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A GORENSTEIN IDEAL OF CODIMENSION 4

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1. Introduction

Let k be an infinite field and let $\mathbb{X} = \{P_1, \cdots, P_s\}$ be a set of s-distinct points in $\mathbb{P}^n$. We denote by $I(\mathbb{X})$ the defining ideal of $\mathbb{X}$ in the polynomial ring $R = k[x_0, \cdots, x_n]$ and by A the homogeneous coordinate ring of $\mathbb{X}$, $A = \sum_{t=0}^{\infty} A_t$. The Hilbert function of $\mathbb{X}$ (or of A) is the function $\mathbf{H} : \mathbb{N} \longrightarrow \mathbb{N}$ described by

$$\mathbf{H}(\mathbb{X}, t) = \mathbf{H}(A, t) = \dim_k A_t = \dim_k R_t - \dim_k I_t.$$

The first difference of the Hilbert function of $\mathbb{X}$ (or of A) is

$$\triangle\mathbf{H}(\mathbb{X}, t) = \begin{cases} 1, & \text{for } t = 0, \\ \mathbf{H}(\mathbb{X}, t) - \mathbf{H}(\mathbb{X}, t-1), & \text{for } t \geq 1. \end{cases}$$

We also denote by $\sigma(\mathbb{X})$ (or $\sigma(A)$) the least integer for which

$$\mathbf{H}(\mathbb{X}, \sigma) = 0 \text{ and } \mathbf{H}(\mathbb{X}, \sigma - 1) \neq 0.$$

In [GPS], we obtained the number and the degrees of the generators of an ideal of a k-configuration in $\mathbb{P}^2$ and the minimal graded free resolution of the ideal.

In [S], we obtained the number and the degrees of the generators of an ideal of a k-configuration in $\mathbb{P}^3$. The aim of this paper is to construct a Gorenstein ideal of codimension 4 from them.

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2. k-configurations in $\mathbb{P}^3$

Roberts and Roitman [6] introduced the following definition:

DEFINITION 2.1. A *k-configuration* is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions:

there exist integers $1 \le d_1 < \cdots < d_m$, and subsets $\mathbb{X}_1, \cdots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \cdots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^m \mathbb{X}_i$;
(2) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \cdots, m$, and;
(3) $\mathbb{L}_i$ $(1 < i \le m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the k-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \cdots, d_m)$.

THEOREM 2.2 ([3]). *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^2$ of type $(d_1, \cdots, d_m)$ and let I be the ideal of $\mathbb{X}$. Then $\nu(I) = m + 1$ and the minimal graded free resolution of I as an R-module is:*

$$\begin{aligned} 0 \longrightarrow & R(-(d_1 + m)) \oplus \cdots \oplus R(-(d_i + m - i + 1)) \oplus \cdots \oplus R(-(d_m + 1)) \\ \longrightarrow & R(-m) \oplus R(-(d_1 + m - 1)) \oplus \cdots \oplus R(-(d_i + m - i)) \\ & \oplus \cdots \oplus R(-d_m) \longrightarrow I \longrightarrow 0 \end{aligned}$$

where $\nu(I)$ is the number of the minimal generators of I.

DEFINITION 2.3 ([4]). A *k-configuration* in $\mathbb{P}^3$ is a finite set of points which satisfies the following conditions:

there exist subsets $\mathbb{X}_1, \cdots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct hyperplanes $\mathbb{H}_1, \cdots, \mathbb{H}_u$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^u \mathbb{X}_i$;
(2) $\mathbb{X}_i \subset \mathbb{H}_i$ for any $i = 1, \cdots, u$;
(3) $\mathbb{H}_i$ $(1 < i \le u)$ does not contain any points of $\mathbb{X}_j$ for any $j < i$, and;
(4) $\mathbb{X}_i$ $(1 \le i \le u)$ is a k-configuration in $\mathbb{H}_i$ of type $(d_{i1}, \cdots, d_{im_i})$ with $d_{im_i} < m_{i+1}$ for every $1 \le i < u$.

In this case, the k-configuration in $\mathbb{P}^3$ is said to be of type

$$(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u}).$$

For simplicity of notation, let (d_{ij}) denote the tuple of integers $(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u})$ with $d_{im_i} < m_{i+1}$ for every $1 \le i < u$.

