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ASYMPTOTIC SEQUENCES AND GENERALIZED FRACTIONS

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1. Introduction

Throughout this note, R is a commutative Noetherian ring (with non-zero identity). For an ideal $\mathfrak{a}$ of R, $\overline{\mathfrak{a}}$ is the integral closure of $\mathfrak{a}$, so

$$\overline{\mathfrak{a}} = \{x \in R : x^n + a_1 x^{n-1} + \cdots + a_n = 0, \text{ for } a_i \in \mathfrak{a}^i\}.$$

In [**5**], Ratliff introduced an asymptotic sequence and a criterion of a locally quasi-unmixed ring. Recall that a *locally quasi-unmixed* R is for each maximal ideal $\mathfrak{m}$ the completion $R^*_{\mathfrak{m}}$ is equidimensional and if R is local then we call it *quasi-unmixed*. The elements $a_1, \ldots, a_n$ of R is said to be a *poor asymptotic sequence* if, for $i = 1, \ldots, n$, $a_i \notin \bigcup_{\mathfrak{p} \in Q} \mathfrak{p}$ where $Q = \text{Ass}\left(R/\overline{(a_1, \ldots, a_{i-1})^t R}\right)$ for all large t; it is said to be an *asymptotic sequence* if, in addition, $(a_1, \ldots, a_n)R \neq R$.

If $\mathfrak{a}$ is an ideal of R, then the *asymptotic grade* of $\mathfrak{a}$, denoted *a-grade*($\mathfrak{a}$), is the common length of all maximal asymptotic sequences in $\mathfrak{a}$ and we interpret *a-grade*$(R) = \infty$.

In [**10**], Yassi showed that the set whose members are poor asymptotic sequences is a triangular subset (see [**7**]). This triangular subset provides a module of generalized fractions by [**7**].

In [**1**], we gave associated prime ideals of modules of generalized fractions defined by some special triangular subsets.

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The purpose of this note is to study associated prime ideals of the module of generalized fractions defined by poor asymptotic sequences and to give an extended criterion of a locally quasi-unmixed ring using this module.

2. Main results

LEMMA 1. [**10**, 2.2.17]. *Let* $(U_a)_n = \{(a_1, \dots, a_n) \in R^n : a_1, \dots, a_n$ *is a poor asymptotic sequence in* R *such that if* $a_i = 1$ *for some* $i = 1, \dots, n-1$ *then* $a_j = 1$ *for all* $j(\geq i)\}$. *Then* $(U_a)_n$ *is a triangular subset of* R^n.

Proof. We show that $(U_a)_n$ is satisfied the three conditions of triangular subset [**7**].

Clearly $(U_a)_n \neq \emptyset$, since $(1, \dots, 1) \in (U_a)_n$.

Next, let $(a_1, \dots, a_n) \in (U_a)_n$. Then we may assume that *a-grade* $(a_1, \dots, a_n)R = n$. Hence by 3.16 of [**5**], $(a_1^{\alpha_1}, \dots, a_n^{\alpha_n}) \in (U_a)_n$ for all choice $\alpha_i \in N$.

Finally, let $(a_1, \dots, a_n)$, $(b_1, \dots b_n) \in (U_a)_n$. Also we may assume that *a-grade*$(a_1, \dots, a_n)R = n$ and *a-grade*$(b_1, \dots, b_n)R = n$. Hence by 3.10 of [**5**], there is $(c_1, \dots, c_n) \in (U_a)_n$ such that for each $i = 1, \dots, n$,

$$(c_1, \dots, c_i)R \subset (a_1, \dots, a_i)R \cap (b_1, \dots, b_i)R. \qquad \square$$

