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ON THE (B, N)-CONSTRUCTION

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1. Introduction

In this paper, k will denote an arbitrary field. If m,n are natural numbers, then $M_{m\times n}(k)$ will denote the set of all $m \times n$ matrices with entries in k. Every k-algebras will be assumed to contain a (multiplicative) identity $1 \neq 0$. A k-subspace R_0 of a k-algebra R will be called a k-subalgebra of R if R_0 is closed under multiplication from R and R_0 contains the identity of R. We will assume all k-algebra homomorphisms take the identity to identity.

Let R be a commutative, k-subalgebra of $M_{n\times n}(k)$. R is called a maximal, commutative, k-subalgebra of $M_{n\times n}(k)$ if R satisfies the following property : If R' is a commutative, k-subalgebra of $M_{n\times n}(k)$ and $R \subseteq R'$, then $R = R'$. Thus, a maximal, commutative, k-subalgebra of $M_{n\times n}(k)$ is a maximal element with respect to inclusion in the set of all maximal, commutative, k-subalgebras of $M_{n\times n}(k)$. We will denote the set of all maximal, commutative, k-subalgebra of $M_{n\times n}(k)$ by $\mathcal{M}_n(k)$.

Maximal, commutative, k-subalgebras of $M_{n\times n}(k)$ come in many different shapes and sizes. Here are a few examples.

Example 1. Let p and q be positive integers such that $\mid p-q \mid \leq 1$. Set $n = p+q$. Let

$$R = \left\{ \begin{pmatrix} xI_p & Z \\ O_{q\times p} & xI_q \end{pmatrix} \mid x \in k, Z \in M_{p\times q}(k) \right\}. \tag{1}$$

In (1), I_p denotes the identity matrix of size p by p and $O_{q\times p}$ denotes the zero matrix of size q by p. Then, R is a commutative, k-subalgebra

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of $M_{n\times n}(k)$ and it is easy to check that $R \in \mathcal{M}_n(k)$. We will call R a Schur algebra of size n. $\square$

EXAMPLE 2 (R.C. COURTER). Let $\mathcal{C} = kI_{14} \bigoplus J \subseteq T_{14}$, where J is the set of all matrices of the following form:

(2)

$$\left(\begin{array}{cccccccccccccc}
\multicolumn{2}{c}{O_{2\times 2}} & \multicolumn{10}{c}{O_{2\times 10}} & \multicolumn{2}{c}{O_{2\times 2}} \\
x_{11} & O & & & & & & & & & & & & \\
O & x_{11} & & & & & & & & & & & & \\
x_{12} & O & & & & & & & & & & & & \\
O & x_{12} & & & & & & & & & & & & \\
x_{21} & O & & & & & & & & & & & & \\
O & x_{21} & \multicolumn{10}{c}{O_{10\times 10}} & \multicolumn{2}{c}{O_{10\times 2}} \\
x_{22} & O & & & & & & & & & & & & \\
O & x_{22} & & & & & & & & & & & & \\
z_{11} & z_{12} & & & & & & & & & & & & \\
z_{21} & z_{22} & & & & & & & & & & & & \\
y_{11} & y_{12} & z_{11} & z_{12} & z_{21} & z_{22} & O & O & O & O & x_{11} & x_{12} & \multicolumn{2}{c}{O_{2\times 2}} \\
y_{21} & y_{22} & O & O & O & O & z_{11} & z_{12} & z_{21} & z_{22} & x_{21} & x_{22} & &
\end{array}\right)$$

In (2), x_{ij}, y_{ij}, and $z_{ij} \in k$. In [3], R.C.Courter showed $\mathcal{C} \in \mathcal{M}_{14}(k)$. Notice $dim_k(\mathcal{C}) = 13$. We will call $\mathcal{C}$ Courter's algebra. $\square$

Let B be a commutative ring and M a right B-module. The direct sum $B \oplus M$ of the B-modules B and M can be given the structure of a commutative ring by defining multiplication in the following way.

$$(b_1, m_1)(b_2, m_2) = (b_1 b_2, m_2 b_1 + m_1 b_2), b_i \in B, m_i \in M, i = 1, 2.$$

The commutative ring thus defined is called the idealization of M and will be denoted by $B \ltimes M$.

Suppose $R \in \mathcal{M}_n(k)$. We say R is a (B, N)-construction if R is k-algebra isomorphic to $B \ltimes N^\ell$ for some finite dimensional, local, commutative, k-algebra B and finitely generated, faithful, B-module N and some positive integer ℓ. Here, N^ℓ denotes the direct sum of ℓ copies of B-module N.

