monoid under the binary operation of concatenation of two words; that is, $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$ for any two words (g_1, h_1) and (g_2, h_2) in $A^* \times A^*$. Without loss of generality, we assume that each word of either $D_{n,p-q}$ or $D_{n,p+q}$ represents a Dyck path whose initial point is the origin $(0, 0)$.

Consider a well-defined homomorphism $\psi : P^*_{n,p,q} \longrightarrow D_{n,p-q} \times D_{n,p+q}$ such that $\psi(1) = (1, 1)$ and the mapping sends

$$x \longrightarrow (a, a), \bar{x} \longrightarrow (\bar{a}, \bar{a}),\ y \longrightarrow (\bar{a},\ a), \text{and}\ \bar{y} \longrightarrow (a, \bar{a})$$

for each letter $x, \bar{x}, y$, or $\bar{y}$ in $f \in P^*_{n,p,q}$. For an example, if $f = x\bar{y}xyxy\bar{x}y \in P^*_{9,3,2}$, then

$$\begin{aligned}\psi(x\bar{y}xyxyxy\bar{x}) &= (a,a)(a,\bar{a})(a,a)(\bar{a},a)(a,a)(\bar{a}. a)(a,a)(\bar{a},a)(\bar{a},\bar{a}) \\ &= (aaa\bar{a}a\bar{a}a\bar{a}\bar{a}, a\bar{a}aaaaaa\bar{a})\end{aligned}$$

in $D_{9,1} \times D_{9,5}$ (see Figure 2.1).

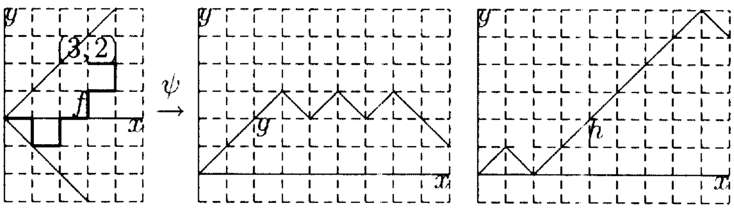

Figure 2.1 $\psi(f) = (g,\ h)$ in $D_{9,1} \times D_{9,5}$

Clearly the mapping ψ is injective by the construction. So it surffices to show that the mapping is surjective. For $(g, h) \in D_{n,p-q} \times D_{n,p+q}$, we will choose a unique word $f \in P^*_{n,p,q}$ such that $\psi(f) = (g, h)$ with $|f| = n$. Consider the left factors f', g', and h' of f, g, and h such that $\psi(f') = (g', h')$. Note that $|g| = |h|$. If $g = 1 = h$, then we are done. Suppose that $g \neq 1 \neq h$. Then, it is not hard to show that

$$\delta_x(f') = \frac{\delta(h') + \delta(g')}{2}$$

and

$$\delta_y(f') = \frac{\delta(h') - \delta(g')}{2}.$$

Thus, $\delta_x(f') = p$ and $\delta_y(f') = q$. Since $\delta(h') \geq 0$ and $\delta(g') \geq 0$,

$$|\frac{\delta(h') - \delta(g')}{2}| \leq |\frac{\delta(h') + \delta(g')}{2}|$$

and therefore $|\delta_y(f')| \le \delta_x(f')$. Thus $f \in P^*_{n,p,q}$.

According to the above facts, ψ is a bijection between $P^*_{n,p,q}$ and $D_{n,p-q} \times D_{n,p+q}$, It follows from this and (2.1) that we have

$$\begin{aligned}|P^*_{n,p,q}| &= |D_{n,p-q}||D_{n,p+q}| \\ &= \frac{(p-q+1)(p+q+1)(n!)^2}{(\frac{n+p-q}{2}+1)!(\frac{n-p+q}{2})!(\frac{n+p+q}{2}+1)!(\frac{n-p-q}{2})!}.\end{aligned}$$

This proved the following theorem.

THEOREM 2.1. *Let p and q be integers with $|q| \le p$. Then, the number $\sigma_\Pi(n,p,q)$ of NSEW-paths of length n in Π, which join the origin with a point (p,q), is*

$$\frac{(p-q+1)(p+q+1)(n!)^2}{(\frac{n+p-q}{2}+1)!(\frac{n-p+q}{2})!(\frac{n+p+q}{2}+1)!(\frac{n-p-q}{2})!}.$$

The next corollary is obvious from Theorem 2.1.

