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A NOTE ON TIGHT CLOSURE AND FROBENIUS MAP

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In recent years M.Hochster and C.Huneke introduced the notions of tight closure of an ideal and of the weak F-regularity of a ring of positive prime characteristic. Here 'F' stands for Frobenius. This notion enabled us to play an important role in a commutative ring theory, and other related topics.

In this paper we study the connections between the Frobenius map and the tight closure.

A weakly F-regular ring is easily seen to be F-pure, but we do not know the converse is true or not in general. We study conditions for an F-pure ring to be weakly F-regular. And as a corollary we give a proof of R. Fedder's conjecture in one dimensional case as follows: "R/xR is F-pur" implies "R is F-pur" whenever R is Cohen-Macaulay ring of dimension one. Finally we study the conditions related to the tight closure that the Cohen-Macaulay ring to be weakly F-regular and Gorenstein.

1. Preliminaries

All rings are commutative, Noetherian with identity of prime characteristic p. And all modules are finitely generated, unless otherwise specified.

DEFINITION 1.1. [Hochster-Huneke] Let $I \subseteq R$ be an ideal and R^o denote the complement of the union of the minimal primes of R and

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let $I^{[q]}$ denote the ideal $(i^q : i \in I)$. We say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$, i.e., for all sufficiently large q of the form p^e. If $I = I^*$, we say that I is *tightly closed* .

DEFINITION 1.2. [Hochster-Huneke] A Noetherian ring is called *weakly F-regular* if every ideal is tightly closed. If every localization of R at a multiplicative subset is weakly F-regular, then we say that R is *F-regular.*

DEFINITION 1.3. [Fedder-Watanabe] A Noetherian local ring of characteristic p is called *F-rational* if every ideal generated by a system of parameter is tightly closed.

Now we introduce the notion of F-purity, relying on the special properties of the Frobenius homomorphism. And we discuss the relationship between the F-purity and the weak F-regularity. Let R be a ring of characteristic p. Denote by ${}^eR$, the ring R viewed as an R-module via the e-th power of the Frobenius map $F(r) = r^q$, where $q = p^e$. Furthermore, for any R-module M, ${}^eM = M \otimes_R {}^eR$ will denote the group M viewed as an R-module via $r \cdot m = r^q m$. $R \xrightarrow{F^e} {}^eR$ is therefore an R-module homomorphism [9].

DEFINITION 1.4. [Hochster-Roberts] A Noetherian ring R of characteristic p is called *F-pure* if for every R-module M,

$$0 \to M \otimes_R R \to M \otimes_R {}^1R$$

is exact. Equivalently, for some $e > 0$, $0 \to M \to M \otimes_R {}^eR$ is exact.

DEFINITION 1.5. [Fedder] We say that a local ring R is *F-contracted* if $R \to {}^1R$ is contracted, which means that every ideal I which is generated by a system of parameter for R satisfies

$$(I \cdot {}^1R) \cap R = I.$$

LEMMA 1.6. *For an F-pure or an F-contracted ring R, the Frobenius map must be injective. Whence R is reduced.*

Proof. If R is F-pure, then the Frobenius map by tensoring with R is also injective from the definition.
If R is F-contracted, and if $F(r) = 0$, then certainly $F(r) \in I \cdot {}^1R$ for every ideal I which is generated by a system of parameter for R. The contractedness hypothesis then guarantees that r lies in the intersection of all ideals of R which are generated by a system of parameter But this intersection is well known to be 0. Thus, $r = 0$ and the Frobenius map is injective. □

When R is reduced, there is a natural identification of maps:

(1) $R \xrightarrow{\text{F}} {}^1R$.
(2) $R \longrightarrow R^{1/p}$ where $R^{1/p}$ denotes the ring of the p-th roots of elements in R.
(3) $R^p \rightarrow R$,where R^p denotes the ring of the p-th powers of elements in R.

Thus, if $I = (a_1, \cdots, a_t)$ is an ideal in R , then ${}^1I$ can be thought of as the ideal $(a_1^{1/p}, \cdots, a_t^{1/p}) \subset R^{1/p}$ under the second identification of maps.