REMARK 2.4. (1) All k-configurations in $\mathbb{P}^3$ of type (d_{ij}) have the same Hilbert function, which will be denoted by $\mathbf{H}^{(d_{ij})}$.

(2) Let $\mathbf{H} = \{b_t\}_{t\geq 0}$ be a zero-dimensional O-sequence with $b_1 = 4$. Applying the procedure of Theorem 4.1 in [GMR], we can get integers $(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u})$ with $d_{im_i} < m_{i+1}$ for every $1 \leq i < u$ such that

$$\mathbf{H} = \mathbf{H}^{(d_{ij})}.$$

THEOREM 2.5. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u})$ and let I be the ideal of $\mathbb{X}$. Then $\nu(I) = \sum_{i=1}^{u} m_i + u + 1$ and the degrees of the minimal generators of I are:*

$$u,\ m_1 + u - 1,\ d_{11} + m_1 + u - 2,\ \cdots,\ d_{1i} + m_1 + u - i - 1,\ \cdots,$$
$$d_{1m_1} + u - 1,$$
$$\vdots$$
$$m_j + u - j,\ d_{j1} + m_j + u - j - 1,\ \cdots,\ d_{ji} + m_j + u - i - j,\ \cdots,$$
$$d_{jm_j} + u - j,$$
$$\vdots$$
$$m_u,\ d_{u1} + m_u - 1,\ \cdots,\ d_{ui} + m_u - i,\ \cdots,\ d_{um_u}.$$

3. The construction of a Gorenstein ideal of codimension 4

In this section, we shall construct some Gorenstein ideals of codimension 4 using k-configurations in $\mathbb{P}^3$ and find the degrees of the minimal generators of these ideals.

DEFINITION 3.1 ([3]). A *weak k-configuration* in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions:

there exist integers $1 \leq d_1 \leq \cdots \leq d_m$, subsets $\mathbb{X}_1, \cdots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \cdots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that:

(1) $i \leq d_i$ for each $i = 1, \cdots, m$;
(2) $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{X}_i$;
(3) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \cdots, m$, and;
(4) $\mathbb{L}_i$ $(1 < i \leq m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the weak k-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \cdots, d_m)$.

THEOREM 3.2 ([3]). *Let $\mathbb{X}$ be a weak k-configuration in $\mathbb{P}^2$ of type $(d_1, \cdots, d_m, \cdots, d_{m+\ell})$ where $d_1 < \cdots < d_m = \cdots = d_{m+\ell}$ and $\ell \geq 1$. Let I be the ideal of $\mathbb{X}$. If $\mathbb{X}$ is a subset of complete intersection in $\mathbb{P}^2$ of type $(m+\ell, d_m)$, then $\nu(I) = m+1$ and the minimal free resolution of I, as an R-module, is:*

$$\begin{aligned} 0 \longrightarrow & R(-(d_1+m+\ell)) \oplus \cdots \oplus R(-(d_i+m+\ell-i+1)) \oplus \cdots \oplus \\ & R(-(d_{m-1}+\ell+2)) \oplus R(-(d_m+\ell+1)) \\ \longrightarrow & R(-(m+\ell)) \oplus R(-(d_1+m+\ell-1)) \oplus \cdots \oplus \\ & R(-(d_i+m+\ell-i)) \oplus \cdots \oplus R(-(d_{m-1}+\ell+1)) \oplus R(-d_m) \\ \longrightarrow & I \longrightarrow 0. \end{aligned}$$

DEFINITION 3.3. A *weak k-configuration* in $\mathbb{P}^3$ is a finite set of points which satisfies the following conditions:

there exist subsets $\mathbb{X}_1, \cdots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct hyperplanes $\mathbb{H}_1, \cdots, \mathbb{H}_u$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$;
(2) $\mathbb{X}_i \subset \mathbb{H}_i$ for any $i = 1, \cdots, u$;
(3) $\mathbb{H}_i$ $(1 < i \leq u)$ does not contain any points of $\mathbb{X}_j$ for any $j < i$, and;
(4) $\mathbb{X}_i$ $(1 \leq i \leq u)$ is a weak k-configuration in $\mathbb{H}_i$ of type $(d_{i1}, \cdots, d_{im_i})$.

In this case, the weak k-configuration in $\mathbb{P}^3$ is said to be of type

$$(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u}).$$

From the Theorem 3.2, we obtain the following theorem.