THEOREM 2. *Fix a non-negative integer* n *such that* $n \leq \sup_{\mathfrak{m} \in \mathrm{Max}(R)}$ *a-grade*$(\mathfrak{m})$. *Put*

$$\begin{aligned}(U_a)_{n+1} = \{(a_1, \dots, a_{n+1}) \in R^{n+1} : \;& a_1, \dots, a_{n+1} \\ & \text{is a poor asymptotic sequence in } R \\ & \text{such that for some } i (1 \leq i < n) \\ & \text{if } a_i = 1 \text{ then } a_j = 1 \text{ for all } j \geq i\}\end{aligned}$$

and

$$\begin{aligned}(U_a)_n = \{(a_1, \dots, a_n) \in R^n : \;& \text{there is } a_{n+1} \in R \\ & \text{such that } (a_1, \dots, a_{n+1}) \in (U_a)_{n+1}\}.\end{aligned}$$

Consider the following conditions.

$$A_n = \{\mathfrak{p} \in \operatorname{Spec}(R) : a\text{-}grade(\mathfrak{p}) = ht\ \mathfrak{p} = n\}.$$
$$A'_n = \{\mathfrak{p} \in \operatorname{Spec}(R) : R_\mathfrak{p} \text{ is quasi-unmixed such that } a\text{-}grade(\mathfrak{p}) = a\text{-}grade(\mathfrak{p}R_\mathfrak{p}) = n\}.$$
$$B_n = \{\mathfrak{p} \in \operatorname{Spec}(R) : a\text{-}grade(\mathfrak{p}) = a\text{-}grade(\mathfrak{p}R_\mathfrak{p}) = n\}.$$
$$B'_n = \{\mathfrak{p} \in \operatorname{Spec}(R) : \text{ for some}(a_1, \dots, a_n) \in (U_a)_n\ \ \mathfrak{p} \in \operatorname{Ass}\left(R/\overline{(a_1, \dots, a_n)^t R}\right) \text{ for all large } t\}.$$

Then we have $A_n = A'_n$, $B_n = B'_n$ *and*

$$A_n \subset \operatorname{Ass}(U_a)_{n+1}^{-n-1} R \subset B_n.$$

Proof. The first and the second assertions follow immediately from 4.5 and 3.8 of [**5**].

For the third assertion, we show that the first inclusion holds. Let $\mathfrak{p} \in A_n$ hence $ht\ \mathfrak{p} = n$. Let $\Phi : R \longrightarrow R_\mathfrak{p}$ be the natural map. Then

$$\Phi(U_a)_{n+1} = \{(\Phi(a_1), \dots, \Phi(a_{n+1})) \in R_\mathfrak{p}^{n+1} : (a_1, \dots, a_{n+1}) \in (U_a)_{n+1}\}$$

is a triangular subset of $R_\mathfrak{p}^{n+1}$ and $((U_a)_{n+1}^{-n-1}R)_\mathfrak{p} = \Phi(U_a)_{n+1}^{-n-1}R_\mathfrak{p}$ by 2.1 of [**3**]. Since $\operatorname{Supp}((U_a)_{n+1}^{-n-1}R) \subset \{\mathfrak{q} \in \operatorname{Spec}(R) : ht\ \mathfrak{q} \geq n\}$ by 3.1 of [**3**], it is enough to show that $\Phi(U_a)_{n+1}^{-n-1}R_\mathfrak{p} \neq 0$. Since $a\text{-}grade(\mathfrak{p}) = a\text{-}grade(\mathfrak{p}R_\mathfrak{p}) = \dim\ R_\mathfrak{p} = n$ and $R_\mathfrak{p}$ is quasi-unmixed, we may assume that

$$\begin{aligned}\Phi(U_a)_{n+1} &= \{(\Phi(a_1), \dots, \Phi(a_{n+1})) \in R_\mathfrak{p}^{n+1} : \text{ there exists } j \text{ with } 0 \leq j \leq n \\ &\quad \text{such that } \Phi(a_1), \dots, \Phi(a_j) \text{ form, a subsystem of parameters for } R_\mathfrak{p} \\ &\quad \text{and } \Phi(a_{j+1}) = \cdots = \Phi(a_{n+1}) = 1\};\end{aligned}$$

for, $ht(a_1, \dots, a_{n+1})R \geq n+1$ and if $\Phi(a_i) = 1$ for some $i < n+1$, then by 3.3 of [**7**] we have $\dfrac{r}{(\Phi(a_1), \dots, \Phi(a_{n+1}))} = 0$.