In [1], W.C. Brown and F.W. Call showed that the Courter's algebra $\mathcal{C}$ is a (B, N)-construction. In fact, $\mathcal{C} \cong \mathcal{B} \ltimes N^2$, where $\mathcal{B}$ is the Schur algebra of size 4, $N = k^4$.

Let $\Omega = \{(R, J(R), k) \in \mathcal{M}_{14}(k) \mid dim_k(R) = 13, i(J(R)) = 3\}$. Here, $i(J(R))$ is the index of the Jacobson radical $J(R)$. Example 2 shows that $\mathcal{C} \in \Omega$. It was conjectured that every $R \in \Omega$ is a (B,N)-construction. If k is the real numbers, then we will construct a k-algebra $(\mathcal{S}, J(\mathcal{S}), k) \in \Omega$ that is not a (B,N)-construction.

2. (B,N)-construction

In this section, we will let

$$\check{E}_{ij} = \begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ E_{ij} & O & O_2 \end{pmatrix}$$

Here, E_{ij} is the (i,j)-th matrix unit in $M_{2\times 2}(k)$ and O_n is the $n \times n$ zero matrix.

Now, we will construct a new k-algebra $(\mathcal{S}, J, k) \in \Omega$ with the following matrices. Let

$$\delta_i = \begin{pmatrix} O_2 & O & O \\ P_i & O_{10} & O \\ O & Q_i & O_2 \end{pmatrix}, \quad i = 1, \ldots, 8. \tag{3}$$

Here,

$$P_1 = \begin{pmatrix} I_2 \\ O_2 \\ O_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} O_2 \\ I_2 \\ O_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} O_2 \\ O_2 \\ I_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad P_4 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ I_2 \\ O_2 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{11} \end{pmatrix}, \quad P_6 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{12} \end{pmatrix}, \quad P_7 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{21} \end{pmatrix}, \quad P_8 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{22} \end{pmatrix},$$

and

$$\begin{aligned} Q_1 &= (I_2\ O_2\ O_2\ O_2\ E_{11}), & Q_2 &= (O_2\ I_2\ O_2\ O_2\ E_{12}) \\ Q_3 &= (O_2\ O_2\ I_2\ O_2\ E_{21}), & Q_4 &= (O_2\ O_2\ O_2\ I_2\ E_{22}) \\ Q_5 &= (E_{11}\ O_2\ E_{21}\ O_2\ O_2), & Q_6 &= (E_{12}\ O_2\ E_{22}\ O_2\ O_2) \\ Q_7 &= (O_2\ E_{11}\ O_2\ E_{21}\ O_2), & Q_8 &= (O_2\ E_{12}\ O_2\ E_{22}\ O_2). \end{aligned}$$

If we let $\mathcal{S} = k[\delta_1, \ldots, \delta_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$, it is easy to check that $\mathcal{S} \in \Omega$.

THEOREM 2.1. *Suppose k is the field of real numbers. Then, $\mathcal{S}$ is not a (B, N)-construction.*

Proof. Suppose $\mathcal{S}$ is a (B,N)-construction. Then, by [1: Theorem 4], $\mathcal{S}$ contains an ideal I which satisfies the following two properties.

(4)
(a) $Ann_{\mathcal{S}}(I) = I$
(b) $0 \longrightarrow I \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}/I \longrightarrow 0$ splits as k-algebras.

Since $\check{E}_{ij} I = 0$ for $i,j = 1,2$, $\check{E}_{ij} \in I$, $i,j = 1,2$ by (a). Notice that $\delta_1 \notin I$. Otherwise, $\delta_1^2 = 0$ by (a). Since $\delta_1^2 = \check{E}_{11} + \check{E}_{22}$, this is impossible. Thus, $\delta_1 \notin I$. Let $\nu : \mathcal{S} \longrightarrow \mathcal{S}/I$ be the natural homomorphism. Let $\theta : \mathcal{S}/I \longrightarrow \mathcal{S}$ be a splitting map. Then, $\theta(\delta_1 + I) = \delta_1 + r$, where $r \in I$. Since θ is a k-algebra homomorphism, we have

$$
\begin{aligned}
\delta_1^2 + 2\delta_1 r &= \delta_1^2 + 2\delta_1 r + r^2 \\
&= (\delta_1 + r)^2 = (\theta(\delta_1 + I))^2 \\
&= \theta((\delta_1 + I)^2) = \theta(\check{E}_{11} + \check{E}_{22} + I) \\
&= \theta(0 + I) = 0. \qquad (5)
\end{aligned}
$$