COROLLARY 2.2. $\sigma_\Pi(n,r,0) = (\lambda_{n,r})^2$, *where n and r have the same parity and $r \le n$.*

REMARK. Let a mapping $\psi : P^*_{n,p,q} \to D_{n,p-q} \times D_{n,p+q}$ be a bijection defined in the proof of Theorem 2.1. For $f \in P^*_{n,p,q}$ and a left factor f' of f, consider the left factors g', h' of g, h such that $\psi(f') = (g', h')$. Then,

(a) if $0 \le \delta_y(f') \le \delta_x(f')$, then $\delta(g') \le \delta(h')$, and
(b) if $-\delta_x(f') \le \delta_y(f') \le 0$, then $\delta(g') \ge \delta(h')$.

If $f = a_1a_2\cdots a_{k-1}a_k$ is a word in A^* for some positive integer k, then we denote $\bar{f} = \bar{a_k}\bar{a_{k-1}}\cdots\bar{a_2}\bar{a_1}$, where

$$\bar{a_j} = \begin{cases} a & , \text{ if } a_j = \bar{a} \\ \bar{a} & , \text{ if } a_j = a \end{cases}$$

for each j with $1 \le j \le k$.

COROLLARY 2.3. *If C_n is the n^{th} Catalan number, then*

$$C_n = \begin{cases} \sum_{j=0}^{n/2} \sigma_\Pi(n,2j,0) & , \text{ if } n \text{ is even} \\ \sum_{j=0}^{(n+1)/2} \sigma_\Pi(n,2j-1,0) & , \text{ if } n \text{ is odd}\end{cases}$$

Proof. Suppose that the given integer n is even. Consider a set α_k of all paths in $D_{2n,0}$ passing through a point (n,k) for each even number k with $0 \leq k \leq n$. Then, we claim that there is a bijection between α_k and $D_{n,k} \times D_{n,k}$ for each fixed k. We define a mapping $\phi : D_{n,k} \times D_{n,k} \to \alpha_k$ by $\phi((g,h)) = g\bar{h}$ for all $(g,h) \in \alpha_k$. $D_{n,k} \times D_{n,k}$ are mutually disjoint for every k with $0 \leq k \leq n$. It is clear that the mapping ϕ is well-defined since there is a unique $g\bar{h} \in \alpha_k$ for every $(g,h) \in D_{n,k} \times D_{n,k}$. The injectiveness of ϕ follows by a straightforward computation. For $f \in \alpha_k$, we can choose two factors f' and f'' of f such that $f = f'f''$ and $|f'| = |f''| = n$. Then, $(f', \bar{f''}) \in D_{n,k} \times D_{n,k}$ and $\phi((f', \bar{f''})) = f$. Thus, ϕ is a bijection. It implies that $|P^*_{n,k,0}| = \sigma_\Pi(n,k,0) = |\alpha_k|$ for all even number k with $0 \leq k \leq n$. It is noted that $D_{2n,0} = \cup_{j=0}^{n/2} \alpha_{2j}$. Hence, if n is even, then

$$\begin{aligned} C_n &= \lambda_{2n,0} = \sum_{j=0}^{n/2} |\alpha_{2j}| \\ &= \sum_{j=0}^{n/2} \sigma_\Pi(n,2j,0). \end{aligned}$$

The other case is easily deduced in the similar way.
This completes the proof of Corollary 2.3. □

3. NSEW-paths in a bounded plane

We now consider NSEW-paths in Π_c, Π_{-c}, or $\Pi_{\pm c}$ as earlier mentioned in section 1. Here, we will investigate the number of paths in all such classes by means of the classical reflection principle and the elementary combinatorial computation; see Comtet [3], pp. 22-23. Throughout this section, integers n and $p+q$ have the same pairity.

Let $\{d_j\}$ be an increasing sequence of integers d_j for $j = 1, 2, \ldots$. Then, for some real number k, we denote by $\mu_k(d_j)$ the largest positive integer t such that $d_t \leq k$.

