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FINITE ELEMENT ANALYSIS FOR A MIXED LAGRANGIAN FORMULATION OF INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

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1. Introduction

This paper is concerned with a mixed Lagrangian formulation of the viscous, stationary, incompressible Navier–Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \tag{1.1}$$

and

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega \tag{1.2}$$

along with inhomogeneous Dirichlet boundary conditions on a portion of the boundary

$$\mathbf{u} = \begin{cases} \mathbf{0} \text{ on } \Gamma_0 \\ \mathbf{g} \text{ on } \Gamma_g\,, \end{cases} \tag{1.3}$$

where Ω is a bounded open domain in $\mathbb{R}^d$, $d = 2$ or 3, with a boundary $\Gamma = \partial\Omega$, which is composed of two disjoint parts Γ_0 and Γ_g. Here, ν denotes the kinematic viscosity in the nondimensional form and $\mathbf{f}$ the given external body force. Note that the constant density has been

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absorbed into the pressure and the body force. For the compatibility and regularity for the solutions, we assume

$$\text{(1.4)} \qquad \text{support of } \mathbf{g} \subset \Gamma_g \text{ and } \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{n}\, d\Gamma = 0\,.$$

One of the main feature of the formulation is the appearance of the stress vector along the inhomogeneous boundary, which is given by

$$\text{(1.5)} \qquad (\mathbf{t})_j = -pn_j + \nu \sum_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

along Γ_g, where n_j denotes j-th component of the outward unit normal vector along Γ_g, u_j and $(\mathbf{t})_j$ components of the velocity $\mathbf{u}$ and the boundary stress $\mathbf{t}$. The boundary stress is an important physical quantity representing the forces exerted by the flow along the boundary of the domain occupied by the flow. In the obstacle problem or various control problems as in [11], boundary stress plays dual role being a Lagrange multiplier supplementing the inhomogeneous boundary conditions as well as a physical factor balancing the state variables. Practically, the variable for the stress was used as a control parameter ([9]) or a balancing factor to achieve optimal solutions ([11],[12],[13]).

One can regard $\mathbf{u}$, p and $\mathbf{t}$ as independent variables. However, since $\mathbf{t}$ is represented in terms of $\mathbf{u}$ and p, it is natural to consider the computation of $\mathbf{t}$ in a post–processing procedure. For the finite element approximation of the mixed Lagrangian formulation, we approximate the boundary data using $\mathbf{L}^2$–projection, instead of boundary interpolants. Using the same meshes for the boundary stress and the trace of the velocity to the inhomogeneous boundary, we derive optimal error estimates for the approximation.

We close this section by introducing some notation and function spaces that will be used in the sequel. Throughout this paper, $\mathcal{I}$ will be used to denote the identity mapping or the identity matrix, and C a generic constant whose value and meaning also vary with context. For Galerkin type variational formulations, we denote by $H^s(\Omega)$, the standard Sobolev space of order s with respect to the set Ω, which is the domain occupied by the flow, or its boundary Γ, or part of its boundary. For vector–valued functions and spaces, we use boldface notation, i.e., $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^d$. We denote the inner product on $H^s(\Omega)$ or $\mathbf{H}^s(\Omega)$ by $(\cdot,\cdot)_s$ and its norm by $\|\cdot\|_s = \sqrt{(\cdot,\cdot)_s}$. For the space of interest to us, we consider the semi-norm defined on $\mathbf{H}^1(\Omega)$

$$|\!|\!|\mathbf{v}|\!|\!| = 2\left(\int_\Omega D(\mathbf{v}) : D(\mathbf{v})\, d\Omega\right)^{1/2} = \frac{1}{2}\Big(\sum_{i,j=1}^{d} \|\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\|_0^2\Big)^{1/2},$$

where $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ denotes the deformation tensor for the flow $\mathbf{u}$.

Let Γ_0 be a subset of Γ with a positive measure. Let us define

$$\mathbf{H}^1_{\Gamma_0}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}\,;$$

$\mathbf{H}^1_{\Gamma_0}(\Omega)$ is the space on which the *homogeneous* boundary conditions are imposed. Note that Korn's inequality leads to $|\!|\!|\mathbf{v}|\!|\!| \geq C\|\mathbf{v}\|_1$ for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$. This implies that the semi-norm $|\!|\!|\cdot|\!|\!|$ is a norm which is equivalent to the norm $\|\cdot\|_{1,\Omega}$ on $\mathbf{H}^1_{\Gamma_0}(\Omega)$. By $< \cdot,\cdot >_{-s}$, we shall denote the duality pairing between $\mathbf{H}^s_{\Gamma_0}(\Omega)$ and its dual space, $\mathbf{H}^{-s}_{\Gamma_0}(\Omega)$.

For the other face Γ_g of a Lipschitz continuous domain Ω, let

$$\mathbf{L}^2_g(\Gamma) = \{\, \mathbf{s} \in \mathbf{L}^2(\Gamma) \mid \mathbf{s} = \mathbf{0} \quad \text{on } \Gamma_0 \,\}$$

and let $\gamma_g : \mathbf{H}^1_{\Gamma_0}(\Omega) \longrightarrow \mathbf{L}^2_g(\Gamma)$ be the trace mapping. Let us define $\mathbf{H}^s(\Gamma_g) = \gamma_g(\mathbf{H}^s_{\Gamma_0}(\Omega))$ and its dual space $\mathbf{H}^{-s}(\Gamma_g)$ for $s \geq 1/2$. The duality between $\mathbf{H}^{-s}(\Gamma_g)$ and $\mathbf{H}^s(\Gamma_g)$ is denoted by $< \cdot,\cdot >_{-s,\Gamma_g}$. For the given boundary force, we take

$$\mathbf{H}^s_0(\Gamma_g) = \{\boldsymbol{\phi} \in \mathbf{H}^s(\Gamma_g) \mid \text{support of } \boldsymbol{\phi} \subset \Gamma_g \text{ and } \int_{\Gamma_g} \boldsymbol{\phi}\cdot\mathbf{n}\, d\Gamma = 0\,\}\,.$$

From the condition (1.4), we assume $\mathbf{g} \in \mathbf{H}^s_0(\Gamma_g)$ for some $s \geq 1/2$.

Since the pressure is determined only up to a constant in the mathematical formulation of the Navier–Stokes equations with velocity boundary conditions, we take the space for the pressure to be

$$L^2_0(\Omega) = \{p \in L^2(\Omega) \mid \int_\Omega p\, d\Omega = 0\}\,,$$

i.e., $L^2_0(\Omega)$ consists of square integrable functions having zero mean over Ω.

When X and Y are Banach spaces, we denote the class of bounded linear operators from X to Y by $\mathcal{L}(X;Y)$.

2. The mixed Lagrangian formulation

For the weak variational formulation, we will use the forms

$$a(\mathbf{u},\mathbf{v}) = 2\int_\Omega D(\mathbf{u}) : D(\mathbf{v})\, d\Omega$$
$$= \frac{1}{2}\sum_{i,j=1}^{d}\int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) d\Omega,\ \forall \mathbf{u},\ \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$b(\mathbf{v},p) = -\int_\Omega p\,\nabla\cdot\mathbf{v}\, d\Omega = -\sum_{i=1}^{d}\int_\Omega p\,\frac{\partial v_i}{\partial x_i}\, d\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), p \in L^2(\Omega)$$

and

$$c(\mathbf{w},\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, d\Omega = \sum_{i,j=1}^{d}\int_\Omega w_j \frac{\partial u_i}{\partial x_j} v_i\, d\Omega$$

over $\mathbf{H}^1(\Omega)$. Obviously $a(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ and $b(\cdot,\cdot)$ is a continuous bilinear form $\mathbf{H}^1(\Omega)\times L^2(\Omega)$; also, $c(\cdot,\cdot,\cdot)$ is a continuous trilinear form on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ which can be verified by the Sobolev embedding of $\mathbf{H}^1(\Omega)\subset\mathbf{L}^4(\Omega)$ and the Hölder's inequality. Moreover, we have the coercivity property

$$a(\mathbf{v},\mathbf{v}) \geq C\|\mathbf{v}\|_1^2 \quad \forall \mathbf{v}\in\mathbf{H}^1_{\Gamma_0}(\Omega)$$

and LBB–condition (or, inf–sup condition) to balance between the velocity and the pressure

$$\inf_{p\in L^2_0(\Omega)}\ \sup_{\mathbf{v}\in\mathbf{H}^1_{\Gamma_0}(\Omega)} \frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_1\|p\|_0} \geq C. \tag{2.1}$$

For details concerning these forms and their properties, one may consult [7], [11] or [14].

Based on this notations, let us first consider the classical weak formulation of the Stokes problem;

For a given $\mathbf{g} \in \mathbf{H}_0^{1/2}(\Gamma_g)$, find $\mathbf{u}$ and p satisfying

$$
(2.2)\qquad
\begin{aligned}
a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &=< \mathbf{f},\mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,,\\
b(\mathbf{u},q) &= 0 \quad \forall q \in L_0^2(\Omega)\,,\\
\mathbf{u} &= \begin{cases} \mathbf{0} \text{ on } \Gamma_0 \\ \mathbf{g} \text{ on } \Gamma_g\,. \end{cases}
\end{aligned}
$$

The well-posedness of the system provides a well-known a priori estimate (cf. [7]);

$$\|\mathbf{u}\|_1 + \|p\|_{0,\Omega} \le C(\|\mathbf{f}\|_{-1} + \|\mathbf{g}\|_{1/2,\Gamma_g})\,.$$

Moreover, if Ω is a convex polyhedral domain and Γ_0 and Γ_g are disjoint faces of $\Gamma = \partial\Omega$, this estimate can be extended to

$$\|\mathbf{u}\|_2 + \|p\|_1 \le C(\|\mathbf{f}\|_0 + \|\mathbf{g}\|_{3/2,\Gamma_g})\,.$$

Lagrange multipliers can be used to relaxate the constraints for the incompressibility constraint and inhomogeneous boundary condition. Let us consider the following saddle point problem:

$$\inf_{(q,\boldsymbol{\eta})} \sup_{\mathbf{v}} \mathcal{E}(\mathbf{v},(q,\boldsymbol{\eta}))\,,$$

where $\mathcal{E}(\cdot,\cdot)$ is a Lagrangian defined by

$$\mathcal{E}(\mathbf{v},(q,\boldsymbol{\eta})) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) + b(\mathbf{v},q) - < \mathbf{f},\mathbf{v} >_{-1} - < \boldsymbol{\eta}, \mathbf{v}-\mathbf{g} >_{-1/2,\Gamma_g}$$

in the space $\mathbf{H}_{\Gamma_0}^1(\Omega) \times (L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma_g))$. Then, the saddle point $(\mathbf{u},(p,\mathbf{t}))$ of $\mathcal{E}$ satisfies

$$
(2.3)\qquad
\begin{aligned}
a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - < \mathbf{t},\mathbf{v} >_{-1/2,\Gamma_g} &=< \mathbf{f},\mathbf{v} >_{-1}\,,\\
b(\mathbf{u},q) - < \boldsymbol{\eta},\mathbf{u} >_{-1/2,\Gamma_g} &=< \boldsymbol{\eta},\mathbf{g} >_{-1/2,\Gamma_g}
\end{aligned}
$$

for all $\mathbf{v} \in \mathbf{H}_{\Gamma_0}^1(\Omega)$ and $(q,\boldsymbol{\eta}) \in L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma_g)$. Note that this can be equivalently written in the form;

$$
(2.4)\qquad
\begin{aligned}
a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - < \mathbf{t},\mathbf{v} >_{-1/2,\Gamma_0} &=< \mathbf{f},\mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_0}^1(\Omega)\,,\\
b(\mathbf{u},q) &= 0 \quad \forall q \in L_0^2(\Omega)\,,\\
< \boldsymbol{\eta},\mathbf{u} >_{-1/2,\Gamma_g} &=< \boldsymbol{\eta},\mathbf{g} >_{-1/2,\Gamma_g} \quad \forall \boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\Gamma_g)\,.
\end{aligned}
$$

Let us first state the existence of a solution for the specified system (2.3) or (2.4) for the Stokes equations, whose existence and uniqueness depends on the following two Lemmas.