THEOREM 3.4. *Let $\mathbb{X}$ be a weak k-configuration in $\mathbb{P}^3$ of type $(d_1, \cdots, d_m, \cdots, d_{m+\ell})$ where $d_1 < \cdots < d_m = \cdots = d_{m+\ell}$ and $\ell \geq 1$. Let I be the ideal of $\mathbb{X}$. If $\mathbb{X}$ is a subset of complete intersection in $\mathbb{P}^3$ of type $(1, m+\ell, d_m)$, then $\nu(I) = m+2$ and the degrees of the minimal generators of I are*

$$1,\ m+\ell,\ d_1+m+\ell-1,\ \cdots,\ d_i+m+\ell-i, \cdots,\ d_{m-1}+\ell+1,\ d_m.$$

DEFINITION 3.5 ([4]). A finite set $\mathbb{Z}$ of points in $\mathbb{P}^n$ is said to be a *basic configuration* in $\mathbb{P}^n$ if there exist integers $r_1, \cdots, r_n$ and distinct hyperplanes $\mathbb{L}_{ij}(1 \leq i \leq n, 1 \leq j \leq r_i)$ such that

$$\mathbb{Z} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_n \text{ as schemes, where } \mathbb{H}_i = \mathbb{L}_{i1} \cup \cdots \cup \mathbb{L}_{ir_i}.$$

In this case $\mathbb{Z}$ is said to be of type $(r_1, \cdots, r_n)$.

REMARK 3.6. Let $\mathbb{Z}$ be a basic configuration in $\mathbb{P}^3$ of type (u, α, β) $(u \leq \alpha < \beta)$. Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i \subset \mathbb{Z}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u})$ where $\mathbb{X}_i$ is a k-configuration in $\mathbb{P}^2$ of type $(d_{i1}, \cdots, d_{im_i})$. Let $m_u < \alpha$ and $d_{um_u} < \beta$. Assume $\mathbb{Z}_i \subset \mathbb{Z}$ is a basic configuration in $\mathbb{P}^3$ of type $(1, \alpha, \beta)$ such that $\mathbb{X}_i \subset \mathbb{Z}_i$ and $\mathbb{Y}_i = \mathbb{Z}_i - \mathbb{X}_i$ is a weak k-configuration $\mathbb{P}^3$ of type $(\beta - d_{im_i}, \cdots, \beta - d_{i1}, \beta, \cdots, \beta)$ for every $i = 1, \cdots, u$. Let $\mathbb{Y} = \bigcup_{i=1}^{u} \mathbb{Y}_i$. Then $\mathbb{Y}$ is a weak k-configuration in $\mathbb{P}^3$.

Moreover,

$$\triangle\mathbf{H}(\mathbb{Z}, t) = \triangle\mathbf{H}(\mathbb{X}, t) + \triangle\mathbf{H}(\mathbb{Y}, \sigma - 1 - t),$$

where $\sigma = \sigma(\mathbb{X}) = u + \alpha + \beta - 2$.

Similarly,

$$\begin{aligned} \triangle\mathbf{H}(\mathbb{Z}_u, t) &= \triangle\mathbf{H}(\mathbb{X}_u, t) + \triangle\mathbf{H}(\mathbb{Y}_u, \sigma' - 1 - t) \\ \triangle\mathbf{H}(\mathbb{Z}', t - 1) &= \triangle\mathbf{H}(\mathbb{X}', t - 1) + \triangle\mathbf{H}(\mathbb{Y}', \sigma - 1 - t), \end{aligned}$$

where $\mathbb{Z}' = \bigcup_{i=1}^{u-1} \mathbb{Z}_i$, $\mathbb{X}' = \bigcup_{i=1}^{u-1} \mathbb{X}_i$, $\mathbb{Y}' = \bigcup_{i=1}^{u-1} \mathbb{Y}_i$, and $\sigma' = \alpha + \beta - 1$. Hence

$$\triangle\mathbf{H}(\mathbb{Y}, \sigma - 1 - t) = \triangle\mathbf{H}(\mathbb{Y}_u, \sigma' - 1 - t) + \triangle\mathbf{H}(\mathbb{Y}', \sigma - 1 - t).$$

Let $s = \sigma - 1 - t$. Since $\sigma' - \sigma = -u + 1$,

$$\triangle\mathbf{H}(\mathbb{Y}, s) = \triangle\mathbf{H}(\mathbb{Y}_u, s - u + 1) + \triangle\mathbf{H}(\mathbb{Y}', s).$$

Hence we obtain the following Lemma.