Lemma 3.1. *Let s_0 be a given point $(p,q) \in \Pi_c$ for some integers p, q, and ℓ_0 the line of the form $y = -x + c$. Let $S = \{s_i = (x_i, y_i) \in \Pi\}$ be the sequence of points lying on the line $y = x - p + q$, where s_i and s_{i+1} are symmetric w.r.t. the line ℓ_i : $y = -x + x_i + y_{i+1}$ for $i = 0, 1, \ldots$.*

Then, the number $\sigma_{\Pi_c}(n,p,q)$ *of NSEW-paths of length* n *in* Π_c *, which join the origin with* (p,q)*, is*

$$\sum_{i=0}^{t}(-1)^i\lambda_{n,p-q}\lambda_{n,x_i+y_i},$$

where $t=\mu_n(x_j+y_j)$.

Proof. It is true that every NSEW-path of length n in Π_c that joins the origin with (p,q) can neither touch nor cross the line $y=-x+c$ for $n<-p-q+2c$, and so $\mu_n(x_j+y_j)=0$ for $j=0,1,\ldots$. It means that $\sigma_{\Pi_c}(n,p,q)=\lambda_{n,p-q}\lambda_{n,p+q}=|P^*_{n,p,q}|$, which is true by Theorem 2.1. Now, we assume that $n\geq -p-q+2c$. Then there is at least one path from the origin to (p,q) in Π which either touch or cross the line $y=-x+c$. If we apply the reflection principle to the point $s_1=(-q+c,-p+c)$ that is symmetric to (p,q) w.r.t. the line $y=-x+c$, then the number of required paths is $\sigma_{\Pi_c}(n,p,q)=|P^*_{n,x_0,y_0}|-|P^*_{n,x_1,y_1}|$ (see Figure 3.1). If there is possibly a path $f\in P^*_{n,x_1,y_1}$ such that f touch the line $y=-x+2c+1$ and (α,β) is the first common point with f and the line , then we set $f=f_1f_2$ where f_1 is the left factor of f from the origin to (α,β). For the reflection f_2' of f_2 w.r.t the line $y=-x+2c+1$, f_1f_2' must exist in Π. However, for the sufficiently large value of $|f|$, f_1f_2' passes through the boundary of Π. Since such paths already were included in P^*_{n,x_1,y_1}, we have

$$\sigma_{\Pi_c}(n,p,q)=|P^*_{n,x_0,y_0}|-\{|P^*_{n,x_1,y_1}|-|P^*_{n,x_2,y_2}|\}$$

by applying again the reflection principle to the symmetric points s_1 and $s_2=(p+c+1,q+c+1)$ w.r.t. the line $y=-x+2c+1$. If we use the reflection principle recursively with the same procedure, then all the possible symmetric points lie on the line $y=x-p+q$ and clearly $x_j+y_j\leq n$ for each j. So, we have $t+1$ symmetric points $s_0,s_1,\ldots,s_t$, where $t=\mu_n(x_j+y_j)$.

According to the above facts,

$$\begin{aligned}\sigma_{\Pi_c}(n,p,q) &= |P^*_{n,x_0,y_0}|-\{|P^*_{n,x_1,y_1}|-\{|P^*_{n,x_2,y_2}|-\{|P^*_{n,x_3,y_3}|-\cdots\cdots\\ &= \lambda_{n,p-q}\{\lambda_{n,x_0+y_0}-\lambda_{n,x_1+y_1}+\lambda_{n,x_2+y_2}-\lambda_{n,x_3+y_3}+\cdots\cdots\\ &\quad +(-1)^t\lambda_{n,x_t+y_t}\}\\ &= \sum_{i=0}^{t}(-1)^i\lambda_{n,p-q}\lambda_{n,x_i+y_i}.\end{aligned}$$

This proved the Lemma 3.1. □

In the above Lemma 3.1, ℓ_i is of the form $y = -x + 2^i(c+1) - 1$ for each $i = 0, 1, 2, \cdots$, and the exact value of $t = \mu_n(x_j + y_j)$ is given in section 4.