LEMMA 2.1. (**Lifting of $\mathbf{H}^{1/2}(\Gamma_g)$**) Let Ω be Lipschitz continuous. Let $(q,\boldsymbol{\eta}) \in L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma_g)$. Then, there exists $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$ such that

$$
(2.5) \qquad \begin{cases} \nabla \cdot \mathbf{w} = q \text{ in } \Omega \\ \quad \mathbf{w} = \mathcal{F}^{-1}(\boldsymbol{\eta}) \text{ on } \Gamma_g \,, \end{cases}
$$

where $\mathcal{F} : \mathbf{H}^{1/2}(\Gamma_g) \longrightarrow \mathbf{H}^{-1/2}(\Gamma_g)$ is the inverse of the Riesz representation mapping, i.e.,

$$
< \boldsymbol{\eta}, \boldsymbol{\xi} >_{-1/2,\Gamma_g} = < \mathcal{F}^{-1}(\boldsymbol{\eta}), \boldsymbol{\xi} >_{1/2,\Gamma_g} \quad \forall \boldsymbol{\eta} \in \mathbf{H}^{1/2}(\Gamma_g)\,,
$$

and $\|\boldsymbol{\eta}\|_{-1/2,\Gamma_g} = \|\mathcal{F}^{-1}(\boldsymbol{\eta})\|_{1/2,\Gamma_g}$.

Moreover, there exists a positive constant C such that

$$
\|\mathbf{w}\|_1 \le C(\|q\|_0 + \|\boldsymbol{\eta}\|_{-1/2,\Gamma_g})\,.
$$

Proof. See, e.g., [11] and [15]. □

Let us state the augmented LBB condition coupling the Lagrange multipliers p and $\mathbf{t}$ along Γ_g. Let us denote $\mathbf{M} = L_0^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma_g)$ and its dual by $\mathbf{M}^*$.

LEMMA 2.2. (**Augmented LBB condition**) For every $(q,\boldsymbol{\eta}) \in \mathbf{M}^*$, there exists a constant $C > 0$ such that

$$
(2.6) \qquad C\|(q,\boldsymbol{\eta})\|_{\mathbf{M}^*} \le \sup_{\mathbf{w}\in\mathbf{H}^1_{\Gamma_0}(\Omega),\, \mathbf{w}\neq \mathbf{0}} \frac{b(\mathbf{w},q) - < \boldsymbol{\eta}, \mathbf{w} >_{-1/2,\Gamma_g}}{\|\mathbf{w}\|_1}\,.
$$

Proof. Let $(q,\boldsymbol{\eta}) \in L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma_g)$ be given. By Lemma 2.1, there exists $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$ such that $\nabla \cdot \mathbf{w} = -q$ in Ω, $\mathbf{w} = -\mathcal{F}^{-1}(\boldsymbol{\eta})$ on Γ_g and $\|\mathbf{w}\|_1 \le C(\|q\|_0 + \|\boldsymbol{\eta}\|_{-1/2,\Gamma_g})$. Hence, it follows that

$$
\begin{aligned}
b(\mathbf{w},q) - < \boldsymbol{\eta}, \mathbf{w} >_{-1/2,\Gamma_g} &= \|q\|_0^2 + \|\mathcal{F}^{-1}(\boldsymbol{\eta})\|_{1/2,\Gamma}^2, \\
&= \|q\|_0^2 + \|\boldsymbol{\eta}\|_{-1/2,\Gamma_g}^2 \\
&\ge C(\|q\|_0 + \|\boldsymbol{\eta}\|_{-1/2,\Gamma_g})\|\mathbf{w}\|_1\,.
\end{aligned}
$$

□

For the sake of brevity, set $\mathbf{V} = \mathbf{H}^1_{\Gamma_0}(\Omega)$ and $\mathbf{M} = L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma_g)$. Let $\mathcal{B} : \mathbf{V} \longrightarrow \mathbf{M}$ be a bounded operator defined by

$$
\begin{aligned}
< \mathcal{B}\mathbf{w}, (q, \boldsymbol{\eta}) >_{\mathbf{M}\times\mathbf{M}^*} &= < \mathbf{w}, \mathcal{B}^T(q, \boldsymbol{\eta}) >_{\mathbf{V}\times\mathbf{V}^*} \\
&= b(\mathbf{w}, q) - < \boldsymbol{\eta}, \mathbf{w} >_{-1/2,\Gamma_g} .
\end{aligned} \tag{2.7}
$$

Lemma 2.2 implies that

$$
\|\mathcal{B}^T(q, \boldsymbol{\eta})\|_{\mathbf{V}^*} \geq C\|(q, \boldsymbol{\eta})\|_{\mathbf{M}^*} \quad \forall (q, \boldsymbol{\eta}) \in \mathbf{M}^* ,
$$

whence $\|\mathcal{B}^T\|_{\mathcal{L}(\mathbf{M}^*;\mathbf{V}^*)} \geq C$.

Using this relation, we easily conclude that $\mathcal{B}$ has a closed range in $\mathbf{M}$ and is surjective due to Lemma 2.1.

Let us show the existence of the solution of the system (2.3) or (2.4).

THEOREM 2.3. *Suppose $b(\cdot,\cdot)$ satisfies the LBB condition* (2.1). *Then, given $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{-1}_{\Gamma_0}(\Omega) \times \mathbf{H}^{1/2}_0(\Gamma_g)$, there exists a unique solution $(\mathbf{u}, (p, \mathbf{t})) \in \mathbf{H}^1_{\Gamma_0}(\Omega) \times (L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma_g))$ satisfying* (2.3) *or* (2.4).

Proof. Using (2.7), (2.3) can be rewritten as

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) + < \mathcal{B}\mathbf{v}, (p, \mathbf{t}) > &= < \mathbf{f}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{V} , \\
< \mathcal{B}\mathbf{u}, (q, \boldsymbol{\eta}) > &= - < \boldsymbol{\eta}, \mathbf{g} >_{-1/2,\Gamma_g} \quad \forall (r, \boldsymbol{\eta}) \in \mathbf{M}^* .
\end{aligned} \tag{2.8}
$$

Since $\mathbf{u} = \mathbf{0}$ on Γ_0, it is easy to check that $\ker \mathcal{B} \subset \mathbf{H}^1_{\Gamma_0}(\Omega)$. Since $\sqrt{a(\cdot,\cdot)} = |||\cdot|||$ is equivalent to $\|\cdot\|_1$, it is obvious that $a(\cdot,\cdot)$ is coercive over $\ker \mathcal{B}$. So, combined with condition (2.6), the existence theorem for the abstract mixed formulation yields the result. □

From the well–posedness of (2.3), the following estimate immediately follows:

$$
\|\mathbf{u}\|_1 + \|p\|_0 + \|\mathbf{t}\|_{-1/2,\Gamma_g} \leq C(\|\mathbf{f}\|_{-1} + \|\mathbf{g}\|_{1/2,\Gamma_g}) .
$$

Note that the saddle point p corresponds to the pressure, i.e., the pressure can be interpreted to be a Lagrange multiplier to relax the incompressibility constraint. To physically interpret the term $\mathbf{t}$ for the Lagrange multiplier, we note from Green's formula that

$$\begin{aligned}\int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega &= \int_\Omega \big(-\Delta\mathbf{u}+\nabla p\big)\cdot\mathbf{v}\,d\Omega\\ &= \frac{1}{2}\int_\Omega (\nabla\mathbf{u}+(\nabla\mathbf{u})^T)\,:\,(\nabla\mathbf{v}+(\nabla\mathbf{v})^T)\,d\Omega-\int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega\\ &\qquad -\int_{\Gamma_g}\mathbf{n}\cdot(-p\mathcal{I}+(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)\cdot\mathbf{v}\,d\Gamma\\ &= a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)-\int_{\Gamma_g}\Big(\Big(-p\mathcal{I}+(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)\Big)\cdot\mathbf{n}\Big)\Big)\cdot\mathbf{v}\,d\Gamma\,.\end{aligned}$$

Hence, compared with (2.3), the Lagrange multiplier $\mathbf{t}$ is given by

$$\mathbf{t}=\big(-p\mathcal{I}+(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)\big)\cdot\mathbf{n}=-p\mathbf{n}+2D(\mathbf{u})\cdot\mathbf{n}\quad\text{on }\Gamma_g\,.$$

Adding the convective term to the first equation of (2.3) or (2.4), we obtain the corresponding formulation of (1.1)–(1.3), which can be written in the form

$$\nu a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)-<\mathbf{t},\mathbf{v}>_{-1/2,\Gamma_g}=<\mathbf{f}-(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}>_{-1}\quad\forall\mathbf{v}\in\mathbf{V}$$

and

$$b(\mathbf{u},q)-<\boldsymbol{\eta},\mathbf{u}>_{-1/2,\Gamma_g}=<\boldsymbol{\eta},\mathbf{g}>_{-1/2,\Gamma_g}\quad\forall(q,\boldsymbol{\eta})\in\mathbf{M}^*\,.$$

Equivalently, it can be written as

$$\begin{aligned}\nu a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)-<\mathbf{t},\mathbf{v}>_{-1/2,\Gamma_g}&\\ =<\mathbf{f},\mathbf{v}>_{-1}-c(\mathbf{u},\mathbf{u},\mathbf{v})&\quad\forall\mathbf{v}\in\mathbf{H}^1_{\Gamma_0}(\Omega)\,,\end{aligned}\tag{2.9}$$

$$b(\mathbf{u},q)=0\quad\forall q\in L^2_0(\Omega)\tag{2.10}$$

and

$$<\boldsymbol{\eta},\mathbf{u}>_{-1/2,\Gamma_g}=<\boldsymbol{\eta},\mathbf{g}>_{-1/2,\Gamma_g}\quad\forall\boldsymbol{\eta}\in\mathbf{H}^{-1/2}(\Gamma_g)\,.\tag{2.11}$$

Note that in this case, the corresponding Lagrange multiplier is given by

$$\mathbf{t} = \big(-p\mathcal{I} + \nu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)\big) \cdot \mathbf{n} = -p\mathbf{n} + 2\nu D(\mathbf{u}) \cdot \mathbf{n} \quad \text{on } \Gamma_g \,,$$

which is the stress vector on Γ_g. Hence the boundary stress along Γ_g plays the role of a Lagrange multiplier in enforcing the essential *homogeneous* boundary condition along Γ_0. For this reason, the variational formulation of the form (2–9)–(2–11) is called the *mixed Lagrangian formulation* for the stationary incompressible Navier–Stokes system incorporating inhomogeneous boundary conditions. In showing (2.9) is a weak formulation of (1.1), it is convenient to replace the viscous term in the latter with $2\nu\nabla \cdot (D(\mathbf{u}))$; the equivalence of the two forms follows from the incompressibility constraint (1.2). Note that the boundary condition on the velocity is enforced weakly through the use of Lagrange multipliers.

3. Split formulation and its approximation

Let us consider the approximation of the system (2.9)–(2.11) in the finite element framework. The major difficulties in the approximation follow from the existence of inhomogeneous boundary data $\mathbf{g}$, the nonlineality of the system and the appearance of the stress term along the inhomogeneous boundary. The space $\mathbf{H}^1_{\Gamma_0}(\Omega)$ for the velocity appeared in the consideration of the physical boundary state, while it provides an additional term for the stress along the part Γ_g of the boundary. Even if this raises additional loads toward the computation for the boundary stress, it makes the system stable as well as physically meaningful. In the Dirichlet boundary value problem, this quantity was appeared in effort to find an efficient method to implement the inhomogeneous essential boundary condition. In [2], the Lagrange multiplier technique was employed to overcome the difficulty of finding stable finite element approximations satisfying the Dirichlet boundary data. In conjunction with the finite element methods and the variational principles, this naturally led us to coupling the pressure and the boundary stress, which are Lagrange multipliers arising from the incompressibility constraint and the inhomogeneous boundary conditions.

Several different approaches have been studied to incorporate the boundary stress (or, the boundary flux, specifically). In [4], the elliptic problem with Dirichlet condition was decomposed into ones with natural boundary conditions by incorporating the approximation of the boundary stress. In this case, the stress was regarded as a parameter to determine the velocity of the flow. In the consideration of the slip boundary condition, [16] and [17] coupled the stress term with the pressure to attain the stability of the resulting mixed formulation. Here, we provide a method that uncouples the computation of the boundary stress. Fundamental steps include the suitable choice of the boundary interpolation for the boundary data and the decoupling of the boundary stress from the velocity and pressure.