LEMMA 3.7. *Let $\mathbb{Y}$, $\mathbb{Y}_u$, and $\mathbb{Y}'$ be as in Remark 3.6. Then*

$$\begin{aligned} \Delta\mathbf{H}(\mathbb{Y},s) &= \Delta\mathbf{H}(\mathbb{Y}_u, s-(u-1)) + \Delta\mathbf{H}(\mathbb{Y}',s), \text{ i.e.,} \\ \mathbf{H}(\mathbb{Y},s) &= \mathbf{H}(\mathbb{Y}_u, s-(u-1)) + \mathbf{H}(\mathbb{Y}',s) \end{aligned} \tag{3.1}$$

for every $s \geq 0$.

REMARK 3.8. Let $\mathbb{Y}$ and $\mathbb{Y}_i$ be as in Remark 3.6 and let $\mathbb{Y}'' = \bigcup_{i=2}^{u} \mathbb{Y}_i$. Then, from (3.1),

$$\mathbf{H}(\mathbb{Y},t) = \mathbf{H}(\mathbb{Y}_1,t) + \mathbf{H}(\mathbb{Y}'',t-1). \tag{3.2}$$

THEOREM 3.9. *Let $\mathbb{Y}$ be as in Remark 3.6. Let $J = I(\mathbb{Y})$. Then $\nu(J) = \sum_{i=1}^{u} m_i + 3$ and the degrees of the minimal generators of J are:*

$$\begin{array}{ccc} \beta - d_{1m_1} + \alpha - 1, & \cdots, & \beta - d_{11} + \alpha - m_1, \\ \beta - d_{2m_2} + \alpha, & \cdots, & \beta - d_{21} + \alpha - m_2 + 1, \\ & \vdots & \\ u, \quad \alpha, \quad \beta - d_{um_u} + \alpha + u - 2, & \cdots, & \beta - d_{u1} + \alpha - m_u + u - 1, \quad \beta. \end{array}$$

Proof. Let $\mathbb{Y}_i$, $\mathbb{Z}$, and $\mathbb{Z}_i$ be as in Remark 3.6. Set $\mathbb{H}_i$ the hyperplane which contains $\mathbb{Z}_i$ and $H_i = I(\mathbb{H}_i)$. We shall prove the theorem by induction on u. If $u = 1$, then we are done by Theorem 3.4.

Now assume $u > 1$. Let $\mathbb{Y}''$ be as in Remark 3.8. Then, by the induction hypothesis, there exist $\sum_{i=2}^{u} m_i + 3$ minimal generators of $I(\mathbb{Y}'')$

$$\begin{array}{c} F'_{21}, \quad \cdots, \quad F'_{2m_2}, \\ \vdots \\ H_2 \cdots H_u, \quad F'_{u0}, \quad F'_{u1}, \quad \cdots, \quad F'_{um_u}, \quad F'_{um_u+1}, \end{array}$$

with degrees

$$\begin{array}{ccc} \beta - d_{2m_2} + \alpha - 1, & \cdots, & \beta - d_{21} + \alpha - m_2, \\ & \vdots & \\ u-1, \quad \alpha, \quad \beta - d_{um_u} + \alpha + u - 3, & \cdots, & \beta - d_{u1} + \alpha - m_u + u - 2, \quad \beta, \end{array}$$

respectively where $F'_{u0} = g$ and $F'_{um_u+1} = h$.

Let $S = R/(H_1)$ and $J' = \frac{J+(H_1)}{(H_1)}$. Then

$$\frac{J}{H_1 \cdot [J : H_1]} = \frac{J}{(H_1) \cap J} \simeq \frac{J + (H_1)}{(H_1)} = J' \subset S.$$

Thus we have an exact sequence of graded modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [J : H_1](-1) & \xrightarrow{\times H_1} & J & \longrightarrow & \frac{J+(H_1)}{(H_1)} & \longrightarrow & 0. \\ & & & & & & \| & & \\ & & & & & & J' & & \end{array} \tag{3.3}$$

Since $[J : H_1] = I(\mathbb{Y}'')$, we can rewrite the exact sequence (3.3) as:

$$0 \longrightarrow I(\mathbb{Y}'')(-1) \xrightarrow{\times H_1} J \longrightarrow J' \longrightarrow 0. \tag{3.4}$$

It follows from (3.4) and (3.2) that

$$\begin{aligned} \mathbf{H}(S/J', t) &= \begin{cases} 1, & \text{for } t = 0 \\ \mathbf{H}(R/J, t) - \mathbf{H}(\mathbb{Y}'', t-1), & \text{for } t \geq 1, \end{cases} \\ &= \mathbf{H}(\mathbb{Y}_1, t), \end{aligned}$$

which implies J' is a saturated ideal, i.e., $J + (H_1) = I(\mathbb{Y}_1)$.