Let $r = \sum_{i=1}^{8} t_i \delta_i + \sum_{j,\ell=1}^{2} s_{j\ell} \check{E}_{j\ell}$, for some real numbers $t_i, s_{j\ell}$. Then, (5) implies

$$(1 + 2t_1 + 2t_5)\check{E}_{11} + 2t_6 \check{E}_{12} + (1 + 2t_1)\check{E}_{22} = 0. \qquad (6)$$

Thus, $t_1 = -\frac{1}{2}$, $t_5 = t_6 = 0$. Hence, we have

$$r = -\frac{1}{2}\delta_1 + t_2\delta_2 + t_3\delta_3 + t_4\delta_4 + t_7\delta_7 + t_8\delta_8 + \sum_{j,\ell=1}^{2} s_{j\ell}\check{E}_{j\ell}. \qquad (7)$$

Since $r \in I$, $r^2 = 0$ by (a). Thus,

$$
\begin{aligned}
&(\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_2t_7)\check{E}_{11} + 2t_2t_8\check{E}_{12} + 2t_4t_7\check{E}_{21} \\
&\quad + (\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_4t_8)\check{E}_{22} = 0. \qquad (8)
\end{aligned}
$$

Therefore, we have the following four equations.

$$
(9) \qquad \begin{aligned}
t_2 t_8 &= 0 \\
t_4 t_7 &= 0 \\
\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_2 t_7 &= 0 \\
\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_4 t_8 &= 0.
\end{aligned}
$$

We will show that there is no real solution of the equations given in (9). Since $t_2 t_8 = 0$, $t_2 = 0$ or $t_8 = 0$. Thus, we have the following two cases to consider.

Case 1: $t_2 = 0$

Case 2: $t_8 = 0$

We will show both cases lead to a contradiction.

Case 1: Suppose $t_2 = 0$. Then, from the third equation in (9), we have $\frac{1}{4} + t_3^2 + t_4^2 = 0$. This is impossible since t_3 and t_4 are real numbers.

Case 2: Suppose $t_8 = 0$. Then, the fourth equation in (9) implies $\frac{1}{4} + t_2^2 + t_3^2 + t_4^2 = 0$. This is again impossible.

Thus, the equations in (9) have no real solutions. This implies that there is no $r \in I$ such that $\theta(\delta_1 + I) = \delta_1 + r$. Thus, there is no splitting map of the exact sequence given in (4). Therefore, $\mathcal{S}$ is not a (B, N)-construction. □

In Theorem 2.1, we proved that $\mathcal{S}$ is not a (B, N)-construction if k is the field of real numbers. But as we will see, if k is the field of complex numbers, then $\mathcal{S}$ is a (B, N)-construction. More generally, we prove $\mathcal{S} \in \Omega$ is a (B, N)-construction if k is an algebraically closed field.

THEOREM 2.2. *Suppose k is an algebraically closed field. Then, $\mathcal{S}$ is a (B, N)-construction.*

Proof. Since k is an algebraically closed field, the polynomial $f(x) = x^2 + 1 \in k[x]$ has a root i. Set $\alpha_1 = \delta_1 - i\delta_2$, $\alpha_2 = \delta_3 - i\delta_4$, $\alpha_3 = \delta_5 - i\delta_7$, and $\alpha_4 = \delta_6 - i\delta_8$. Then, $\alpha_1\delta_5 = \check{E}_{11}$, $\alpha_1\delta_6 = \check{E}_{12}$, $\alpha_2\delta_5 = \check{E}_{21}$, $\alpha_2\delta_6 = \check{E}_{22}$. Thus, the ideal I generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ contains $\check{E}_{mn}$ for all $m, n = 1, 2$. It is easy to check $I = L(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22})$. Thus, $\dim_k(I) = 8$.

Let $\beta \in Ann_{\mathcal{S}}(I)$. Then, $\beta = \sum_{n=1}^{8} t_n\delta_n + \sum_{m,n=1}^{2} s_{mn}\check{E}_{mn}$ for some $t_n, s_{mn} \in k$. Since $\alpha_n \in I$ for all $n = 1,2,3,4$, $\alpha_n\beta = 0$ for all $n = 1,2,3,4$. From $\alpha_1\beta = \alpha_2\beta = 0$, we have

$$\begin{aligned}(t_1 - it_2 + t_5 - it_7)\check{E}_{11} + (t_6 - it_8)\check{E}_{12} + (t_1 - it_2)\check{E}_{22} = 0\\ (t_3 - it_4)\check{E}_{11} + (t_5 - it_7)\check{E}_{21} + (t_3 - it_4 + t_6 - it_8)\check{E}_{22} = 0.\end{aligned} \tag{10}$$