3.1. Split formulation

Before presenting the computational procedure, some assumptions for the choice of spaces $\mathbf{V}^h$, S^h and $\mathbf{P}^h$ are in order. One may choose any pair of subspaces $\mathbf{V}^h$ and S^h that can be used to the finite element approximation for the velocity and pressure in the Navier–Stokes equations. For practical use, we choose a pair $\mathbf{V}^h \subset \mathbf{H}^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and $S^h \subset \mathbf{L}^2(\Omega)$, and then simply set $\mathbf{V}^h_{\Gamma_0} = \mathbf{V}^h \cap \mathbf{H}^1_{\Gamma_0}(\Omega)$, $\mathbf{V}^h_0 = \mathbf{V}^h \cap \mathbf{H}^1_0(\Omega)$, $S^h_0 = S^h \cap L^2_0(\Omega)$ and $\mathbf{P}^h_{\Gamma_g} = \mathbf{V}^h_{\Gamma_0}\Big|_{\Gamma_g}$. Since $\mathbf{P}^h_{\Gamma_g} \subset \mathcal{C}(\overline{\Gamma_g})$, we have $\mathbf{P}^h_{\Gamma_g} \subset \mathbf{H}^1(\Gamma_g)$. In this choice of approximation spaces, the finite element analogy corresponding to the mixed Lagrangian formulation (2.9)–(2.11) reads:

Seek $(\mathbf{u}^h, p^h, \mathbf{t}^h) \in \mathbf{V}^h_{\Gamma_0} \times S^h_0 \times \mathbf{P}^h_{\Gamma_g}$ satisfying

$$
\begin{aligned}
\nu a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) - < \mathbf{t}^h, \mathbf{v}^h >_{\Gamma_g} = &< \mathbf{f}, \mathbf{v}^h > \\
&- c(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) \ \forall \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0}, \\
b(\mathbf{u}^h, q^h) = 0 \quad \forall q^h \in S^h_0 &, \\
< \boldsymbol{\xi}^h, \mathbf{u}^h >_{\Gamma_g} = < \boldsymbol{\xi}^h, \mathbf{g} >_{\Gamma_g} &\quad \forall \boldsymbol{\xi}^h \in \mathbf{P}^h_{\Gamma_g}.
\end{aligned}
\tag{3.1}
$$

We notice that since $\mathbf{V}^h_{\Gamma_0} \subset \mathbf{H}^1_{\Gamma_0}(\Omega)$, $a(\cdot,\cdot)\Big|_{\mathbf{V}^h_{\Gamma_0} \times \mathbf{V}^h_{\Gamma_0}}$ is continuous and coercive. In order to secure the stability and convergence, of the approximation (3.1) to the solution of the Navier–Stokes equations we assume discrete LBB condition on elements; there exists a constant C

which is independent of h such that

$$(3.2) \qquad \inf_{0\neq q^h\in S_0^h} \sup_{0\neq \mathbf{v}^h\in \mathbf{V}_{\Gamma_0}^h} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1\|q^h\|_0} \geq C\,.$$

This condition is needed to keep the balance between velocity and pressure, whence to allow the stability of the approximation.

REMARK. The condition (3.2) is somewhat stronger. In practice, the generic constant C of (3.2) may be taken to depend on the size of the mesh. Even in such cases, LBB condition may still imply the convergence of the chosen elements, provided that the infimum of the constant $C(h)$ decays to zero not too fast (see, e.g. [15]).

For general applications, we assume that $\mathbf{V}^h$ and S^h satisfy the following approximation properties: there exists an integer k and a constant C which is independent of h, $\mathbf{v}$ and q such that for $1\leq m\leq k$,

$$(3.3) \qquad \inf_{q^h\in S_0^h} \|q-q^h\|_0 \leq Ch^{m-1}\|q\|_{m-1} \quad \forall q\in H^{m-1}(\Omega)\cap L_0^2(\Omega)$$

and for $s=0,1$,

$$(3.4) \qquad \inf_{\mathbf{v}^h\in \mathbf{V}_{\Gamma_0}^h} \|\mathbf{v}-\mathbf{v}^h\|_s \leq Ch^{m-s}\|\mathbf{v}\|_m \; \forall \mathbf{v}\in \mathbf{H}^m(\Omega)\cap \mathbf{H}_{\Gamma_0}^1(\Omega).$$

Moreover, we assume the following inverse inequality for $\mathbf{P}_{\Gamma_g}^h \subset \mathbf{H}^1(\Gamma_g)$:

$$(3.5) \qquad \|\boldsymbol{\xi}^h\|_{s,\Gamma_g} \leq Ch^{t-s}\|\boldsymbol{\xi}^h\|_{t,\Gamma_g} \quad \forall \boldsymbol{\xi}^h\in \mathbf{P}_{\Gamma_g}^h\,,\; -1/2\leq t\leq s\leq 1\,,$$

where C is independent of h and $\boldsymbol{\xi}^h$. In (3.3)–(3.5), the integer k is related to the degree of the polynomial approximation.

For the choice of the boundary interpolation, we notice from the third equation of (3.1) that $<\mathbf{s}^h, \mathbf{u}^h-\mathbf{g}>_{1/2,\Gamma_g}=0$ for all $\mathbf{s}^h\in \mathbf{P}_{\Gamma_g}^h$, i.e., $\mathbf{u}^h$ cannot exactly approximate the boundary data $\mathbf{g}$ along Γ_g by merely taking $\mathbf{u}^h\big|_{\Gamma_g}$. This in general spoils the accuracy for the approximation. To circumvent it, we take $\mathbf{g}^h$ as the $\mathbf{L}^2(\Gamma_g)$–projection of $\mathbf{g}$, i.e., $\mathbf{g}^h=\mathcal{P}_{\Gamma_g}^h(\mathbf{g})$, where $\mathcal{P}_{\Gamma_g}^h$ denotes the $\mathbf{L}^2(\Gamma_g)$–projection from

$\mathbf{H}^{1/2}(\Gamma_g)$ onto $\mathbf{P}^h_{\Gamma_g}$. Then, the following orthogonal property for the projection holds;

$$\int_{\Gamma_g} (\mathcal{I} - \mathcal{P}^h_{\Gamma_g})\,\mathbf{g} \cdot \mathcal{P}^h_{\Gamma_g}(\gamma_g \mathbf{v})\, d\Gamma = 0\,, \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega)\,. \tag{3.6}$$

Based on this structure, one can approximate the system (3.1) in the following manner:

[-] Given $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma_g)$, evaluate $\mathbf{g}^h = \mathcal{P}^h_{\Gamma_g}(\mathbf{g})$.

[-] Solve for $(\mathbf{u}^h, p^h) \in \mathbf{V}^h_{\Gamma_0} \times S^h_0$ such that

$$\begin{aligned} \nu\, a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) &= < \mathbf{f}, \mathbf{v}^h > \\ &\quad - c(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{V}^h_0\,, \\ b(\mathbf{u}^h, q^h) &= 0 \quad \forall q^h \in S^h_0\,, \\ < \boldsymbol{\xi}^h, \mathbf{u}^h >_{\Gamma_g} &= < \boldsymbol{\xi}^h, \mathbf{g}^h >_{\Gamma_g} \quad \forall \mathbf{s}^h \in \mathbf{P}^h_{\Gamma_g}\,. \end{aligned} \tag{3.7}$$

[-] Solve for $\mathbf{t}^h \in \mathbf{P}^h_{\Gamma_g}$ such that

$$\begin{aligned} < \mathbf{t}^h, \mathbf{v}^h >_{\Gamma_g} &= \nu a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) + c(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) \\ &\quad - < \mathbf{f}, \mathbf{v}^h > \quad \forall \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0} - \mathbf{V}^h_0\,. \end{aligned} \tag{3.8}$$

In this approach, the computation of the boundary stress is attained in terms of known velocity and pressure. It is interesting that, despite the stress not being a boundary condition for the problem considered, a natural postprocessing procedure for the boundary stress nevertheless ensues.

Note that using this split formulation, the weak primal version (2.9)-(2.11) can be rewritten in the form:

[-] Solve for $(\mathbf{u}, p) \in \mathbf{H}^1_{\Gamma_0} \times L^2_0(\Omega)$ such that

$$\begin{aligned} \nu\, a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= < \mathbf{f}, \mathbf{v} >_{-1} - c(\mathbf{u}, \mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\,, \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in L^2_0(\Omega)\,, \\ < \boldsymbol{\xi}, \mathbf{u} >_{\Gamma_g} &= < \boldsymbol{\xi}, \mathbf{g} >_{\Gamma_g} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_g)\,. \end{aligned} \tag{3.9}$$

[–] Solve for $\mathbf{t} \in \mathbf{H}^{-1/2}(\Gamma_g)$ such that

$$(3.10) \qquad \begin{aligned} <\mathbf{t}, \mathbf{v}>_{\Gamma_g} = \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) \\ - <\mathbf{f}, \mathbf{v}>_{-1} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega) - \mathbf{H}^1_0(\Omega). \end{aligned}$$

3.2. Approximation $\mathbf{g}^h$ to $\mathbf{g}$

Since the approximation properties are fulfilled for the familiar regular finite elements (see, e.g., [4], [6] and [16]), the accuracy of the scheme mainly depends on how good an approximation $\mathbf{g}^h$ is to $\mathbf{g}$. For this purpose, we introduce the $\mathbf{H}^1_{\Gamma_0}$-projection $\mathcal{Q}^h$ from $\mathbf{H}^1_{\Gamma_0}(\Omega)$ onto $\mathbf{V}^h_{\Gamma_0}$, i.e. for $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$,

$$(3.11) \qquad ((\mathcal{I} - \mathcal{Q}^h)\mathbf{w}, \mathbf{v}^h))_1 = a((\mathcal{I} - \mathcal{Q}^h)\mathbf{w}, \mathbf{v}^h) = 0 \quad \forall \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0}.$$

We need some preliminary results for the approximation to $\mathbf{g}$.

LEMMA 3.1. *There exists a constant $C > 0$ such that*

$$(3.12) \qquad \|\gamma_g \mathbf{v}\|^2_{0,\Gamma_g} \leq C(\delta^{1/2}\|\!|\mathbf{v}|\!\|^2 + \delta^{-1/2}\|\mathbf{v}\|^2_0),$$

for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$ and $0 < \delta < 1$.

Proof. This is an immediate consequence of [8] (Theorem 1.5.1.10) and the continuity of the trace mapping. □

LEMMA 3.2. *Let $\mathbf{w}$ be an element of $\mathbf{H}^1_{\Gamma_0}(\Omega)$ with $\gamma_g \mathbf{w} = \mathbf{g}$. Then*

$$(3.13) \qquad \left\|\gamma_g\Big((\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\Big)\right\|_{0,\Gamma_g} \leq Ch^{1/2}\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1.$$

Proof. For the proof, one may recourse to the so-called *Aubin–Nitsche trick* (see [6]). Let $\mathbf{e} = \mathbf{w} - \mathcal{Q}^h\mathbf{w}$. Since $\mathbf{V}^h_{\Gamma_0} \subset \mathbf{H}^1_{\Gamma_0}(\Omega)$, $\mathbf{e} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$. Let us assume $\boldsymbol{\eta} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{\Gamma_0}(\Omega)$ be the solution of the following auxiliary problem:

$$(3.14) \qquad a(\boldsymbol{\eta}, \mathbf{v}) = (\mathbf{z}, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega),$$

where $\mathbf{z} \in \mathbf{H}^{-1}_{\Gamma_0}(\Omega) \cap \mathbf{L}^2(\Omega)$ is given. The regularity of (3.14) yields $\|\boldsymbol{\eta}\|_2 \leq C\|\mathbf{z}\|_0$. Now let us take $\mathbf{v} = \mathbf{e}$ in (3.14). Then, using the symmetry of $a(\cdot,\cdot)$ and (3.14), we have

$$\begin{aligned}\|\mathbf{e}\|_0 &= \sup_{\mathbf{z}} \frac{(\mathbf{z},\mathbf{e})_0}{\|\mathbf{z}\|_0} \\ &= \sup_{\mathbf{z}} \frac{a(\boldsymbol{\eta},\mathbf{e})}{\|\mathbf{z}\|_0} \\ &= \sup_{\mathbf{z}} \frac{a(\boldsymbol{\eta}-\boldsymbol{\eta}^h,\mathbf{e})}{\|\mathbf{z}\|_0},\end{aligned}$$

where $\boldsymbol{\eta}^h$ denotes a $\mathbf{V}^h_{\Gamma_0}$–interpolant of $\boldsymbol{\eta}$. Hence,

$$\|\mathbf{e}\|_0 \leq C|\!|\!|\boldsymbol{\eta}-\boldsymbol{\eta}^h|\!|\!| \sup_{\mathbf{z}} \frac{|\!|\!|\mathbf{e}|\!|\!|}{\|\mathbf{z}\|_0}.$$