DEFINITION 1.7. [Hochster] The map $R \xrightarrow{\phi} S$ is called *cyclically pure* if for every ideal $I \subset R$, $\{x \in R \mid \phi(x) \in IS \} = I$.

Note that the fact that ϕ must be injective follows from the case when $I = 0$. Let $S = {}^1R$ and ϕ be the Frobenius map. Then, since $I \cdot {}^1R = {}^1(I^{[p]}R)$, it follows that $R \rightarrow {}^1R$ is cyclically F-pure if and only if $f^p \in I^{[p]}$ implies $f \in I$. Clearly if $R \rightarrow {}^1R$ is F-pure, then this map is cyclically F-pure. But the converse is true only when R is approximately Gorenstein [6].

2. Weak F-regularity and F-purity

PROPOSITION 2.1. *A weakly F-regular ring R is F-pure.*

Proof. In fact, the weak F-regularity always implies that the map $R \rightarrow {}^1R$ is cyclically pure because $f^p \in I^{[p]}$ implies $1 \cdot f^q \in I^{[q]}$ for every $q = p^e$. Whence, $f \in I^* = I$. But if R is approximately Gorenstein, then $R \rightarrow S$ is cyclically pure if and only if it is pure. Since weakly F-regular rings are normal, and so approximately Gorenstein. It follows that R is F-pure. □

But the converse of Proposition 2.1., that is, the F-purity implies the weak F-regularity, remains open. However, for the zero dimensional case, we have an affirmative answer.

THEOREM 2.2. *A zero dimensional F-pure ring R is weakly F-regular.*

Proof. We may assume that R is local with the maximal ideal $\underline{m}$. Let I be an ideal of R and let $x \in I^*$. Then there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q = p^e$ because an F-pure ring is reduced. But $\bigcup \{P : P \text{ is a minimal prime ideal of } R\} = \underline{m}$, and $R^o = R - \underline{m}$ is the units of R, we have $x^q \in I^{[q]}$. Thus $x \in I$ by F-purity. Hence $I = I^*$ and R is weakly F-regular. □

Now we can prove an one dimensional case of an important conjecture, which is raised by R. Fedder in his paper [2], by using Theorem 2.2.

Fedder's Conjecture : "R/fR is F-pure" should imply "R is F-pure", whenever R is Cohen-Macaulay ring and $f \notin Z(R)$.

COROLLARY 2.3. *Let R be a one dimensional ring, and let $f \notin Z(R)$. If R/fR is F-pure, then R is F-pure.*

Proof. Since R/fR is a zero dimensional F-pure ring, R/fR is weakly F-regular by Theorem 2.2. Since $\dim R = 1$, R is also weakly F-regular[1]. Thus R is F-pure. □

Now we prove that an F-pure ring is weakly F-regular for the higher dimensional case under additional conditions.

DEFINITION 2.4. Let R be a Noetherian reduced ring of characteristic p, and let M be an R-module. We say that M is *F-unstable* if for every nonzero $x \in M$,

$$\bigcap_{e>0} \mathrm{Ann}_R(F^e(x)) = (0),$$

where $F^e(x)$ denotes the image of $x = x \otimes 1$ in $F^e(M) = M \otimes_R {}^eR$.

Lemma 2.5. *For every ideal I of a domain R, $I = I^*$ if and only if R/I is F-unstable as an R-module.*

Proof. Assume that R/I is F-unstable and I is not tightly closed. Let $y \in I^* - I$. Then there exists $c \neq 0 \in R$ such that $cy^q \in I^{[q]}$ for all $q = p^e$. Let x be the image of y in R/I. Then $F^e(x) \in F^e(R/I) = R/I^{[q]}$, and $cF^e(x) = 0$ in $F^e(R/I)$ for every $e > 0$. Thus $0 \neq c \in \bigcap_{e>0} \mathrm{Ann}_R(F^e(x))$, a contradiction.
Conversely, assume R/I is not F-unstable. Then there exist a nonzero $x \in R/I$ and nonzero $c \in \mathrm{Ann}_R(F^e(x))$ for every $e > 0$. Thus $cy^q \in I^{[q]}$ for every $q = p^e$, where y is the representative of x in R. Hence $y \in I^*$, but $y \notin I$. That is, I is not tightly closed. □