Equation (10) implies

$$\begin{aligned} t_1 - it_2 + t_5 - it_7 &= 0\\ t_3 - it_4 + t_6 - it_8 &= 0\\ t_6 - it_8 &= 0\\ t_1 - it_2 &= 0\\ t_3 - it_4 &= 0\\ t_5 - it_7 &= 0\end{aligned} \tag{11}$$

Thus, we have $t_1 = it_2, t_3 = it_4, t_5 = it_7$, and $t_6 = it_8$. Hence,

$$\begin{aligned}\beta &= it_2\delta_1 + t_2\delta_2 + it_4\delta_3 + t_4\delta_4 + it_7\delta_5 + it_8\delta_6\\ &\quad + t_7\delta_7 + t_8\delta_8 + \textstyle\sum_{m,n=1}^{2} s_{mn}\check{E}_{mn}\\ &= it_2\alpha_1 + it_4\alpha_2 + it_7\alpha_3 + it_8\alpha_4 + \textstyle\sum_{n,n=1}^{2} s_{mn}\check{E}_{mn}.\end{aligned} \tag{12}$$

Therefore, $\beta \in I$ and hence $Ann_{\mathcal{S}}(I) \subseteq I$. Since $I^2 = 0$, $I \subseteq Ann_{\mathcal{S}}(I)$. Thus, $Ann_{\mathcal{S}}(I) = I$.

Notice that $\Delta = \{I_{14} + I, \delta_1 + I, \delta_3 + I, \delta_5 + I, \delta_6 + I\}$ is a k-vector space basis of $\mathcal{S}/I$. Since $dim_k(I) = 8$ and $dim_k(\mathcal{S}) = 13$, we have $dim_k(\mathcal{S}/I) = 5$. Since $i\alpha_n \in I$ for all $n = 1,2,3,4$ we have

$$\begin{aligned}\delta_2 + I &= -i\delta_1 + I\\ \delta_4 + I &= -i\delta_3 + I\\ \delta_7 + I &= -i\delta_5 + I\\ \delta_8 + I &= -i\delta_6 + I.\end{aligned} \tag{13}$$

Let θ be the k-vector space homomorphism from $\mathcal{S}/I$ to $\mathcal{S}$ defined as follows:

$$\begin{aligned}\theta(I_{14} + I) &= I_{14}\\ \theta(\delta_1 + I) &= \tfrac{1}{2}\delta_1 + \tfrac{1}{2}i\delta_2\\ \theta(\delta_3 + I) &= \tfrac{1}{2}\delta_3 + \tfrac{1}{2}i\delta_4\\ \theta(\delta_5 + I) &= \tfrac{1}{2}\delta_5 + \tfrac{1}{2}i\delta_7\\ \theta(\delta_6 + I) &= \tfrac{1}{2}\delta_6 + \tfrac{1}{2}i\delta_8.\end{aligned} \tag{14}$$

Then,

$$(15)\qquad \begin{aligned} \theta(\delta_2 + I) &= \theta(-i\delta_1 + I) = -i\theta(\delta_1 + I) = \tfrac{1}{2}\delta_2 - \tfrac{1}{2}i\delta_1 \\ \theta(\delta_4 + I) &= \theta(-i\delta_3 + I) = -i\theta(\delta_3 + I) = \tfrac{1}{2}\delta_4 - \tfrac{1}{2}i\delta_3 \\ \theta(\delta_7 + I) &= \theta(-i\delta_5 + I) = -i\theta(\delta_5 + I) = \tfrac{1}{2}\delta_7 - \tfrac{1}{2}i\delta_5 \\ \theta(\delta_8 + I) &= \theta(-i\delta_6 + I) = -i\theta(\delta_6 + I) = \tfrac{1}{2}\delta_8 - \tfrac{1}{2}i\delta_6 \end{aligned}$$

It is easy to show that θ is a k-algebra homomorphism.

Recall $\nu : \mathcal{S} \longrightarrow \mathcal{S}/I$ is the natural homomorphism defined by $v(r) = r + I$ for $r \in \mathcal{S}$. Then, easy computations show that $\nu\theta(r + I) = r + I$ for all $r \in \mathcal{S}$. This implies the exact sequence

$$(16)\qquad 0 \longrightarrow I \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}/I \longrightarrow 0$$

splits as k-algebras. Therefore, the ideal I of $\mathcal{S}$ satisfies the two conditions in [1:Theorem 4] and $\mathcal{S}$ is a (B, N)-construction. □

Theorem 2.1 and 2.2 show that the question: "When is $(R, J(R), k) \in \Omega$ a (B, N)-construction?" depends on the field k. From Theorem 2.2, one could ask that every $(R, J, k) \in \Omega$ is a (B, N)-construction if k is an algebraically closed field. At present, this conjecture is opened.

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