From the approximation assumption (3.4), we have

$$|\!|\!|\boldsymbol{\eta}-\boldsymbol{\eta}^h|\!|\!| \leq \|\boldsymbol{\eta}-\boldsymbol{\eta}^h\|_1 \leq Ch\|\boldsymbol{\eta}\|_2 \leq Ch\|\mathbf{z}\|_0,$$

whence

$$\|\mathbf{e}\|_0 = \|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_0 \leq Ch|\!|\!|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}|\!|\!| \leq Ch\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1.$$

Applying the inequality (3.12) by taking $\delta = h^2$, it follows that

$$\begin{aligned}\left\|\gamma_g\Big((\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\Big)\right\|^2_{0,\Gamma_g} &\leq C\Big(h|\!|\!|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}|\!|\!|^2 + h^{-1}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_0^2\Big) \\ &\leq Ch\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1^2.\end{aligned}$$

□

REMARK. In the auxiliary problem (3.14), we assumed the $\mathbf{H}^2$–regularity for the solution $\boldsymbol{\eta}$. On a convex domain with $\mathcal{C}^{0,1}$–boundary, the elliptic problem always guarantees such an order of regularity by the classical regularity theory ([8]). For a nonconvex domain, however, this may not hold. One may get at most $\mathbf{H}^{1+\rho}$–regularity for some $0 < \rho < 1$. Nevertheless, the estimate (3.13) does not change, for

(3.12) holds for all $\delta \in (0,1)$, i.e., the minor adjusting of δ and the index of h will yield the same conclusion.

The preliminary estimates for the computation $\mathbf{g}^h = \mathcal{P}^h_{\Gamma_g}(\mathbf{g})$ can be obtained by refining the result of Lemma 3.2, using the inverse inequality (3.5). Our approach is illustrated in the following (non-commutative) diagram:

$$\begin{array}{ccc}
\mathbf{H}^1_{\Gamma_0}(\Omega) \subset \mathbf{H}^1(\Omega) & \xrightarrow{\mathcal{Q}^h} & \mathbf{V}^h_{\Gamma_0} \subset \mathbf{V}^h \\
\gamma_g \big\downarrow & & \big\downarrow \gamma_g \\
\mathbf{H}^{1/2}(\Gamma_g) \subset \mathbf{L}^2(\Gamma_g) & \xrightarrow{\mathcal{P}^h_{\Gamma_g}} & \mathbf{P}^h_{\Gamma_g} \subset \mathbf{P}^h
\end{array}$$

Let $\mathbf{w}$ be an element of $\mathbf{H}^1_{\Gamma_0}(\Omega)$ such that $\gamma_g \mathbf{w} = \mathbf{g}$. Then, preliminary result includes estimations of $\mathbf{g} - \mathcal{P}^h_{\Gamma_g}\mathbf{g}$ with respect to $(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}$ (see also [3] and [4]). Once the basis of the discrete space $\mathbf{P}^h_{\Gamma_g}$ is known, it is easy to build the approximation $\mathbf{g}^h$.

LEMMA 3.3. *For a given* $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma_g)$, *let* $\mathbf{g}^h = \mathcal{P}^h_{\Gamma_g}\mathbf{g} \in \mathbf{P}^h_{\Gamma_g}$. *It holds that*

$$\|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_g} \leq C h^{1/2} \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 , \tag{3.15}$$

$$\|\mathbf{g} - \mathbf{g}^h\|_{-1/2,\Gamma_g} \leq C h \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 \tag{3.16}$$

and

$$\|\mathbf{g} - \mathbf{g}^h\|_{1/2,\Gamma_g} \leq C \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 . \tag{3.17}$$

Proof. Note that $\displaystyle\int_{\Gamma_g} (\mathbf{g} - \mathbf{g}^h) \cdot \boldsymbol{\phi}^h \, d\Gamma = 0 \ \ \forall \boldsymbol{\phi}^h \in \mathbf{P}^h_{\Gamma_g}$ by (3.6). Since $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$ is chosen so that $\gamma_g \mathbf{w} = \mathbf{g}$ and $\gamma_g(\mathcal{Q}^h \mathbf{w}) = \mathbf{g}^h \in \mathbf{P}^h_{\Gamma_g}$, we have

$$\begin{aligned}
\|\mathbf{g} - \mathbf{g}^h\|^2_{0,\Gamma_g} &= \int_{\Gamma_g} (\mathbf{g} - \mathbf{g}^h) \cdot (\mathbf{g} - \gamma_g(\mathcal{Q}^h \mathbf{w}) + \gamma_g(\mathcal{Q}^h \mathbf{w}) - \mathbf{g}^h) \, d\Gamma \\
&= \int_{\Gamma_g} (\mathbf{g} - \mathbf{g}^h) \cdot (\mathbf{g} - \gamma_g(\mathcal{Q}^h \mathbf{w})) \, d\Gamma \\
&\leq \|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_g} \|\gamma_g(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_g} .
\end{aligned}$$

Hence (3.13) yields

$$\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_g} \le \|\gamma_g(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_g} \le Ch^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 ,$$

so that (3.15) is obtained.

For the proof of (3.16), we need some preliminary facts: For each $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_g)$, let $\mathbf{v}_{\boldsymbol{\phi}} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$ be a lifting of $\boldsymbol{\phi}$ for the trace, i.e., $\gamma_g \mathbf{v}_{\boldsymbol{\phi}} = \boldsymbol{\phi}$ and $\|\mathbf{v}_{\boldsymbol{\phi}}\|_1 \le C\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}$. From the orthogonality of the projection $\mathcal{Q}^h$, we obtain

$$\|\mathbf{v}_{\boldsymbol{\phi}}\|_1^2 = \|(\mathcal{I}-\mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1^2 + \|\mathcal{Q}^h\mathbf{v}_{\boldsymbol{\phi}}\|_1^2 .$$

Hence, it follows that

$$\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1 \le \|\mathbf{v}_{\boldsymbol{\phi}}\|_1 \le C\|\boldsymbol{\phi}\|_{1/2,\Gamma_g} . \tag{3.18}$$

The inequality (3.16) is the composite result of (3.6), (3.15) and (3.18);

$$\begin{aligned}
\|\mathbf{g}-\mathbf{g}^h\|_{-1/2,\Gamma_g} &= \sup_{0\neq\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma_g)} \frac{\int_{\Gamma_g}(\mathbf{g}-\mathbf{g}^h)\cdot\boldsymbol{\phi}\,d\Gamma}{\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}} \\
&= \sup_{0\neq\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma_g)} \frac{\int_{\Gamma_g}(\mathbf{g}-\mathbf{g}^h)\cdot(\boldsymbol{\phi}-\mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi})\,d\Gamma}{\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}} \\
&\le C\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_g} \sup_{0\neq\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma_g)} \frac{\|\boldsymbol{\phi}-\mathcal{P}^h_{\Gamma_g}\|_{0,\Gamma_g}}{\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}} \\
&\le Ch^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 \sup \frac{h^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1}{\|\mathbf{v}_{\boldsymbol{\phi}}\|_1} \\
&\le Ch\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 .
\end{aligned}$$

In order to show (3.17), we notice that

$$\begin{aligned}
\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_g} &\le \|\mathbf{g}-\gamma_g(\mathcal{Q}^h\mathbf{w})\|_{0,\Gamma_g} = \|\gamma_g(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_g} \\
&\le Ch^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 ,
\end{aligned}$$

which follows from the orthogonality of $\mathcal{P}^h_{\Gamma_g}$ and (3.3). Then, using the inverse inequality and the continuity of the trace mapping, we have

$$\begin{aligned}
&\|\mathbf{g}-\mathbf{g}^h\|_{1/2,\Gamma_g}\\
&\le \|\gamma_g(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_{1/2,\Gamma_g}+\|\gamma_g(\mathcal{Q}^h\mathbf{w})-\mathbf{g}^h\|_{1/2,\Gamma_g}\\
&\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1+h^{-1/2}\|\gamma_g(\mathcal{Q}^h\mathbf{w})-\mathbf{g}^h\|_{0,\Gamma_g}\,]\\
&\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1+h^{-1/2}(\|\mathbf{g}-\gamma_g(\mathcal{Q}^h\mathbf{w})\|_{0,\Gamma_g}+\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_g})\,]\\
&\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1+h^{-1/2}(\|\gamma_g(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_g}+\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_g})\,]\\
&\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1+h^{-1/2}h^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1\,]\\
&\le C\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1\,. \qquad \square
\end{aligned}$$

Using a similar technique, the results of Lemma 3.3 can be generalized as follows:
Suppose $\mathbf{g}\in\mathbf{H}^s(\Gamma_g)$ for all $s\in[0,1/2]$ and $\mathbf{g}^h=\mathcal{P}^h_{\Gamma_g}\mathbf{g}$. Let $\mathbf{v}_\mathbf{g}$ be a lifting of $\mathbf{g}$. Then, it follows for $0\le s\le 1/2$

$$\begin{aligned}\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_g}&\le Ch^s\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{v}_\mathbf{g}\|_{s+1/2}\\&\le Ch^s\|\mathbf{g}\|_s\,.\end{aligned}\tag{3.19}$$

Moreover, for $\boldsymbol{\phi}\in\mathbf{H}^s(\Gamma_g)$, since

$$\begin{aligned}\int_{\Gamma_g}(\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\mathbf{g}\cdot\boldsymbol{\phi}\,d\Gamma&=\int_{\Gamma_g}(\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\mathbf{g}\cdot(\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi}\,d\Gamma\\&\le\|(\mathbf{g}-\mathbf{g}^h)\|_0\|(\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi}\|_0\\&\le Ch^t\|\mathbf{g}\|_t h^s\|\boldsymbol{\phi}\|_s\quad\text{for } 0\le s,\ t\le 1/2\,,\end{aligned}$$

we obtain

$$\|\mathbf{g}-\mathbf{g}^h\|_{-s,\Gamma_g}=\sup_{\boldsymbol{\phi}\in\mathbf{H}^s(\Gamma_g),\|\boldsymbol{\phi}\|_{s,\Gamma_g}\le 1}\int_{\Gamma_g}(\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\mathbf{g}\cdot\boldsymbol{\phi}\,d\Gamma\le Ch^{s+t}\|\mathbf{g}\|_t\,.$$

So, this naturally leads us to

$$\|\mathbf{g}-\mathbf{g}^h\|_{-s,\Gamma_g}\le Ch^{1/2+s}\|\mathbf{g}\|_{1/2}\quad\text{for } 0\le s\le 1/2\,.\tag{3.20}$$

Now, we state the main result for the estimation of $\mathbf{g}^h$ to the $\mathcal{P}^h_{\Gamma_g}$-projection of $\mathbf{g}$.

THEOREM 3.4. *Let* $\mathbf{g}$ *and* $\mathbf{g}^h$ *be defined as in Lemma 3.3. Then, as* $h \to 0^+$, *we have*

$$h^{-1/2}\|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_g} \longrightarrow 0\,, \tag{3.21}$$

$$h^{-1}\|\mathbf{g} - \mathbf{g}^h\|_{-1/2,\Gamma_g} \longrightarrow 0\,, \tag{3.22}$$

and

$$\|\mathbf{g} - \mathbf{g}^h\|_{1/2,\Gamma_g} \longrightarrow 0\,. \tag{3.23}$$

Proof. Since $\mathbf{w}$ is a lifting of $\mathbf{g}$ taken arbitrary. (3.20) implies that

$$h^{-1/2}\|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_g} \leq \inf_{\mathbf{w}\in \mathbf{H}^1_{\Gamma_0}(\Omega)\,,\gamma_g \mathbf{w}=\mathbf{g}} \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1\,.$$

Since $\mathcal{C}^\infty(\overline{\Omega}) \cap \mathbf{H}^1_{\Gamma_0}(\Omega)$ is dense in $\mathbf{H}^1_{\Gamma_0}(\Omega)$, one can deduce (3.21) from the approximation property of $\mathbf{V}^h_{\Gamma_0}$. (3.22) and (3.23) can be shown in the similar manner. □

If sufficient regularity is allowed for the domain and the data, the estimates of Lemma 3.3 for the approximation can be sharpened.