Theorem 2.6. *Let R be a complete F-pure domain. Then the followings are equivalent:*

(1) *Every ideal of R is tightly closed.*
(2) *Every finite R-module is F-unstable.*
(3) *If R is a local ring with the unique maximal ideal $\underline{m}$, then $E_R(R/\underline{m})$, the injective hull of $R/\underline{m}$, is F-unstable.*

Proof. (1) implies (2) ; We will prove by induction on the number n of generators of M.
(i) $n = 1$; M is a cyclic R-module, let $M = Rx, x \in M$. Then M is isomorphic to R/I, where $I = \mathrm{Ann}_R(x)$. Since I is tightly closed by the weak F-regularity of R, M is F-unstable by Lemma 2.5.
(ii) $n > 1$; Let $M_1 = \sum_{i=1}^{k-1} Rx_i$, $M = \sum_{i=1}^{k} Rx_i$, and $M_2 = M/M_1$, where $x_i \in M$ for every $i = 1, \cdots, k$. Then the induction hypothesis implies that M_1 and M_2 are F-unstable. From the following commutative diagram follows that M is F-unstable.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 \otimes_R {}^eR & \longrightarrow & M \otimes_R {}^eR & \longrightarrow & M_2 \otimes_R {}^eR & \longrightarrow & 0
\end{array}$$

(2) implies (3) ; we can write $E = E_R(R/\underline{m})$ as a direct limit of finite R-modules. That is, $E = \varinjlim M_i$, where M_i are finite R-modules. Here, each M_i is F-unstable by the hypothesis. Then,

$$E \otimes_R {}^eR = (\varinjlim M_i) \otimes_R {}^eR = \varinjlim(M_i \otimes_R {}^eR).$$

The second equality follows from the fact that the tensoring, $\otimes_R {}^eR$, commutes with the direct limit. Thus (3) is also true.
(3) implies (1) ; Assume that $I^* \supsetneq I$ for any ideal I of R.
Then $(0 :_E I^*) \subsetneq (0 :_E I) = \{r \in E \mid rI = 0\}$, and $I^* \cdot (0 :_E I) \neq 0$. For, if $(0 :_E I^*) = (0 :_E I)$, then $\mathrm{Ann}_R((0 :_E I^*)) = \mathrm{Ann}_R((0 :_E I))$. But $\mathrm{Ann}_R((0 :_E J)) = J$ for any ideal J of R [8]. This implies that $I = I^*$, which is a a contradiction.
We can therefore choose $y \in I^*$ and $x \in (0 :_E I)$ such that $z = yx$ is a nonzero element of E. Since $y \in I^*$, there exists $c \neq 0 \in R$ such that $cy^q \in I^{[q]}$ for all $q = p^e$. Since $x \in (0 :_E I)$, $F^e(x) \in (0 :_E I^{[q]})$. Thus, $0 = (cy^q)F^e(x) = cF^e(yx) = cF^e(z)$ for every $e > 0$. That is, $c \in \bigcap_{e>0} (0 :_R F^e(z)) = \bigcap_{e>0} \mathrm{Ann}_R(F^e(z))$. Since $c \neq 0$, E is not F-unstable, which is a a contradiction. □

3. The Frobenius Map and the Weak F-regularity

Recall that a ring R of characteristic p is *F-contracted* if every ideal generated by a system of parameter is contracted with respect to the Frobenius map $F : R \to {}^1R$, that is, $(I \cdot {}^1R) \cap R = I$.

PROPOSITION 3.1. *Let $(R, \underline{m})$ be a Cohen-Macaulay local ring with the maximal ideal $\underline{m}$. Then the followings are equivalent:*

(1) *The map from $H^n_{\underline{m}}(R)$ to $H^n_{\underline{m}}({}^1R)$, induced by the Frobenius map from R to ${}^1R$, is injective.*
(2) *R is F-contracted.*
(3) *There exists a system of parameter which is contracted with respect to the Frobenius map from R to ${}^1R$.*