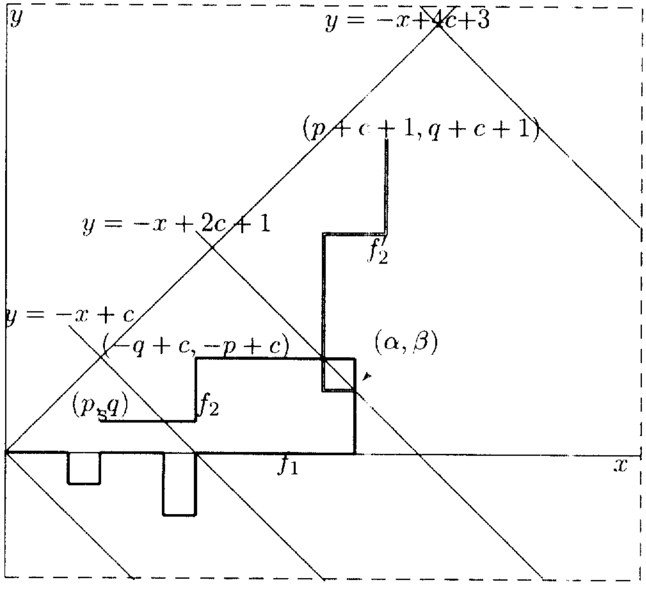

Figure 3.1 NSEW-paths in Π_c

A simple form may be obtained for $\eta = \sum_{i=0}^{t}(-1)^i \lambda_{n,x_i+y_i}$ given in Lemma 3.1. But even for $i = 2$ a simple form seems difficult to get. However, one may prove that η is equal to the number of Dyck paths of length n and height p, which do not cross nor touch the horizontal line $y = c$.

The following Corollary 3.2 produce an interesting formula which can reformulate $\sigma_{\Pi_c}(n, p, q)$ given in Lemma 3.1. The proof is obvious since $\sigma_{\Pi_c}(n, p, q) = \sigma_{\Pi_{-c}}(n, p, -q)$ for the fixed point $(p, q) \in \Pi_c$.

COROLLARY 3.2. *Let s_0 be a given point $(p, q) \in \Pi_{-c}$ for some integers p, q, and ℓ_0 the line of the form $y = x - c$. Let $S = \{s_i = (x_i, y_i) \in \Pi_{-c}\}$ be the sequence of symmetric points lying on the line $y = -x + p + q$, where s_i and s_{i+1} are symmetric w.r.t. the line $\ell_i : y = x - x_i + y_{i+1}$ for $i = 0, 1, \cdots$. Then,*

$$\sigma_{\Pi_{-c}}(n, p, q) = \sum_{i=0}^{t}(-1)^i \lambda_{n,p+q} \lambda_{n,x_i - y_i},$$

where $t = \mu_n(x_j - y_j)$ *for* $j = 0, 1, \cdots$.

THEOREM 3.3. *Let* (p, q) *be a point in* $\Pi_{\pm c}$ *for some integers* p *and* q. *Then, the number* $\sigma_{\Pi_{\pm c}}(n, p, q)$ *of NSEW-paths of length* n *in* $\Pi_{\pm c}$, *which join the origin with* (p, q), *is*

$$\frac{\sigma_{\Pi_c}(n, p, q) \cdot \sigma_{\Pi_{-c}}(n, p, q)}{\sigma_{\Pi}(n, p, q)}.$$

Proof. Suppose that there is at least one NSEW-path of fixed length n, which join the origin with the given point (p, q) and cross both the line $y = -x + c$ and also the line $y = x - c$. Let $f = f'f''$ where f' is a left factor of f from $(0, 0)$ to the point A meeting with the line $y = -x + c$ at first, and f''_c the path symmetric to f'' w.r.t. the line $y = -x + c$. If the point B is the first common point with the line $y = x - c$ whenever f''_c passes through the line $y = x - c$, then we have the path $g = g'g''_{-c}$ where g' is the path from $(0, 0)$ to B in $f'f''_c$ and g''_{-c} is the path symmetric to the path g'' from B to $(-q + c, -p + c)$ in f''_c (see Figure 3.2). Thus we have a path of length n from $(0, 0)$ to $(-p + 2c, -q)$ in Π. It is noted that $(-p + 2c, q)$ is symmetric to (p, q) w.r.t. the line $y = -x + c$ and the line $y = x - c$ simultaneously. So, by using the reflection principle, if all such paths g are in $\Pi_{\mp 2c \mp 1}$, then

$$\begin{aligned} \sigma_{\Pi_{\pm c}}(n, p, q) &= +\{|P^*_{n,p,q}| - |P^*_{n,-q+c,-p+c}|\} \\ &\quad -\{|P^*_{n,q+c,p-c}| - |P^*_{n,-p+2c,-q}|\}. \end{aligned}$$