THEOREM 3.5. *Suppose* $\mathbf{g} \in \mathbf{H}^{m-1/2}(\Gamma_g)$ *for* $1 \leq m \leq k$ *and* $\mathbf{w} \in \mathbf{H}^m(\Omega) \cap \mathbf{H}^1_{\Gamma_0}(\Omega)$ *be a lifting of* $\mathbf{g}$. *Then, under the same condition with Lemma 3.3., we have*

$$\|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_g} \leq Ch^{m-1/2}\|\mathbf{w}\|_m\,, \tag{3.24}$$

$$\|\mathbf{g} - \mathbf{g}^h\|_{-1/2,\Gamma_g} \leq Ch^{m}\|\mathbf{w}\|_m\,, \tag{3.25}$$

$$\|\mathbf{g} - \mathbf{g}^h\|_{1/2,\Gamma_g} \leq Ch^{m-1}\|\mathbf{w}\|_m\,. \tag{3.26}$$

Proof. Since $Q^h\mathbf{w} \in \mathbf{V}^h_{\Gamma_0}$, from the approximation property (3.4), (3.10) yields

$$\begin{aligned}\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_g} &\le Ch^{1/2}\|(\mathcal{I}-Q^h)\mathbf{w}\|_1\\ &\le Ch^{1/2}h^{m-1}\|\mathbf{w}\|_m = Ch^{m-1/2}\|\mathbf{w}\|_m\,,\end{aligned}$$

so that (3.24) holds. Similarly, for (3.25) and (3.26). □

REMARK. The regularity problem we encounter can be simply stated as follows; For $\mathbf{g} \in \mathbf{H}^{m-1/2}(\Gamma_g)$, can we find $\mathbf{w} \in \mathbf{H}^m(\Omega) \cap \mathbf{H}^1_{\Gamma_0}(\Omega)$ such that $a(\mathbf{w},\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$ with $\mathbf{w} = \mathbf{g}$ on Γ_g? This is always true on the smooth domain by the Lax–Milgram Lemma and the regularity results for elliptic problems (see [6]). However, in a polyhedral domain, sufficient regularity may not be available, m will be at best 2 (see [8]). That explains why we have taken the $\mathbf{L}^2$–projection for the approximation of the boundary data instead of boundary interpolants. Methods using conventional boundary interpolants do not ensure the convergence nor optimal $\mathbf{L}^2$–error estimates without supplying a sufficient regularity assumption for the solution.

In conjunction with the existence of continuous lifting of $\mathbf{H}^{1/2}(\Gamma_g)$ in $\mathbf{H}^1(\Omega)$ as in Lemma 2.1, it is useful to mention its discrete counterpart for later use.

LEMMA 3.6. **(Lifting of $\mathbf{P}^h_{\Gamma_g}$)** Given $\boldsymbol{\xi}^h \in \mathbf{P}^h_{\Gamma_g}$, there exists $\mathbf{v}^h \in \mathbf{V}^h_{\Gamma_g}$ such that $\gamma_g(\mathbf{v}^h) = \boldsymbol{\xi}^h$ and $\|\mathbf{v}^h\|_1 \le C\|\boldsymbol{\xi}^h\|_{1/2,\Gamma_g}$.

Proof. For our purpose, we only show the basic idea in 2D-case. Let us consider

$$-\Delta\mathbf{v} = \mathbf{0} \text{ in } \Omega,$$

and

$$\mathbf{v} = \begin{cases} \mathbf{0} \text{ on } \Gamma_0 \\ \boldsymbol{\xi}^h \text{ on } \Gamma_g. \end{cases}$$

Since $\mathbf{V}^h_{\Gamma_0} \subset \mathbf{H}^1_{\Gamma_0}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, we have $\boldsymbol{\xi}^h \in \mathbf{H}^{1/2+\delta}(\Gamma_g)$ for $0 < \delta \le \frac{1}{2}$. Sobolev embedding and elliptic regularity yield that $\mathbf{v} \in \mathbf{H}^{1+\delta}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ and $\|\mathbf{v}\|_{1+\delta} \le C\|\boldsymbol{\xi}^h\|_{1/2+\delta,\Gamma_g}$. Now, take $\mathbf{v}^h$ to be a

$\mathbf{V}^h_{\Gamma_0}$-interpolant of $\mathbf{v}$ so that $\gamma_g \mathbf{v}^h = \boldsymbol{\xi}^h$. By Bramble-Hilbert Lemma, we have $\|\mathbf{v}-\mathbf{v}^h\|_0 \le Ch^{1+\delta}\|\mathbf{v}\|_{1+\delta}$ and $\|\!|\mathbf{v}-\mathbf{v}^h|\!\| \le Ch^{\delta}\|\mathbf{v}\|_{1+\delta}$. Applying inverse inequality, one can conclude that

$$\begin{aligned}\|\mathbf{v}^h\|_1 &\le \|\mathbf{v}-\mathbf{v}^h\|_1 + \|\mathbf{v}\|_1 \le C(h^{\delta}\|\mathbf{v}\|_{1+\delta} + \|\mathbf{v}\|_1)\\ &\le C(h^{\delta}\|\boldsymbol{\xi}^h\|_{1/2+\delta} + \|\boldsymbol{\xi}^h\|_{1/2,\Gamma_g}) \le C\|\boldsymbol{\xi}^h\|_{1/2,\Gamma_g}. \qquad \square\end{aligned}$$

3.3. Intermediate operator for the boundary stress

For error estimates of the approximation (3.8) for the stress along the inhomogeneous boundary, some comments are in order for the choice of an approximation space $\mathbf{P}^h_{\Gamma_g}$ for the stress. $\mathbf{P}^h_{\Gamma_g}$ has been chosen to accommodate both the trace of the velocity and the boundary stress. For strict computation, one may consider taking $\mathbf{P}^h_{\Gamma_g}$ independently of the velocity space; two different spaces $\mathbf{P}^{h_1}_{\Gamma_g}$ and $\gamma_g(\mathbf{V}^{h_2}_{\Gamma_0})$ with different meshes may be taken to approximate the stress and the trace of the velocity. However, in order to sustain the stability of the scheme, this approach necessitates an additional requirement for meshes such that $h_1 \ge Kh_2$, where K is a positive constant dependent on the domain (see [2], [4] and Remark of this section). This levies an additional difficulty of determining K. Our approach is simply $\gamma_g(\mathbf{V}^h_{\Gamma_0}) = \mathbf{P}^h_{\Gamma_g}$ and $\mathbf{P}^h_{\Gamma_g}$, as an approximation space of the stress, is understood to be embedded in $\mathbf{H}^{-1/2}(\Gamma_g)$. [17] studied similar structure relating the interior mesh to the boundary for the Lagrange multiplier, which is represented by the normal derivative terms of state variables. However, his approach consists of taking the approximation space in $\mathbf{L}^2(\Gamma)$.

Let us begin our discussion by introducing an intermediate operator to interpret $\mathbf{P}^h_{\Gamma_g}$ in $\mathbf{H}^{-1/2}(\Gamma_g)$. We consider the operator $\mathcal{R}^h_{\Gamma_g}$: $\mathbf{H}^{-1/2}(\Gamma_g) \to \mathbf{P}^h_{\Gamma_g} \subset \mathbf{H}^{-1/2}(\Gamma_g)$ defined by

$$(3.27) \qquad < \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, \mathbf{v}^h >_{\Gamma_g} = < \boldsymbol{\xi}, \mathbf{v}^h >_{\Gamma_g} \quad \forall \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0}.$$

This operator is an extension of $\mathcal{P}^h_{\Gamma_g}$ in the sense that $\mathcal{R}^h_{\Gamma_g}\Big|_{\mathbf{H}^s(\Gamma_g)} = \mathcal{P}^h_{\Gamma_g}$ for all $s \ge 1/2$. For our purpose, we call attention to particular properties of $\mathcal{R}^h_{\Gamma_g}$.

LEMMA 3.7. *The operator* $\mathcal{R}^h_{\Gamma_g}$ *defined in (3.27) satisfies the following properties:*

(i) $\mathcal{R}^h_{\Gamma_g}$ *is a bounded operator in* $\mathbf{H}_{\Gamma_g}^{-1/2}$.

(ii) *For* $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma_g)$, *we have*

$$\|(\mathcal{I} - \mathcal{R}^h_{\Gamma_g})\boldsymbol{\xi}\|_{-1/2,\Gamma_g} \to 0 \quad \text{as } h \to 0^+ . \tag{3.28}$$

(iii) *We suppose that* $\boldsymbol{\xi} \in \mathbf{H}^s(\Gamma_g)$ *for* $-1/2 \le s \le 1/2$. *Then we have*

$$\|(\mathcal{I} - \mathcal{R}^h_{\Gamma_g})\boldsymbol{\xi}\|_{-1/2,\Gamma_g} \le C h^{s+1/2} \|\boldsymbol{\xi}\|_{s,\Gamma_g} . \tag{3.29}$$

Proof. (i); Let $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_g)$ and $\mathbf{v}_{\boldsymbol{\phi}}$ be a lifting of $\boldsymbol{\phi}$ in $\mathbf{H}^1_{\Gamma_0}(\Omega)$ so that $\|\mathbf{v}_{\boldsymbol{\phi}}\|_1 \le C\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}$. Using (3.17) and (3.18), we have

$$\begin{aligned}
\|\mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi}\|_{1/2,\Gamma_g} &\le \|(\mathcal{I} - \mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi}\|_{1/2,\Gamma_g} + \|\boldsymbol{\phi}\|_{1/2,\Gamma_g} \\
&\le C\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1 + \|\boldsymbol{\phi}\|_{1/2,\Gamma_g} \\
&\le C\|\boldsymbol{\phi}\|_{1/2,\Gamma_g} .
\end{aligned}$$

Since $\mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi} \in \mathbf{P}^h_{\Gamma_g}$, we deduce from (3.6) and (3.27) that

$$\begin{aligned}
< \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, \boldsymbol{\phi} >_{\Gamma_g} &=< \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, (\mathcal{I} - \mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi} + \mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi} >_{\Gamma_g} \\
&=< \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, \mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi} >_{\Gamma_g} \\
&=< \boldsymbol{\xi}, \mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi} >_{\Gamma_g} \\
&\le \|\boldsymbol{\xi}\|_{-1/2,\Gamma_g} \|\mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi}\|_{1/2,\Gamma_g} \\
&\le C\|\boldsymbol{\xi}\|_{-1/2,\Gamma_g} \|\boldsymbol{\phi}\|_{1/2,\Gamma_g} .
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\|\mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}\|_{-1/2,\Gamma_g} &= \sup_{\mathbf{0}\neq\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma_g),\, \|\boldsymbol{\phi}\|_{1/2,\Gamma_g}\le 1} < \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, \boldsymbol{\phi} >_{\Gamma_g} \\
&\le C\|\boldsymbol{\xi}\|_{-1/2,\Gamma_g} ;
\end{aligned}$$

so that $\|\mathcal{R}^h_{\Gamma_g}\|_{\mathcal{L}(\mathbf{H}^{-1/2}(\Gamma_g))} \leq C$.

(ii); We use the fact that $\mathcal{C}^\infty(\Gamma_g)$ is dense in $\mathbf{H}^{-1/2}(\Gamma_g)$ and $\mathcal{R}^h_{\Gamma_g}(\mathcal{C}^\infty(\Gamma_g)) = \mathcal{P}^h_{\Gamma_g}(\mathcal{C}^\infty(\Gamma_g))$. For $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma_g)$, take $\hat{\boldsymbol{\xi}} \in \mathcal{C}^\infty(\Gamma_g)$. Since $\mathcal{R}^h_{\Gamma_g}$ is bounded in $\mathbf{H}^{-1/2}(\Gamma_g)$, using (3.20) one can carry out

$$\begin{aligned}
&\|(\mathcal{I} - \mathcal{R}^h_{\Gamma_g})\boldsymbol{\xi}\|_{-1/2,\Gamma_g} \\
&= \|(\mathcal{I} - \mathcal{R}^h_{\Gamma_g})((\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}) + \hat{\boldsymbol{\xi}})\|_{-1/2,\Gamma_g} \\
&\leq \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_{-1/2,\Gamma_g} + \|\mathcal{R}^h_{\Gamma_g}(\boldsymbol{\xi} - \hat{\boldsymbol{\xi}})\|_{-1/2,\Gamma_g} + \|(\mathcal{I} - \mathcal{P}^h_{\Gamma_g})\hat{\boldsymbol{\xi}}\|_{-1/2,\Gamma_g} \\
&\leq C[\,\|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_{-1/2,\Gamma_g} + h\|\hat{\boldsymbol{\xi}}\|_{1/2,\Gamma_g}].
\end{aligned}$$

Therefore, (3.28) follows from the denseness of $\mathcal{C}^\infty(\Gamma_g)$ in $\mathbf{H}^{-1/2}(\Gamma_g)$.