By Theorem 3.4, there exist $F_{10}, F_{11}, \cdots, F_{1m_1}, F_{1m_1+1} \in J$ with degrees

$$\begin{aligned} &\deg F_{10} = \alpha, \ \deg F_{11} = \beta - d_{1m_1} + \alpha - 1, \ \cdots, \\ &\deg F_{1m_1} = \beta - d_{11} + \alpha - m_1, \ \deg F_{1m_1+1} = \beta \end{aligned}$$

such that $\overline{F}_{10}, \overline{F}_{11}, \cdots, \overline{F}_{1m_1}, \overline{F}_{1m_1+1}$ are the minimal generators of J'. Moreover, $F_{10} = g$ and $F_{1m_1+1} = h$. Let $\{F'_{ij}\}$ be the minimal generators of $I(\mathbb{Y}'')$ and $\{F_{ij}\} = \{F'_{ij}H_1\} \cup \{F_{10}, F_{11}, \cdots, F_{1m_1}, F_{1m_1+1}\}$.

Claim : $J = \langle \{F_{ij}\} \rangle$.

Proof of Claim. Clearly, $\langle \{F_{ij}\} \rangle \subseteq J$. Conversely, for every $F \in J$, $\overline{F} \in J'$. Hence

$$F = F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + H_1K$$

for some $N_0, N_1, \cdots, N_{m_1}, N_{m_1+1}, K \in R$. Since $K \in [J : H_1] = I(\mathbb{Y}'')$,

$$K = \sum F'_{ij} M_{ij}$$

for some $M_{ij} \in R$. Hence

$$\begin{aligned} F &= F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + H_1 K \\ &= F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + H_1 \sum F'_{ij} M_{ij} \\ &= F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + \sum (F'_{ij} H_1) M_{ij} \\ &\in \langle \{F_{ij}\} \rangle. \end{aligned}$$

Since $F'_{u0} = F_{10} = g$, $F'_{um_u+1} = F_{1m_1+1} = h$, $\nu(J) = \sum_{i=1}^{u} m_i + 3$ where the degrees of the minimal generators of J are

$$\begin{array}{cccc} & \beta - d_{1m_1} + \alpha - 1, & \cdots, & \beta - d_{11} + \alpha - m_1, \\ & \beta - d_{2m_2} + \alpha, & \cdots, & \beta - d_{21} + \alpha - m_2 + 1, \\ & & \vdots & \\ u, \quad \alpha, & \beta - d_{um_u} + \alpha + u - 2, & \cdots, & \beta - d_{u1} + \alpha - m_u + u - 1, \quad \beta. \end{array}$$

Hence we are done.

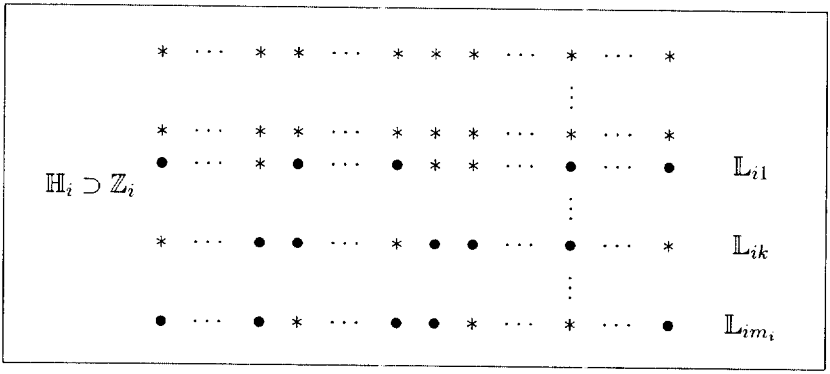
REMARK 3.10. Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u})$ where $\mathbb{X}_i$ is a k-configuration in $\mathbb{P}^3$ of type $(d_{i1}, \cdots, d_{im_i})$ contained in the hyperplane $\mathbb{H}_i$. Assume that the hyperplanes $\mathbb{H}_i$ are parallel to each other. Since $\mathbb{X}_i$ is a k-configuration in $\mathbb{H}_i$ of type $(d_{i1}, \cdots, d_{im_i})$, there exist subsets $\mathbb{X}_{i1}, \cdots, \mathbb{X}_{im_i}$ and distinct lines $\mathbb{L}_{i1}, \cdots, \mathbb{L}_{im_i}$ which are contained in $\mathbb{H}_i$ such that:

(1) $\mathbb{X}_i = \bigcup_{k=1}^{m_i} \mathbb{X}_{ik}$;
(2) $|\mathbb{X}_{ik}| = d_{ik}$ and $\mathbb{X}_{ik} \subset \mathbb{L}_{ik}$ for each $k = 1, \cdots, m_i$.

Choose α and β such that $m_u < \alpha$, $d_{um_u} < \beta$, and $\alpha < \beta$. Let $\mathbb{Y}_i$ be the weak k-configuration in $\mathbb{P}^3$ of type $(\beta - d_{im_i}, \cdots, \beta - d_{i1}, \beta, \cdots, \beta)$ which is obtained by taking the complement of $\mathbb{X}_i$ in a set $\mathbb{Z}_i$ where $\mathbb{Z}_i$ is constructed as follows.

To each line $\mathbb{L}_{ik}$ of $\mathbb{H}_i$, add $\beta - d_{ik}$ distinct new points. Further add $\alpha - m_i$ new lines each containing β distinct points. (This set will then contain $\alpha\beta$ distinct points. See Fig. 1)

Let $\mathbb{Y} := \bigcup_{i=1}^{u} \mathbb{Y}_i$ and $J = I(\mathbb{Y})$. Then $\mathbb{Y}$ is a weak k-configuration in $\mathbb{P}^3$ of type $(\beta - d_{um_u}, \cdots, \beta - d_{u1}, \beta, \cdots, \beta; \cdots; \beta - d_{1m_1}, \cdots, \beta - d_{11}, \beta, \cdots, \beta)$. From the proof of Theorem 3.9, we can see that $\nu(J) \leq \sum_{i=1}^{u} m_i + 2u + 1$. The following example shows that each case of the above inequality can occur.



$\mathbb{X}_i$ is the set of all $\bullet$'s.
$\mathbb{Y}_i$ is the set of all $*$'s.

FIGURE 1

EXAMPLE 3.11 (MACAULAY [1]). Consider the following examples.

(1) Let $\mathbb{Z}$ be a basic configuration in $\mathbb{P}^3$ of type $(2, 3, 5)$ and $\mathbb{Y}_1 \subset \mathbb{Z}$ be a weak k-configuration in $\mathbb{P}^3$ of type $(3, 4, 5; 4, 5, 5)$. Then the number of minimal generators of the ideal of $\mathbb{Y}_1$ is 6 by Theorem 3.9.

(2) Let

$$\begin{aligned}\mathbb{Y}_2 = \ & \{(1,2,1,1), \ (2,4,1,1), \ (3,6,1,1), \ (0,1,1,1), \ (1,3,1,1), \\ & (2,5,1,1), \ (3,7,1,1), \ (0,-1,1,1), \ (1,1,1,1), \ (2,3,1,1), \\ & (3,5,1,1), \ (0,1,0,1), \ (0,2,0,1), \ (0,3,0,1), \ (1,2,0,1), \\ & (1,3,0,1), \ (2,0,0,1), \ (2,1,0,1), \ (2,2,0,1), \ (2,3,0,1)\}.\end{aligned}$$

Then $\mathbb{Y}_2$ is a weak k-configuration in $\mathbb{P}^3$ of type $(2, 3, 4; 3, 4, 4)$, and the number of minimal generators of the ideal of $\mathbb{Y}_2$ is 7 from Macaulay [BS].

(3) Let

$$\mathbb{Y}_3' = \{(4,8,1,1), (4,9,1,1), (4,7,1,1), (0,4,0,1), (1,4,0,1), (2,4,0,1)\}$$

and $\mathbb{Y}_3 = \mathbb{Y}_2 \cup \mathbb{Y}_3'$. Then $\mathbb{Y}_3$ is a weak k-configuration in $\mathbb{P}^3$ of type $(3,4,5;4,5,5)$, and the number of minimal generators of the ideal of $\mathbb{Y}_3$ is 8 from Macaulay [BS].