Therefore by 3.5 of [8] we have

$$\Phi(U_a)_{n+1}^{-n-1}R_{\mathfrak{p}} \cong \mathbf{H}^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0.$$

Next for the second inclusion, let $\mathfrak{p} \in \mathrm{Ass}((U_a)_{n+1}^{-n-1}R)$. Then by 5.1 of [9] there is $\dfrac{r}{(a_1, \ldots, a_n, 1)} \in (U_a)_{n+1}^{-n-1}R$ such that

$$\left(0 : \frac{r}{(a_1, \ldots, a_n, 1)}\right) = \mathfrak{p}.$$

Since $(a_1, \ldots, a_n)R \subset \mathfrak{p}$ by 3.3 of [7], we have $a\text{-}grade(\mathfrak{p}) \geq n$. Suppose that $a\text{-}grade(\mathfrak{p}) > n$. Let $Q = \mathrm{Ass}\left(R/\overline{(a_1, \ldots, a_n)^t R}\right)$ for all large t. Then for all $\mathfrak{q} \in Q$, $a\text{-}grade(\mathfrak{q}) = n$ by 3.7 of [5]. Hence there is $a_{n+1} \in \mathfrak{p} \setminus \bigcup_{\mathfrak{q}\in Q} \mathfrak{q}$ such that

$$(a_1, \ldots, a_{n+1}) \in (U_a)_{n+1}.$$

Therefore we have

$$\left(0 : \frac{r}{(a_1, \ldots, a_n, 1)}\right) = \left(0 : \frac{r}{(a_1, \ldots, a_{n+1})}\right) = \mathfrak{p}$$

by 5.1 of [9] again. Hence we have the following contradiction.

$$\frac{a_{n+1}r}{(a_1, \ldots, a_{n+1})} = \frac{r}{(a_1, \ldots, a_n, 1)} = 0.$$

On the other hand, by Corollary(p. 38) of [4]

$$\mathfrak{p} \in \mathrm{Ass}((U_a)_{n+1}^{-n-1}R) \Longleftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}(\Phi(U_a)_{n+1}^{-n-1}R_{\mathfrak{p}}).$$

Note that, for all $(a_1, \ldots, a_{n+1}) \in (U_a)_{n+1}$, $a\text{-}grade\,((\Phi(a_1), \ldots, \Phi(a_i))R) \geq i$ for $i = 1, \ldots, n$ by 2.9 of [5]. Hence replacing R with $R_{\mathfrak{p}}$, we have $a\text{-}grade(\mathfrak{p}R_{\mathfrak{p}}) = n$, using the same argument as above. □

COROLLARY 3. *The following statements are equivalent.*

(1) *R is locally quasi-unmixed.*
(2) *a-$grade(I) = ht\ I$ for all ideals I in R.*
(3) *a-$grade(\mathfrak{m}) = ht\ \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ in R.*
(4) *If I is an ideal of the principal class in R, then a-$grade(I) = ht\ I$.*
(5) $\mathrm{Ass}((U_a)_{n+1}^{-n-1}R) = \{\mathfrak{p} \in \mathrm{Spec}(R) : ht\ \mathfrak{p} = n\}$ *for all* $n = 0, 1, 2, \ldots$.
(6) *The following complex defined in* [**6**, p. 52]

$$0 \longrightarrow R \longrightarrow (U_a)_1^{-1}R \longrightarrow \cdots \longrightarrow (U_a)_i^{-i}R \longrightarrow \cdots$$

is of Cousin type for R with respect to the height filtration $\mathcal{F} = (F_i)_{i\geq 0}$ where $F_i = \{\mathfrak{p} \in \mathrm{Spec}(R) : ht\ \mathfrak{p} \geq i\}$ (see [**6**, 3.1]).