(iii); Employing interpolation arguments for the Sobolev space (see [7]), (3.29) follows from (3.16) and (3.20). □

REMARK. When $\mathbf{P}^{h_1}_{\Gamma_g}$ and $\gamma_g(\mathbf{V}^{h_2}_{\Gamma_0})$ are taken to approximate the stress and the trace of the velocity, respectively, and the meshes are taken to satisfy the relation $h_1 \geq K h_2$ for some sufficiently large constant K, we can show that there exists a positive constant C such that

$$\|\mathcal{B}^{h^T}\|_{\mathcal{L}(\mathbf{M}^*,\mathbf{V}^*)} \geq C\,, \tag{3.30}$$

where $\mathcal{B}^h : \mathbf{V}^{h_2}_{\Gamma_0} \to S^{h_2}_0 \times \mathbf{P}^{h_1}_{\Gamma_g}$ is the restriction of the operator (2.7), i.e.,

$$< \mathcal{B}^h \mathbf{w}^h, (q^h, \boldsymbol{\xi}^h) >= b(\mathbf{w}^h, q^h) - < \boldsymbol{\xi}^h, \mathbf{w}^h >_{\Gamma_g}$$

It can be achieved by simply modifying the auxiliary problem in [10] as follows: Given $(\boldsymbol{\xi}^h, q^h) \in \mathbf{P}^{h_1}_{\Gamma_g} \times S^{h_2}_0$, find $(\boldsymbol{\phi}, r) \times \mathbf{H}^1_{\Gamma_0}(\Omega) \times L^2_0(\Omega)$ satisfying

$$\begin{aligned}
-\Delta\boldsymbol{\phi} + \nabla r &= \mathbf{0} \quad \text{in } \Omega\,, \\
\nabla \cdot \boldsymbol{\phi} &= 0 \quad \text{in } \Omega\,, \\
-r\mathbf{n} + \mathbf{n} \cdot (\nabla\boldsymbol{\phi} + (\nabla\boldsymbol{\phi})^T) &= -\boldsymbol{\xi}^h \quad \text{on } \Gamma_g\,.
\end{aligned}$$

Note that (3.30) is tantamount to the discrete augmented LBB condition for the discrete Stokes problem

$$\begin{aligned}
a(\mathbf{u}^h, \mathbf{v}^h) + < \mathcal{B}^h \mathbf{v}^h, (p^h, \mathbf{t}^h) > &= < \mathbf{f}, \mathbf{v}^h > \quad \forall \mathbf{v}^h \in \mathbf{V}^{h_2}_{\Gamma_0}\,, \\
< \mathcal{B}^h \mathbf{u}^h, (q^h, \boldsymbol{\eta}^h) > &= - < \boldsymbol{\eta}^h, \mathbf{g} >_{\Gamma_g} \ \forall (q^h, \boldsymbol{\eta}^h) \in S^{h_2}_0 \times \mathbf{P}^{h_1}_{\Gamma_g}\,.
\end{aligned}$$

In [17], the Stokes problem with no-slip boundary conditions was considered in a similar manner. [17] used the same mesh $h_1 = h_2$ and enhanced the approximation for the velocity by adding bubble functions along the boundary to achieve the augmented LBB condition.

4. Error estimates

In this section, we derive error estimates for the approximation of $(\mathbf{u}^h, p^h, \mathbf{t}^h)$ to $(\mathbf{u}, p, \mathbf{t})$, where $(\mathbf{u}^h, p^h, \mathbf{t}^h)$ and $(\mathbf{u}, p, \mathbf{t})$ are solutions of the systems (3.7)–(3.10).

To study the approximation, we first invoke the corresponding nonlinear function formulation as in [5] and [8]. The study of the approximation may be reduced to the analysis of the corresponding approximation to the solution of the Stokes formulation whose boundary stress is decoupled from the velocity and pressure.

4.1. Brezzi-Rappaz-Raviart framework

We take $\mathbf{X} = \mathbf{H}^1_{\Gamma_0}(\Omega) \times L^2_0(\Omega)$, $\mathbf{Y} = \mathbf{H}^{-1}_{\Gamma_0}(\Omega) \times \mathbf{H}^{1/2}_0(\Gamma_g)$ and $\mathbf{Z} = \mathbf{L}^{3/2}(\Omega) \times \{\mathbf{0}\}$. We define the solution operator $T \in \mathcal{L}(\mathbf{Y}; \mathbf{X})$ for the Stokes problem with inhomogeneous boundary conditions by $T(\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) = (\widehat{\mathbf{u}}, \widehat{\mathbf{p}})$ if and only if

$$a(\widehat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widehat{\mathbf{p}}) = < \widehat{\mathbf{f}}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\,,$$

$$b(\widehat{\mathbf{u}}, q) = 0 \quad \forall q \in L^2_0(\Omega)\,,$$

and

$$< \mathbf{s}, \widehat{\mathbf{u}} >_{-1/2,\Gamma_g} = < \mathbf{s}, \widehat{\mathbf{g}} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_g)\,.$$

Analogously, for the solution operator of the approximate Stokes problem with $\mathbf{L}^2$-projection of the boundary data, we define $T^h(\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) = (\widehat{\mathbf{u}}^h, \widehat{\mathbf{p}}^h) \in \mathbf{X}^h = \mathbf{V}^h_{\Gamma_0} \times S^h_0$ if and only if

$$\widehat{\mathbf{g}}^h = \mathcal{P}^h_{\Gamma_g}(\widehat{\mathbf{g}}) \quad \forall \widehat{\mathbf{g}} \in \mathbf{H}^{1/2}_0(\Gamma_g)\,,$$

$$a(\widehat{\mathbf{u}}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \widehat{\mathbf{p}}^h) = < \widehat{\mathbf{f}}, \mathbf{v}^h > \quad \forall \mathbf{v}^h \in \mathbf{V}^h_0\,,$$

$$b(\widehat{\mathbf{u}}^h, q^h) = 0 \quad \forall q^h \in S^h_0\,,$$

and

$$< \mathbf{s}^h, \widehat{\mathbf{u}}^h >_{\Gamma_g} = < \mathbf{s}^h, \widehat{\mathbf{g}}^h >_{\Gamma_g} \quad \forall \mathbf{s}^h \in \mathbf{P}^h_{\Gamma_g} .$$

Since $\mathbf{X}^h$ is a dense subspace of $\mathbf{X}$, T^h is a bounded lineal operator from $\mathbf{Y}$ to $\mathbf{X}$. To cover the nonlinear part, we take Λ to be a compact subset of $\mathbb{R}^+$ and define the nonlinear operator G from $\Lambda \times \mathbf{X}$ to $\mathbf{Y}$ by $G(\lambda, (\mathbf{u}, p)) = (\boldsymbol{\eta}, \boldsymbol{\kappa})$, for $\lambda = \dfrac{1}{\nu} \in \Lambda$ if and only if

$$\begin{cases} < \boldsymbol{\eta}, \mathbf{v} >_{-1} = \lambda c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda < \mathbf{f}, \mathbf{v} >_{-1} & \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega) , \\ < \mathbf{s}, \boldsymbol{\kappa} >_{-1/2, \Gamma_g} = - < \mathbf{s}, \mathbf{g} >_{-1/2, \Gamma_g} & \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_g) . \end{cases}$$

Then, (3.7) and (3.9) can be written as

$$(\mathbf{u}, \lambda p) + TG(\lambda, (\mathbf{u}, \lambda p)) = 0$$

and

$$(\mathbf{u}^h, \lambda p^h) + T^h G(\lambda, (\mathbf{u}^h, \lambda p^h)) = 0 ,$$

respectively.

It is easy to check that the first and second Fréchet derivatives $D_{\boldsymbol{\phi}} G$ and $D_{\boldsymbol{\phi}\boldsymbol{\phi}} G$ belong to $\mathcal{L}(\mathbf{X}; \mathbf{Y})$, where $\boldsymbol{\phi} = (\mathbf{u}, p) \in \mathbf{X}$. Moreover, since $\mathbf{L}^{3/2}(\Omega)$ is continuously and compactly embedded in $\mathbf{H}^{-1}_{\Gamma_0}(\Omega)$, $\mathbf{Z}$ is compactly embedded in $\mathbf{Y}$. Hence due to Brezzi-Rappaz-Raviart framework ([5]), the analysis of the convergence turns into that of the approximation of T^h to T.

4.2. Error estimates for velocity–pressure

Let $\mathbf{y} = (\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) \in \mathbf{Y}$ be given. The approximation depends on the estimate for $\|(T^h - T)\mathbf{y}\|_{\mathbf{X}}$. For this purpose, we introduce the discrete operator T^h of T defined by $T^h \mathbf{y} = (\widehat{\mathbf{u}}^h, \widehat{\mathbf{p}}^h) \in \mathbf{X}^h$ if and only if

$$\widehat{\mathbf{g}}^h = \mathcal{P}^h_{\Gamma_g}(\widehat{\mathbf{g}}) \quad \forall \widehat{\mathbf{g}} \in \mathbf{H}^{1/2}_0(\Gamma_g)$$

$$a(\widehat{\mathbf{u}}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \widehat{\mathbf{p}}^h) = < \widehat{\mathbf{f}}, \mathbf{v}^h > \quad \forall \mathbf{v}^h \in \mathbf{V}^h_0 ,$$

$$b(\widehat{\mathbf{u}}^h, q^h) = 0 \quad \forall q^h \in S^h_0 ,$$

and

$$< \mathbf{s}^h, \widehat{\mathbf{u}}^h >_{\Gamma_g} = < \mathbf{s}^h, \widehat{\mathbf{g}}^h >_{\Gamma_g} \quad \forall \mathbf{s}^h \in \mathbf{P}^h_{\Gamma_g} .$$

We now state estimates for $\|(T^h - T)\mathbf{y}\|_{\mathbf{X}}$.

LEMMA 4.1. *Let* $\mathbf{y} = (\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) \in \mathbf{Y}$. *Let* $T\mathbf{y} = (\widehat{\mathbf{u}}, \widehat{\mathbf{p}})$. *Then, it holds that*

$$\|(T^h - T)\mathbf{y}\|_{\mathbf{X}} \le C \inf_{(\check{\mathbf{u}}^h, \check{p}^h) \in \check{\mathbf{X}}^h} \|(\widehat{\mathbf{u}}, \widehat{\mathbf{p}}) - (\check{\mathbf{u}}^h, \check{p}^h)\|_{\mathbf{X}}, \tag{4.1}$$

where $\check{\mathbf{X}}^h = \{(\check{\mathbf{u}}^h, \check{p}^h) \in \mathbf{V}^h_{\Gamma_0} \times S^h_0 \mid \gamma_g \check{\mathbf{u}}^h = \widehat{\mathbf{g}}^h\}$.

Proof. This follows directly from the application of the result due to [11]. □

One can combine (4.1) with the approximation result for $\widehat{\mathbf{g}}^h$ to obtain the following result.