Proof. See [3, Proposition 1.4]. □

For a Gorenstein local ring R of dimension n, it is a well-known fact from local duality theory that $\mathrm{H}^n_{\underline{m}}(R)$ is isomorphic to E, the injective hull of $R/\underline{m}$, where $\underline{m}$ is the unique maximal ideal of R. Hochster and Roberts proved that R is F-pure if and only if $E \to E \otimes {}^1R$ is injective [7]. Hence we have the following:

PROPOSITION 3.2. *Let R be a local Gorenstein ring with the maximal ideal $\underline{m}$. Then the followings are equivalent:*

(1) *R is F-pure.*
(2) *R is F-contracted.*
(3) *There exists asystem of parameter which is contracted with respect to the Frobenius map.*
(4) *$H^n_{\underline{m}}(R) \to H^n_{\underline{m}}({}^1R)$ is injective, where $\dim R = n$.*

Proof. (2), (3), and (4) are equivalent by Proposition 3.1. And the implication of (1) to (2) is clear. Now it remains only to prove that (4) implies (1). But $H^n_{\underline{m}}(R) \cong E$, the injective hull of $R/\underline{m}$, implies that $E \to E \otimes {}^1R$ is injective. Thus, R is F-pure. □

Now we discuss the relationship between the F-contractedness and the weak F-regularity, and characterize the Gorenstein ring of dimension zero.

PROPOSITION 3.3. *Let R be a local Gorenstein ring and let $x_1, \cdots, x_d$ be a system of parameter If the image of I in R/I is contracted with respect to the Frobenius map*

$$F : R/I \to {}^1(R/I),$$

where $I = (x_1, \cdots, x_d)R$, then R is weakly F-regular.

Proof. R/I is a zero-dimensional Gorenstein F-pure ring by the hypothesis and Proposition 3.2. Thus R/I is weakly F-regular by Theorem 2. Since $x_1, \cdots, x_d$ is a regular sequence in R, R is also Gorenstein. Thus R is weakly F-regular. □

In Proposition 3.3, the condition that R is Gorenstein can be replaced by the condition that R is Cohen-Macaulay.

LEMMA 3.4. *Let R be a reduced ring of dimension zero. Then R is Gorenstein, and weakly F-regular.*

Proof. We may assume that R is local. Since R is a direct product of finite number of fields, R is normal. But we know that any normal local ring is approximately Gorenstein. Since R is zero-dimensional local, R is Gorenstein. Now we need only to show that R is weakly F-regular. It is enough to show that (0), a system of parameterideal of R, is tightly closed. Let $r \in (0)^*$. Then there exists $c \in R^o$ such that $cr^q = 0$ for all $q = p^e$. But $R^o = R \backslash Z(R)$, since R is Noetherian reduced. We have $r^q = 0$ and $r = 0$. Thus, $(0) = (0)^*$, as required. □

THEOREM 3.5. *Let R be a Cohen-Macaulay local ring of dimension d and let I be an ideal of R which is generated by a system of parameter If R/I is reduced, then R is Gorenstein and R is F-regular.*

Proof. Since R/I is zero-dimensional and reduced, R/I is Gorenstein and (weakly) F-regular by Lemma 3.4. And since a system of parameter for R is a regular sequence in R, R is also Gorenstein. We know that if $x_1, \cdots, x_d$ form a regular sequence in a Gorenstein local ring and $R/(x_1, \cdots, x_d)R$ is weakly F-regular, then R is weakly F-regular [1]. □

Now we prove that Proposition 3.3. is still true when R is Cohen-Macaulay.

PROPOSITION 3.6. *Let R be a Cohen-Macaulay local ring of dimension d, and let $x_1, \cdots, x_d$ be a system of parameter If the image of $I = (x_1, \cdots, x_d)R$ in R/I is contracted with respect to the Frobenius map*

$$F : R/I \to {}^1(R/I),$$

then R is weakly F-regular and Gorenstein.

Proof. The condition that the image of $I = (x_1, \cdots, x_d)R$ in R/I is contracted with respect to the Frobenius map implies that R/I is F-contracted, and R/I is reduced. Thus R is Gorenstein and F-regular by Theorem 3.5. □

COROLLARY 3.7. *Let R be a Cohen-Macaulay local ring of dimension d. If R/I is F-pure and I is an ideal generated by an s.o.p., then R is F-regular.*

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