In the above equallity, we note that the first parenthesis of right handside is dependent only on the alternating signs and the symmetric points in Π_{-c}, and the second parenthesis is dependent only on the alternating signs and the symmetric points in $\Pi_{-2c-1} \backslash \Pi_{-c}$. It is not hard to show that the general case still holds in Π if we apply the reflection principle recursively to each lines $y = \pm x \mp c$, $y = \mp x \mp 2c \mp 1$, $y = \pm x \mp 4c \mp 3, \ldots$, with the associated symmetric points for sufficiently large n. Thus, by using Theorem 2.1 and Lemma 3.1, we obtain

$$\begin{aligned}
&\sigma_{\Pi_{\pm c}}(n,p,q)\\
=\ &+\{|P^*_{n,p,q}| - |P^*_{n,-q+c,-p+c}| + |P^*_{n,p+c+1,q+c+1}| - \cdots\}\\
&-\{|P^*_{n,q+c,p-c}| - |P^*_{n,-p+2c,-q}| + |P^*_{n,q+2c+1,p+1}| - \cdots\}\\
&+\{|P^*_{n,p+c+1,q-c-1}| - |P^*_{n,-q+2c+1,-p-1}| + |P^*_{n,p+2c+2,q}| - \cdots\}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
=\ &\frac{\sigma_{\Pi_c}(n,p,q)}{\lambda_{n,p-q}}\{\lambda_{n,p-q} - \lambda_{n,q-p+2c} + \lambda_{n,p-q+2c+2} - \cdots\cdots\}\\
=\ &\frac{\sigma_{\Pi_c}(n,p,q)}{\lambda_{n,p-q}} \cdot \frac{\sigma_{\Pi_c}(n,p,-q)}{\lambda_{n,p+q}}\\
=\ &\frac{\sigma_{\Pi_c}(n,p,q)\cdot\sigma_{\Pi_c}(n,p,-q)}{\sigma_{\Pi}(n,p,q)}.
\end{aligned}$$

Therefore, the required number can be derived from the above equation. □

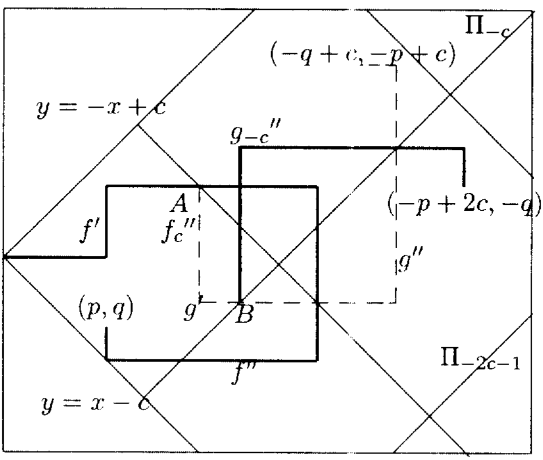

Figure 3.2 NSEW-paths in $\Pi_{\pm c}$

Next, we have the following corollaries that are clear from Lemma 3.1 and Corollary 2.2.

COROLLARY 3.4. $\sigma_{\Pi}(n,p,q) = \dfrac{\sigma_{\Pi_c}(n,p,q)\cdot\sigma_{\Pi_{-c}}(n,p,q)}{\sigma_{\Pi_{\pm c}}(n,p,q)}$ *for the fixed point* $(p,q)\in\Pi_{\pm c}$.

COROLLARY 3.5. *Let k be a positive integer and r a nonnegative integer with $r < k$. Then, the number of Dyck paths with length n and height r, is*

$$\frac{\sigma_{\Pi_k}(n, r, 0)}{\sqrt{\sigma_{\Pi_{\pm k}}(n, r, 0)}},$$

where n and r have the same parity.