LEMMA 4.2. *Under the same conditions with Lemma 4.1, we have*

$$\|(T^h - T)\mathbf{y}\|_{\mathbf{X}} \le C \inf_{(\boldsymbol{\eta}^h, r^h) \in \mathbf{X}^h} \|(\widehat{\mathbf{u}}, \widehat{\mathbf{p}}) - (\boldsymbol{\eta}^h, r^h)\|_{\mathbf{X}}. \tag{4.2}$$

Furthermore, if we assume that $(\widehat{\mathbf{u}}, \widehat{\mathbf{p}}) \in \mathbf{X} \cap (\mathbf{H}^m(\Omega) \times H^{m-1}(\Omega))$ *for* $1 \le m \le k$, *then there exists a positive constant* C, *independent of* h, *such that*

$$\|(T^h - T)\mathbf{y}\|_{\mathbf{X}} \le C h^{m-1} \left[\|\widehat{\mathbf{u}}\|_m + \|\widehat{\mathbf{p}}\|_{m-1} \right]. \tag{4.3}$$

Proof. We first show that

$$\begin{aligned} &\|(T^h - T)\mathbf{y}\|_{\mathbf{X}} \\ &\quad \le C\Big[\inf_{(\boldsymbol{\eta}^h, r^h) \in \mathbf{X}^h} \|(\widehat{\mathbf{u}}, \widehat{\mathbf{p}}) - (\boldsymbol{\eta}^h, r^h)\|_{\mathbf{X}} + \|\widehat{\mathbf{g}} - \widehat{\mathbf{g}}^h\|_{1/2, \Gamma_g} \Big]. \end{aligned} \tag{4.4}$$

We observe that

$$|||(\mathcal{I} - Q^h)\widehat{\mathbf{u}}||| = \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0}} |||\widehat{\mathbf{u}} - \boldsymbol{\eta}^h||| \le \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0}} \|\widehat{\mathbf{u}} - \boldsymbol{\eta}^h\|_1.$$

Hence, from Korn's inequality we have

$$\|(\mathcal{I} - Q^h)\widehat{\mathbf{u}}\|_1 \le C |||(\mathcal{I} - Q^h)\widehat{\mathbf{u}}||| \le C \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0}} \|\widehat{\mathbf{u}} - \boldsymbol{\eta}^h\|_1. \tag{4.5}$$

Let $\mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0}$ be a lifting of $\widehat{\mathbf{g}}^h - \gamma_g(\mathcal{Q}^h\widehat{\mathbf{u}}) \in \mathbf{P}^h_{\Gamma_g}$ (see Lemma 3.6). Take $\boldsymbol{\eta}^h = \mathbf{v}^h + \mathcal{Q}^h\widehat{\mathbf{u}}$. Applying the continuity of the trace and (4.5), we deduce that

$$\begin{aligned}
\|\widehat{\mathbf{u}} - \boldsymbol{\eta}^h\|_1 &\le \|\widehat{\mathbf{u}} - \mathcal{Q}^h\widehat{\mathbf{u}}\|_1 + \|\mathbf{v}^h\|_1 \\
&\le \|\widehat{\mathbf{u}} - \mathcal{Q}^h\widehat{\mathbf{u}}\|_1 + C\|\widehat{\mathbf{g}}^h - \gamma_g(\mathcal{Q}^h\widehat{\mathbf{u}})\|_{1/2,\Gamma_g} \\
&\le \|\widehat{\mathbf{u}} - \mathcal{Q}^h\widehat{\mathbf{u}}\|_1 + C(\,\|\widehat{\mathbf{g}} - \widehat{\mathbf{g}}^h\|_{1/2,\Gamma_g} + \|\gamma_g(\mathcal{I} - \mathcal{Q}^h)\widehat{\mathbf{u}}\|_{1/2,\Gamma_g}) \\
&\le C(\,\|\widehat{\mathbf{u}} - \mathcal{Q}^h\widehat{\mathbf{u}}\|_1 + \|\widehat{\mathbf{g}} - \widehat{\mathbf{g}}^h\|_{1/2,\Gamma_g}) \\
&\le C(\,|\!|\!|\widehat{\mathbf{u}} - \mathcal{Q}^h\widehat{\mathbf{u}}|\!|\!| + \|\widehat{\mathbf{g}} - \widehat{\mathbf{g}}^h\|_{1/2,\Gamma_g}) \\
&\le C[\inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0}} \|\widehat{\mathbf{u}} - \boldsymbol{\eta}^h\|_1 + \|\widehat{\mathbf{g}} - \widehat{\mathbf{g}}^h\|_{1/2,\Gamma_g}]\,.
\end{aligned}$$

Hence, (4.4) is obtained from (4.2). We further note from (3.16) and (4.5) that

$$\|\widehat{\mathbf{g}} - \widehat{\mathbf{g}}^h\|_{1/2,\Gamma_g} \le C\|(\mathcal{I} - \mathcal{Q}^h)\widehat{\mathbf{u}}\|_1 \le C \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0}} \|\widehat{\mathbf{u}} - \boldsymbol{\eta}^h\|_1\,. \tag{4.6}$$

Therefore, combined with (4.4), the estimate (4.2) is a composite result of (4.1) and (4.6).

Next, we turn to showing (4.3). From the regularity result for the Stokes operator (cf. [7]), we have; if $(\widehat{\mathbf{u}}, \widehat{\mathbf{p}}) \in \mathbf{H}^m(\Omega) \times (H^{m-1}(\Omega) \cap L^2_0(\Omega))$ for $1 \le m \le k$, $\widehat{\mathbf{g}} \in \mathbf{H}^{m-1/2}_0(\Gamma_g)$. So, it follows from (4.7) that $\|\widehat{\mathbf{g}} - \widehat{\mathbf{g}}^h\|_{1/2,\Gamma_g} \le Ch^{m-1}\|\widehat{\mathbf{u}}\|_m$. Then, applying the approximation properties (3.3)–(3.5) to (4.4), (4.3) is obtained. □

As a supplementary result to Lemma 4.2, one can obtain the main estimates for $\|(\mathbf{u}^h, p^h) - (\mathbf{u}, p)\|_{\mathbf{X}}$.

THEOREM 4.3. *Assume that* Λ *be a compact subset of* $\mathbb{R}^+$. *Let* $\mathbf{X} = \mathbf{H}^1_{\Gamma_0}(\Omega) \times L^2_0(\Omega)$.

Suppose $\{(\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda)) \mid \lambda = \dfrac{1}{\nu} \in \Lambda\}$ *be a branch of regular solutions of* (3.7). *Then, for* h *small enough, there exists a unique regular branch* $\{(\lambda, (\mathbf{u}^h(\lambda), \lambda p^h(\lambda))) \mid \lambda \in \Lambda\}$ *of solutions of* (3.9) *in the neighborhood of* $(\mathbf{u}(\lambda), p(\lambda))$ *in* $\mathbf{X}$ *and a positive constant* C, *independent of* h *and* $\lambda \in \Lambda$, *such that*

$$\|(\mathbf{u}^h(\lambda), p^h(\lambda)) - (\mathbf{u}(\lambda), p(\lambda))\|_{\mathbf{X}} \longrightarrow 0 \tag{4.7}$$

as $h \to 0^+$, uniformly in $\lambda = \dfrac{1}{\nu} \in \Lambda$.

In addition, if we assume that $\{(\mathbf{u}(\lambda), p(\lambda)) \mid \lambda \in \Lambda\}$ belongs to $\mathbf{X} \cap (\mathbf{H}^m(\Omega) \times H^{m-1}(\Omega))$ for $1 \le m \le k$, there exists positive constants C which is independent of $\lambda \in \Lambda$ and h such that

$$\begin{aligned} \|(\mathbf{u}^h(\lambda), p^h(\lambda)) - (\mathbf{u}(\lambda), p(\lambda))\|_{\mathbf{X}} \\ \le Ch^{m-1}[\,\|\mathbf{u}(\lambda)\|_m + \|p(\lambda)\|_{m-1}], \end{aligned} \tag{4.8}$$

for all $\lambda \in \Lambda$.

4.3. Error estimates for the boundary stress

We are now concerned with the error estimates for the stress $\mathbf{t}$ in (3.8) and (3.10). We recall that even though $\mathbf{g} \in \mathbf{H}_0^s(\Gamma_g)$ for $s \ge \dfrac{3}{2}$, if Ω is Lipschitz continuous, the solution $(\mathbf{u}, p)$ of the system (3.9) belongs to $\mathbf{X} \cap (\mathbf{H}^{3/2-\delta}(\Omega) \times H^{1/2-\delta}(\Omega))$ for some small $\delta > 0$ from the loss of the convexity and regularity of the domain (cf. [10]). We provide general perspective for the operator $\mathcal{R}_{\Gamma_g}^h$.

LEMMA 4.4. *Let $\mathbf{t} \in \mathbf{H}^{-1/2}(\Gamma_g)$ be a solution of the scheme (3.9) – –(3.10). The following properties hold for $\mathcal{R}_{\Gamma_g}^h$:*

(i) $\|\mathbf{t} - \mathcal{R}_{\Gamma_g}^h \mathbf{t}\|_{-1/2,\Gamma_g} \to 0$ *as $h \to 0^+$.*

(ii) *If Ω is Lipschitz continuous and $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma_g)$,*

$$\|\mathbf{t} - \mathcal{R}_{\Gamma_g}^h \mathbf{t}\|_{-1/2,\Gamma_g} \le Ch^{1/2-\delta}[\,\|\mathbf{u}\|_{3/2-\delta} + \|p\|_{1/2-\delta}\,] \tag{4.9}$$

for some small $\delta > 0$.

Furthermore, provided that the solution $(\mathbf{u}, p)$ of the system (3.9) belongs to $\mathbf{X} \cap (\mathbf{H}^m(\Omega) \times H^{m-1}(\Omega))$ for $1 \le m \le k$, we have more general estimates:

(iii) *If $1 \le m < 2$, we have*

$$\|\mathbf{t} - \mathcal{R}_{\Gamma_g}^h \mathbf{t}\|_{-1/2,\Gamma_g} \le Ch^{m-1}[\,\|\mathbf{u}\|_m + \|p\|_{m-1}\,]. \tag{4.10}$$

(iv) *If $m = 2$ and Ω is a convex polyhedral domain, it holds that*

$$\|\mathbf{t} - \mathcal{R}_{\Gamma_g}^h \mathbf{t}\|_{-1/2,\Gamma_g} \le Ch^{1-\delta}[\,\|\mathbf{u}\|_2 + \|p\|_1\,] \tag{4.11}$$

for all $0 < \delta < \dfrac{1}{2}$.

(v) *If* $2 \le m \le k$ *and* $\mathbf{t} \in \gamma_g(\mathbf{H}^1_{\Gamma_0}(\Omega) \cap \mathbf{H}^{m-1}(\Omega))$, *we obtain*

$$\|\mathbf{t} - \mathcal{R}^h_{\Gamma_g}\mathbf{t}\|_{-1/2,\Gamma_g} \le Ch^{m-1} \inf_{\mathbf{v_t}\in\mathcal{V}} \|\mathbf{v_t}\|_{m-1}, \tag{4.12}$$

where $\mathcal{V} = \{\mathbf{v_t} \in (\mathbf{H}^1_{\Gamma_0}(\Omega) \cap \mathbf{H}^m) \mid \gamma_g \mathbf{v_t} = \mathbf{t}\}$.

Proof. (i) follows from Lemma 3.6.

(ii); Since $(\mathbf{u}, p) \in \mathbf{X} \cap (\mathbf{H}^{3/2-\delta} \times H^{1/2-\delta})$, $\mathbf{t} \in \mathbf{H}^{-1/2}(\Gamma_g) \cap \mathbf{H}^{-\delta}(\Gamma_g)$. Hence, from (3.28), we obtain

$$\begin{aligned}\|\mathbf{t} - \mathcal{R}^h_{\Gamma_g}\mathbf{t}\|_{-1/2,\Gamma_g} &\le Ch^{1/2-\delta}[\,\|\mathbf{t}\|_{-\delta,\Gamma_g}\,] \\ &\le Ch^{1/2-\delta}[\,\|\mathbf{u}\|_{3/2-\delta} + \|p\|_{1/2-\delta}\,].\end{aligned}$$

Similar argument also results in (iii).

(iv); Suppose $m = 2$ and the domain is convex polyhedral. Since $\mathbf{t} = -p\mathbf{n} + \nu\mathbf{n}\cdot(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, $\mathbf{t}$ is not continuous along the boundary and $\mathbf{t} \in \mathbf{H}^{1/2-\delta}(\Gamma_g)$ for $0 < \delta < \dfrac{1}{2}$. So, from (3.28) it follows that

$$\begin{aligned}\|\mathbf{t}\|_{-1/2,\Gamma_g} &\le Ch^{1-\delta}\|\mathbf{t}\|_{1-\delta,\Gamma_g} \\ &\le Ch^{1-\delta}[\,\|\mathbf{u}\|_2 + \|p\|_1\,].\end{aligned}$$

(v); Since $\mathbf{t} \in \mathbf{H}^s(\Gamma_g)$ for $s \ge \dfrac{1}{2}$, $\mathcal{R}^h_{\Gamma_g} = \mathcal{P}^h_{\Gamma_g}$ so that

$$\|\mathbf{t} - \mathcal{R}^h_{\Gamma_g}\mathbf{t}\|_{-1/2,\Gamma_g} \le Ch^{m-1}\|\mathbf{v_t}\|_{m-1},$$

where $\mathbf{v_t}$ is a lifting of $\mathbf{t}$. Since this holds for all $\mathbf{v_t}$, (4.12) follows. □

The main estimates for $\|\mathbf{t} - \mathbf{t}^h\|_{-1/2,\Gamma_g}$ are found by Theorem 4.3 and Lemma 4.4.