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) See 4.1 of [**5**].

(2) $\Rightarrow$ (5) By the hypothesis we have a-$grade(\mathfrak{p}) = ht\ \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Therefore Theorem 2 completes the proof.

(5) $\Rightarrow$ (3) Suppose that $ht\ \mathfrak{m} = n$ for some $\mathfrak{m} \in \mathrm{Max}(R)$. Then $\mathfrak{m} \in \mathrm{Ass}((U_a)_{n+1}^{-n-1}R)$ by the hypothesis. Hence by Theorem 2 we have a-$grade(\mathfrak{m}) = n$.

(1) $\Rightarrow$ (6) Using (5) and the following Corollary 4, the assertion follows from 3.2 of [**2**].

(6) $\Rightarrow$ (5) This follows immediately from 3.4 of [**2**]. □

COROLLARY 4. *If R is locally quasi-unmixed, then there is an isomorphic R-homomorphism*

$$\Theta : (U_a)_{n+1}^{-n-1}R \longrightarrow \bigoplus_{ht\mathfrak{p}=n} ((U_a)_{n+1}^{-n-1}R)_\mathfrak{p} \quad \text{for all } n = 0, 1, 2, \ldots$$

where , for all $x \in (U_a)_{n+1}^{-n-1}R$ and $\mathfrak{p}$ of height n, the component of $\Theta(x)$ in the summand $((U_a)_{n+1}^{-n-1}R)_\mathfrak{p}$ is $x/1$.

Proof. By Corollary 3(5) and 3.2 of [**1**] it is sufficient to show that, for some $(a_1, \ldots, a_n) \in (U_a)_n$ and an ideal $\mathfrak{a}$ of R containing $(a_1, \ldots, a_n)$ R and not contained in any $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $ht\ \mathfrak{p} = n$, there is $a_{n+1} \in \mathfrak{a}$ such that $(a_1, \ldots, a_{n+1}) \in (U_a)_{n+1}$. But this is clear by the definition of a poor asymptotic sequence. □

REMARK 5. If $\mathrm{Ass}((U_a)_{n+1}^{-n-1}R) = B_n$ for all $n = 0, 1, 2, \ldots$, then the converse of Corollary 4 holds.

Proof. We show that Corollary 3(3) is satisfied. Assume that *a-grade* $(\mathfrak{m}) = n$ for some $\mathfrak{m} \in \mathrm{Max}(R)$. Then $a\text{-}grade(\mathfrak{m}R_{\mathfrak{m}}) = n$ by 3.5 of [**5**]. Hence we have $\mathfrak{m} \in \mathrm{Ass}((U_a)_{n+1}^{-n-1}R)$ by the hypothesis.

On the other hand, by the assumption we have

$$\begin{aligned}&\mathrm{Ass}((U_a)_{n+1}^{-n-1}R)\\ &\qquad = \mathrm{Ass}(\bigoplus_{ht\mathfrak{p}=n}((U_a)_{n+1}^{-n-1}R)_{\mathfrak{p}}) \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : ht\ \mathfrak{p} = n\}\end{aligned}$$

by 3.1 of [**3**]. Hence we have $ht\ \mathfrak{m} = n$. □

EXAMPLE 6. Let $R = k[\![X, Y, Z]\!]/(X) \cap (Y, Z) = k[\![x, y, z]\!]$ and $\mathfrak{m} = (x, y, z)$. Then R is not quasi-unmixed and $a\text{-}grade(x, y) = a\text{-}grade(\mathfrak{m}) = a\text{-}grade(\mathfrak{m}R_{\mathfrak{m}}) = 1$. But $ht(x, y) = 1$ and $ht(\mathfrak{m}) = 2$. Hence we have $A_n \neq B_n$ in Theorem 2.

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