COROLLARY 3.12. *Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and J be as in Remark 3.6 and let $I = I(\mathbb{X})$. Then $I + J$ is a Gorenstein ideal of codimension 4 and*

$$\nu(I+J) = 2\sum_{i=1}^{u} m_i + u + 1.$$

Proof. By Remarque 1.4 in [PS], $I+J$ is a Gorenstein ideal of codimension 4. Let H_i be as in the proof of Theorem 3.9 and $H = \prod_{i=1}^{u} H_i$. Let $\{H, F_{10}, F_{11}, \cdots, F_{1m_1}; \cdots; F_{u0}, F_{u1}, \cdots, F_{um_u}\}$ be the set of the minimal generators of I and let $\{H, G_{10}, \cdots, G_{1m_1}, G_{1m_1+1}; G_{21}, \cdots, G_{2m_2}; \cdots; G_{u1}, \cdots, G_{um_u}\}$ be the set of the minimal generators of J where $F_{u0}|G_{10}$ and $F_{um_u}|G_{1m_1+1}$. (This is always possible.) So we have that

$$\begin{array}{l} H,\ F_{10},\ F_{11},\ \cdots,\ F_{1m_1},\ \cdots,\ F_{u0},\ F_{u1},\ \cdots,\ F_{um_u}, \\ G_{11},\ \cdots,\ G_{1m_1},\ G_{1m_1+1},\ G_{21},\ \cdots,\ G_{2m_2},\ \cdots,\ G_{u1},\ \cdots,\ G_{um_u} \end{array}$$

certainly generate $I+J$.

We first show that no other F_{ij} can be eliminated from the set. If $F_{ij}\langle H, \cdots, \widehat{F_{ij}}, \cdots, F_{um_u}; G_{11}, \cdots, G_{um_u}\rangle$ (where $\hat{*}$ means that $*$ is omitted), then

$$\begin{aligned} F_{ij} = \alpha H + \alpha_{10}F_{10} + \cdots + \widehat{\alpha_{ij}F_{ij}} + \cdots \\ + \alpha_{um_u}F_{um_u} + \beta_{11}G_{11} + \cdots + \beta_{um_u}G_{um_u} \end{aligned}$$

for some $\alpha, \alpha_{10}, \cdots, \widehat{\alpha}_{ij}, \cdots, \alpha_{um_u}, \beta_{11}, \cdots, \beta_{um_u} \in R$. Thus

$$\begin{aligned} & \alpha_{10}F_{10} + \cdots - F_{ij} + \cdots + \alpha_{um_u}F_{um_u} \\ = & -(\alpha H + \beta_{11}G_{11} + \cdots + \beta_{um_u}G_{um_u}) \\ \in & I \cap J = \langle H, G_{10}, G_{1m_1+1}\rangle. \end{aligned}$$

Hence there exist $\alpha', \alpha'', \alpha''' \in R$ such that

$$\alpha_{10}F_{10} + \cdots - F_{ij} + \cdots + \alpha_{um_u}F_{um_u} = -(\alpha' H + \alpha'' G_{10} + \alpha''' G_{1m_1+1}),$$

i.e.,

$$
\begin{aligned}
F_{ij} = &\alpha' H + \alpha_{10}F_{10} + \cdots + \widehat{\alpha_{ij}F_{ij}} + \cdots \\
&+ \alpha_{um_u}F_{um_u} + \alpha'' G_{10} + \alpha''' G_{1m_1+1} \\
\in &\langle H, F_{10}, \cdots, \widehat{F}_{ij}, \cdots, F_{um_u}\rangle,
\end{aligned}
$$

a contradiction.
Hence $F_{ij} \notin \langle H, F_{10}, \cdots, \widehat{F}_{ij}, \cdots, F_{um_u}, G_{11}, \cdots, G_{um_u}\rangle$.