THEOREM 4.5. *Assume that* Λ *be a compact subset of* $\mathbb{R}^+$. *Suppose* $\{(\lambda, \boldsymbol{\phi} = (\mathbf{u}(\lambda), \lambda p(\lambda))) \mid \lambda = \dfrac{1}{\nu} \in \Lambda\}$ *be a regular branch of solutions of* (3.9) *and let* $(\mathbf{u}^h(\lambda), p^h(\lambda)) \in \mathbf{X}$ *be the solution of* (3.7) *in* *the neighborhood of* $\boldsymbol{\phi}$ *in* $\mathbf{X}$. *Let* $\mathbf{t}(\lambda) \in \mathbf{H}^{-1/2}(\Gamma_g)$ *and* $\mathbf{t}^h(\lambda) \in \mathbf{P}^h_{\Gamma_g} \subset \mathbf{H}^{-1/2}(\Gamma_g)$ *be corresponding solutions of the system* (3.10) *and* (3.8), *respectively. Then, we have the following error estimates on the same branch :*

(i) $\|\mathbf{t}^h(\lambda) - \mathbf{t}(\lambda)\|_{-1/2,\Gamma_g} \longrightarrow 0$, *uniformly as* $h \to 0^+$.

(ii) *If* Ω *is Lipschitz continuous and* $\mathbf{g} \in \mathbf{H}_0^{3/2}(\Gamma_g)$, *we have*

$$\|\mathbf{t}^h(\lambda) - \mathbf{t}(\lambda)\|_{-1/2,\Gamma_g} \le Ch^{1/2-\delta}(1 + \|\mathbf{u}\|_1)\big(\|\mathbf{u}\|_{3/2-\delta} + \|p\|_{1/2-\delta}\big)$$

for some small $\delta > 0$.

Moreover, let $(\mathbf{u}(\lambda), p(\lambda)) \in \mathbf{X} \cap (\mathbf{H}^m(\Omega) \times H^{m-1}(\Omega))$ *for* $1 \le m \le k$ *and let* $(\mathbf{u}^h(\lambda), p^h(\lambda))$ *be a corresponding approximate solution on the same branch. Then, the error estimates can be sharpened as followings:*

(iii) *If* $1 \le m < 2$, *we have*

$$\|\mathbf{t}(\lambda) - \mathcal{R}^h_{\Gamma_g}\mathbf{t}(\lambda)\|_{-1/2,\Gamma_g} \le Ch^{m-1}(1 + \|\mathbf{u}\|_1)\big(\|\mathbf{u}\|_m + \|p\|_{m-1}\big)\,.$$

(iv) *If* $m = 2$ *and* Ω *is a convex polyhedral domain, it holds that*

$$\|\mathbf{t}(\lambda) - \mathcal{R}^h_{\Gamma_g}\mathbf{t}(\lambda)\|_{-1/2,\Gamma_g} \le Ch^{1-\delta}(1 + \|\mathbf{u}\|_1)\big(\|\mathbf{u}\|_2 + \|p\|_1\big)$$

for all $0 < \delta < \dfrac{1}{2}$.

(v) *If* $2 \le m \le k$ *and* $\mathbf{t}(\lambda) \in \gamma_g(\mathbf{H}^1_{\Gamma_0}(\Omega) \cap \mathbf{H}^{m-1}(\Omega))$, *we obtain*

$$\|\mathbf{t}(\lambda) - \mathcal{R}^h_{\Gamma_g}\mathbf{t}(\lambda)\|_{-1/2,\Gamma_g} \le Ch^{m-1}(1 + \|\mathbf{u}\|_1) \inf_{\mathbf{v_t} \in \mathcal{V}} \|\mathbf{v_t}\|_{m-1}\,,$$

where $\mathcal{V} = \{\mathbf{v_t} \in \mathbf{H}^1_{\Gamma_0}(\Omega) \cap \mathbf{H}^m) \mid \gamma_g \mathbf{v_t} = \mathbf{t}\}$. *Here,* C *is taken independent of* h *and* $\lambda \in \Lambda$.

Proof. From the triangle inequality, we have

$$\text{(4.13)} \quad \|\mathbf{t} - \mathbf{t}^h\|_{-1/2,\Gamma_g} \le \|\mathbf{t} - \mathcal{R}^h_{\Gamma_g}\mathbf{t}\|_{-1/2,\Gamma_g} + \|\mathcal{R}^h_{\Gamma_g}\mathbf{t} - \mathbf{t}^h\|_{-1/2,\Gamma_g}\,.$$

Since the estimates for $\|\mathbf{t} - \mathcal{R}^h_{\Gamma_g}\mathbf{t}\|_{-1/2,\Gamma_g}$ are given in Lemma 4.4, it is sufficient to find estimates for $\|\mathcal{R}^h_{\Gamma_g}\mathbf{t}\|_{-1/2,\Gamma_g}$. By the definition of $\mathcal{R}^h_{\Gamma_g}\mathbf{t}$, we have

$$\text{(4.14)} \quad \begin{aligned} &< \mathcal{R}^h_{\Gamma_g}\mathbf{t}, \mathbf{v}^h >_{\Gamma_g} = < \mathbf{t}, \mathbf{v}^h >_{\Gamma_g} \\ &\qquad = \nu a(\mathbf{u}, \mathbf{v}^h) + b(\mathbf{v}^h, p) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - < \mathbf{f}, \mathbf{v}^h >_{-1} \end{aligned}$$

for all $\mathbf{v}^h \in \mathbf{V}^h_{\Gamma_g}$. Since $\mathbf{V}^h_{\Gamma_0} \subset \mathbf{H}^1_{\Gamma_0}(\Omega)$ and $(\mathbf{u}, p) \in \mathbf{X}$ is the solution of the system (3.9), (4.10) is justified in the same sense with (3.10). By subtracting (3.8) from (4.14), we obtain

$$
\begin{aligned}
&< \mathcal{R}^h_{\Gamma_g}\mathbf{t} - \mathbf{t}^h, \mathbf{v}^h >_{\Gamma_g} \\
(4.15) \qquad &= \nu a(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p - p^h) + c(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) \\
&\qquad + c(\mathbf{u} - \mathbf{u}^h, \mathbf{u}, \mathbf{v}^h) + c(\mathbf{u}, \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h)\,.
\end{aligned}
$$

Let $\boldsymbol{\xi}^h = \mathcal{R}^h_{\Gamma_g}\mathbf{t} - \mathbf{t}^h \in \mathbf{P}^h_{\Gamma_g}$ and $\mathbf{v}^h_{\boldsymbol{\xi}^h}$ be a lifting of $\boldsymbol{\xi}^h$ such that

$$
\|\mathbf{v}^h_{\boldsymbol{\xi}^h}\|_1 \le C\|\boldsymbol{\xi}^h\|_{1/2,\Gamma_g} \le Ch^{-1/2}\|\mathcal{R}^h_{\Gamma_g}\mathbf{t} - \mathbf{t}^h\|_{0,\Gamma_g}\,,
$$

which is followed by the inverse inequality. Hence, by setting $\mathbf{v}^h = \mathbf{v}^h_{\boldsymbol{\xi}^h}$ in (4.15), we have that

$$
\begin{aligned}
&\|\mathcal{R}^h_{\Gamma_g}\mathbf{t} - \mathbf{t}^h\|^2_{0,\Gamma_g} \\
&\le (\nu\,\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\mathbf{u} - \mathbf{u}^h\|_1^2 + 2\|\mathbf{u}\|_1\|\mathbf{u} - \mathbf{u}^h\|_1)\|\mathbf{v}^h_{\boldsymbol{\xi}^h}\|_1 \\
&\le Ch^{-1/2}\mathbf{E}(\mathbf{u}, p, \mathbf{u}^h, p^h)\|\mathcal{R}^h_{\Gamma_g}\mathbf{t} - \mathbf{t}^h\|_{0,\Gamma_g}\,,
\end{aligned}
$$

where

$$
\mathbf{E}(\mathbf{u}, p, \mathbf{u}^h, p^h) = \nu\,\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\mathbf{u} - \mathbf{u}^h\|_1^2 + 2\|\mathbf{u}\|_1\|\mathbf{u} - \mathbf{u}^h\|_1\,.
$$

Thus, we obtain that

$$
(4.16) \qquad \|\mathcal{R}^h_{\Gamma_g}\mathbf{t} - \mathbf{t}^h\|_{0,\Gamma_g} \le Ch^{-1/2}\mathbf{E}(\mathbf{u}, p, \mathbf{u}^h, p^h)\,.
$$

Let $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_g)$ and $\mathcal{P}^h_{\Gamma_g}(\boldsymbol{\phi}) = \boldsymbol{\phi}^h$. Let $\mathbf{v}^h_{\boldsymbol{\phi}^h}$ be a lifting of $\boldsymbol{\phi}^h$ in $\mathbf{V}^h_{\Gamma_0}$ such that

$$
\|\mathbf{v}^h_{\boldsymbol{\phi}^h}\|_1 \le C\|\mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi}\|_{1/2,\Gamma_g} \le C\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}\,.
$$

We also note that $\|(\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi}\|_{0,\Gamma_g} \le Ch^{1/2}\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}$. Then, using (4.16), we have that

$$\begin{aligned}
&< \mathcal{R}^h_{\Gamma_g}\mathbf{t}-\mathbf{t}^h, \boldsymbol{\phi} >_{\Gamma_g} \\
&=< \mathcal{R}^h_{\Gamma_g}\mathbf{t}-\mathbf{t}^h, (\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi} + \mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi} >_{\Gamma_g} \\
&=< \mathcal{R}^h_{\Gamma_g}\mathbf{t}-\mathbf{t}^h, (\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi} >_{\Gamma_g} +\nu a(\mathbf{u}-\mathbf{u}^h, \mathbf{v}^h_{\boldsymbol{\phi}^h}) + b(\mathbf{v}^h_{\boldsymbol{\phi}^h}, p-p^h) \\
&\qquad + c(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{u}^h, \mathbf{v}^h_{\boldsymbol{\phi}^h}) + c(\mathbf{u}-\mathbf{u}^h, \mathbf{u}, \mathbf{v}^h_{\boldsymbol{\phi}^h}) + c(\mathbf{u}, \mathbf{u}-\mathbf{u}^h, \mathbf{v}^h_{\boldsymbol{\phi}^h}) \\
&\le \|\mathcal{R}^h_{\Gamma_g}\mathbf{t}-\mathbf{t}^h\|_0 \|(\mathcal{I}-\mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi}\|_0 + C\mathbf{E}(\mathbf{u},p,\mathbf{u}^h,p^h)\|\mathbf{v}^h_{\boldsymbol{\phi}^h}\|_1 \\
&\le Ch^{-1/2}\mathbf{E}(\mathbf{u},p,\mathbf{u}^h,p^h)Ch^{1/2}\|\boldsymbol{\phi}\|_{1/2,\Gamma_g} + C\mathbf{E}(\mathbf{u},p,\mathbf{u}^h,p^h)\|\boldsymbol{\phi}\|_{1/2,\Gamma_g} \\
&\le C\mathbf{E}(\mathbf{u},p,\mathbf{u}^h,p^h)\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathcal{R}^h_{\Gamma_g}\mathbf{t}-\mathbf{t}^h\|_{-1/2,\Gamma_g} &= \sup_{\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma_g),\|\boldsymbol{\phi}\|_{1/2,\Gamma_g}\le 1} < \mathcal{R}^h_{\Gamma_g}\mathbf{t}-\mathbf{t}^h, \boldsymbol{\phi} >_{\Gamma_g} \\
&\le C\mathbf{E}(\mathbf{u},p,\mathbf{u}^h,p^h)\,.
\end{aligned}$$

Hence, combined (4.13) with Lemma 4.4, (i) is followed. The other estimates are also obtained in a similar manner. □

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