4. Examples

In this section, we give several problems related to NSEW-path counting in the plane. To compute the required number, we need to determine the exact value of $t = \mu_n(x_j + y_j)$ given in Lemma 3.1. It is true that $t = \mu_n(x_{2j}+y_{2j})+\mu_n(x_{2j-1}+y_{2j-1})$. Let $z_{2j} = 2+2^3+2^5+\cdots+2^{2j-1}$ and $z_{2j-1} = 2+2^2+2^4+\cdots+2^{2j-2}$ for each fixed j. Clearly, $z_{2j-1} = (2^{2j}+2)/3$ and $z_{2j} = (2^{2j+1} - 2)/3$ for $j = 1, 2, \ldots$. It says that

$$\begin{aligned} x_{2j} + y_{2j} &= p + q + z_{2j}(c + 1), \text{ and} \\ x_{2j-1} + y_{2j-1} &= -p - q + z_{2j-1}(c + 1) - 2. \end{aligned} \tag{2}$$

Hence, we have

$$t = \mu_{c_1}(z_{2j}) + \mu_{c_2}(z_{2j-1}), \tag{3}$$

where $c_1 = (n - p - q)/(c + 1)$ and $c_2 = (n + p + q + 2)/(c + 1)$.

EXAMPLE 1. If $n = 6$, $(p, q) = (0, 0)$, and $c = 2$, then $\mu_2(z_{2j}) = 1$ and $\mu_{8/3}(z_{2j} - 1) = 1$. So, $t = \mu_2(z_{2j}) + \mu_{8/3}(z_{2j-1}) = 2$ and $x_0 + y_0 = 0$, $x_1 + y_1 = 4$, and $x_2 + y_2 = 6$ by using (4.1) and (4.2). Thus, $\sigma_{\Pi_2}(6, 0, 0)$ is equal to $\lambda_{6,0}\{\lambda_{6,0} - \lambda_{6,4} + \lambda_{6,6}\} = 5\{5 - 5 + 1\} = 5$ by Lemma 3.1. In fact, there are only 5 NSEW-paths of length 6 in Π_2 which join $(0, 0)$ to itself as follows:

$$\begin{aligned} f_1 &= x\bar{x}x\bar{x}x\bar{x} : (0,0) \to (1,0) \to (0, 0) \to (1, 0) \to (0, 0) \\ &\to (1,0) \to (0,0) \end{aligned}$$

$$
\begin{aligned}
f_2 &= x\bar{x}x\bar{y}y\bar{x} : (0,0) \to (1,0) \to (0,\ 0) \to (1,\ 0) \to (1,-1) \\
&\qquad \to (1,0) \to (0,0) \\
f_3 &= x\bar{y}y\bar{y}y\bar{x} : (0,0) \to (1,0) \to (1,-1) \to (1,\ 0) \to (1,-1) \\
&\qquad \to (1,0) \to (0,0) \\
f_4 &= x\bar{y}y\bar{x}x\bar{x} : (0,0) \to (1,0) \to (1,-1) \to (1,\ 0) \to (0,\ 0) \\
&\qquad \to (1,0) \to (0,0) \\
f_5 &= x\bar{y}x\bar{x}y\bar{x} : (0,0) \to (1,0) \to (1,-1) \to (2,-1) \to (1,-1) \\
&\qquad \to (1,0) \to (0,0)
\end{aligned}
$$

EXAMPLE 2. If $(p, q) = (4, -2)$, $n = 22$, and $c = 6$, then $\mu_6(x_i+y_i) = 2$ and $x_0 + y_0 = 2$, $x_1 + y_1 = 6$, and $x_2 + y_2 = 12$. Thus,

$$
\begin{aligned}
\sigma_{\Pi_6}(22, 4, -2) &= \lambda_{22,6}\{\lambda_{22,2} - \lambda_{22,6} + \lambda_{22,12}\} \\
&= 149,226\{149,226 - 149,226 + 19,019\} \\
&= 2,838,129,294.
\end{aligned}
$$

The number is equal to $\sigma_{\Pi_{-6}}(22, 4, 2)$.

EXAMPLE 3. By Theorem 3.3, the number of NSEW-paths of length 6, which join the origin to itself in $\Pi_{\pm 2}$, is

$$
\sigma_{\Pi_{\pm 2}}(6, 0, 0) = \frac{\sigma_{\Pi_2}(6,0,0) \cdot \sigma_{\Pi_{-2}}(6,0,0)}{[\lambda_{6,0}]^2} = \frac{5 \cdot 5}{5^2} = 1.
$$

It makes sense because there is only one such an NSEW-path of length 6 in the plane and the path is $x\bar{x}x\bar{x}x\bar{x}$.

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H. G. Park and D. S. Yoon
Department of Mathematics
Hanyang University
Seoul 133-791, Korea

S. H. Choi
Department of Mathematics
Jeonju University
Jeonju 560-759, Korea