We now show that no G_{kl} can be eliminated from this set. Assume $G_{kl} \in \langle H, F_{10}, \cdots, F_{um_u}, G_{11}, \cdots, \widehat{G}_{kl}, \cdots, G_{um_u}\rangle$. Then

$$
\begin{aligned}
G_{kl} = &\alpha H + \alpha_{10}F_{10} + \cdots + \alpha_{um_u}F_{um_u} + \beta_{11}G_{11} + \cdots \\
&+ \widehat{\beta_{kl}G_{kl}} + \cdots + \beta_{um_u}G_{um_u}
\end{aligned}
$$

for some $\alpha, \alpha_{10}, \cdots, \alpha_{um_u}, \beta_{11}, \cdots, \widehat{\beta}_{kl}, \cdots, \beta_{um_u} \in R$. Thus

$$
\begin{aligned}
&-(\alpha H + \alpha_{10}F_{10} + \cdots + \alpha_{um_u}F_{um_u}) \\
=&\beta_{11}G_{11} + \cdots - G_{kl} + \cdots + \beta_{um_u}G_{um_u} \\
\in &I \cap J = \langle H, G_{10}, G_{1m_1+1}\rangle.
\end{aligned}
$$

Hence

$$
\beta_{11}G_{11} + \cdots - G_{kl} + \cdots + \beta_{um_u}G_{um_u} = -(\beta H + \beta' G_{10} + \beta'' G_{1m_1+1})
$$

for some $\beta, \beta', \beta'' \in R$. It follows that

$$
\begin{aligned}
G_{kl} = &\beta H + \beta' G_{10} + \beta_{11}G_{11} + \cdots \\
&+ \widehat{\beta_{kl}G_{kl}} + \cdots + \beta_{um_u}G_{um_u} + \beta'' G_{1m_1+1} \\
\in &\langle H, G_{10}, \cdots, \widehat{G}_{kl}, \cdots, G_{um_u}\rangle,
\end{aligned}
$$

a contradiction. Thus $G_{kl} \notin \langle H, F_{10}, \cdots, F_{um_u}, G_{11}, \cdots, \widehat{G}_{kl}, \cdots, G_{um_u}\rangle$, we are done.

REMARK 3.13. Let $\mathbb{Z} = \bigcup_{i=1}^{u} \mathbb{Z}_i$ be a basic configuration in $\mathbb{P}^3$ of type (u, m_u, d_{um_u}) $(u \geq 2)$ where $\mathbb{Z}_i$ is a basic configuration in $\mathbb{P}^3$ of type $(1, m_u, d_{um_u})$ and $\mathbb{X} \subset \mathbb{Z}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots,$
$d_{um_u})$. Let $\mathbb{X}_i = \mathbb{Z}_i \cap \mathbb{X}$. Then $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$. Let $\mathbb{Y}_i = \mathbb{Z}_i - \mathbb{X}_i$. Then $\mathbb{Y} =: \mathbb{Z} - \mathbb{X} = \bigcup_{i=1}^{u} \mathbb{Y}_i$ and $\mathbb{Y}$ is a weak k-configuration in $\mathbb{P}^3$. Assume that $\mathbb{Y}_i$ is a weak k-configuration in $\mathbb{P}^2$ of type $(d_{um_u} - d_{im_i}, \cdots, d_{um_u} - d_{i1}, d_{um_u}, \cdots, d_{um_u})$.

The proof of the following theorem is the same as that of Theorem 3.9, so we shall omit it.

THEOREM 3.14. *Let $\mathbb{Y}$ be as in Remark 3.13. Let $J = I(\mathbb{Y})$. Then $\nu(J) = \sum_{i=1}^{u} m_i + 3$ and the degrees of the minimal generators of J are:*

$$
\begin{gathered}
u,\ d_{um_u} - d_{1m_1} + m_u - 1,\ \cdots,\ d_{um_u} - d_{11} + m_u - m_1, \\
\vdots \\
m_u,\ d_{um_u} - d_{u-1m_{u-1}} + m_u + u - 3,\ \cdots, \\
d_{um_u} - d_{u-11} + m_u - m_{u-1} + u - 2,\ d_{um_u}, \\
m_u + u - 2,\ d_{um_u} - d_{um_u - 1} + m_u + u - 3,\ \cdots,\ d_{um_u} - d_{u1} + u - 1.
\end{gathered}
$$

We also get the following corollary by the same method as in the proof of Corollary 3.12.

COROLLARY 3.15. *Let $\mathbb{X}$ and $\mathbb{Y}$ be as in Remark 3.13. Let $I = I(\mathbb{X})$ and $J = I(\mathbb{Y})$. Then $I + J$ is a Gorenstein ideal of codimension 4 and*

$$\nu(I + J) = 2\sum_{i=1}^{u} m_i + u + 1.$$

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