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ON FREE PRODUCT IN $V(ZS_3)$

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ABSTRACT. The group $V(ZS_3)$ of units of augmentation 1 in the integral group ring ZS_3 is characterized as the free product of C_2 and S_3, where C_2 is the cyclic group of order 2.

1. Introduction

In [2], Hughes and Pearson characterized $U(ZS_3)$, the group of units of the integral group ring ZS_3. One of the main results is :

$$U(ZS_3) \cong \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,Z) | a+c \equiv b+d (\text{mod } 3)\}.$$

From their work we can easily get the following characterization of $V(ZS_3)$, the group of units of augmentation 1 in ZS_3.

$$V(ZS_3) \cong G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,Z) | a+c \equiv b+d \equiv 1 (\text{mod } 3)\}.$$

On the other hand, in [5] Taussky gives two nontrivial units of order 2 in $V(ZS_3)$. Consider $A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, one of the units given by Taussky, $B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ in G. Clearly $\langle B, C\rangle \cong S_3$. In this note we shall prove that

$$V(ZS_3) = \langle A, B, C\rangle = \langle A\rangle * \langle B, C\rangle \cong C_2 * S_3,$$

where C_2 is the cyclic group of order 2, and $X * Y$ denotes the free product of the groups X and Y.

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2. Theorems

First we prove that A, B and C generate G.

THEOREM 2.1. $\langle A, B, C\rangle = G$.

Proof. Let $F = \langle A, B, C\rangle$ and we will show that $F = G$. Since A, B, C are integral matrices with determinant ± 1 so are A^{-1}, B^{-1}, C^{-1}, and hence the elements of F are integral matrices. If $F \neq G$ then there exists $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \backslash F$ with $|a| + |c|$ minimal. For the moment suppose that $a \neq 0$ and $c \neq 0$. Since $(BC)A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$,

$$((BC)A)^r X = \begin{pmatrix} 1 & 0 \\ 3r & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 3ra + c & 3rb + d \end{pmatrix} \notin F.$$

If $2|a| \leq |c|$ then the integer r can be chosen so that $|3ra + c| < |c|$, and hence $|3ra + c| + |a| < |a| + |c|$, contradicting the choice of $|a| + |c|$. Hence $2|a| > |c|$. So $|a| < |c| < 2|a|$, $|c| < |a|$, or $|a| = |c|$. First suppose that $|a| < |c| < 2|a|$ and a, c have same sign. Then

$$BX = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - c & b - d \\ -c & -d \end{pmatrix} \notin F.$$

But $|a - c| < |a|$. This gives a contradiction to the choice of $|a| + |c|$. If a, c have different signs then

$$AX = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ 2a + c & 2b + d \end{pmatrix} \notin F.$$

But $|2a + c| < |c|$. So we also have a contradiction. Now suppose that $|c| < |a|$ and a, c have the same sign then

$$BX = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - c & b - d \\ -c & -d \end{pmatrix} \notin F.$$

But $|a - c| < |a|$. This contradicts the choice of a and c. If a and c have different signs then

$$(ABC^2)X = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a + 2c & b + 2d \end{pmatrix} \notin F.$$

But $|a+2c| < |a|$, and hence $|a+2c| + |c| < |a| + |c|$. This is also a contradiction to the choice of a and c. Finally we suppose that $|a| = |c|$. Then a and c may have same or different sign. In each case, we can get a contradiction by same arguments as above. Therefore $a = 0$ or $c = 0$. If $a = 0$, then since $G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,Z) | a+c \equiv b+d \equiv 1 (\text{mod } 3)\}$, we have two cases $X = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}$. If $X = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$ then $d = 3k$ for some integer k. Now

$$\begin{pmatrix} 0 & 1 \\ 1 & 3k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3k & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (BCA)^k BC^2 \in F.$$

However this is a contradiction. If $X = \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}$ then $d = 3k' + 2$ for some integer k'. Now

$$\begin{aligned} X = \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 3k'+2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -3k' & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = (BCA)^{-k'}(ABC^2) \in F, \end{aligned}$$

and this is also a contradiction. Now if $c = 0$ then $X = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ or $X = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$. If $X = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ then $b = 3k$ for some $k \in Z$. So,

$$\begin{aligned} CX &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 3k-1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 3k'+2 \end{pmatrix} = (BCA)^{k'}(ABC^2). \end{aligned}$$

Therefore $X = C^2(BCA)^{k'}(ABC^2) \in F$, a contradiction. If $X = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$ then $b = 3k+2$ for some $k \in Z$ and $(ABC^2)X = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 3k+2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 3k \end{pmatrix} = (BCA)^k BC^2$ Hence $X = (ABC^2)^{-1}$ $(BCA)^k BC^2 \in F$, the final contradiction which proves the theorem. □

Now any element $\langle A, B, C\rangle$ can be expressed as $B^\alpha C^\beta A^\gamma$ or $B^\alpha C^\beta$ $(AB^{\epsilon_1}C^{\delta_1})\cdots(AB^{\epsilon_n}C^{\delta_n})A^\gamma$, where $\alpha, \gamma, \epsilon_i = 0, 1$, and $\beta, \delta_i = 0, 1, 2$, and $n \geqslant 1$, and $\epsilon_i + \delta_i \neq 0, i = 1, \ldots, n$. Consider

$$\Gamma = \{(AB^{\epsilon_1}C^{\delta_1})\cdots(AB^{\epsilon_n}C^{\delta_n}) | \epsilon_i = 0, 1, \\ \text{and } \delta_i = 0, 1, 2, \text{ and } \epsilon_i + \delta_i \neq 0, \text{ and } n \geqslant 1\}.$$

If $n = 1$, there are five cases :

$$AC = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}, AC^2 = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}, AB = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix},$$
$$ABC = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, ABC^2 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

DEFINITION. $X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2, Z)$ is called (i, j)- cornered, $1 \leqslant i, j \leqslant 2$, if

$|a_{ij}| = \max\{|a_{11}|, |a_{12}|, |a_{21}|, |a_{22}|\}$ and $|a_{3-i3-j}| = \min\{|a_{11}|, |a_{12}|, |a_{21}|, |a_{22}|\}$,

and the absolute values of the sum and difference of the row and column containing a_{ij} is greater than or equal to those of the row and column containing a_{3-i3-j}, respectively.

We can see that the above five matrices are $(2, 1)$ or $(2, 2)$-cornered.

THEOREM 2.2. *Any* $X = (AB^{\epsilon_1}C^{\delta_1})\cdots(AB^{\epsilon_n}C^{\delta_n}) \in \Gamma$ *is one of the following three forms :*

$$T_1 : (-1)^m \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}, (2,1) - \textit{cornered},$$

$$T_2 : (-1)^m \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}, (2,2) - \textit{cornered, and}$$

$$T_3 : (-1)^m \begin{pmatrix} \alpha & \beta \\ -\gamma & -\delta \end{pmatrix}, (2,2) - \textit{cornered}, \ \alpha, \beta, \gamma, \delta \geqslant 0.$$

Proof. We prove this by induction on n. As mentioned before it is obvious when $n = 1$. Now assume the theorem for $n - 1$. We need to consider

$$T_i AB^{\epsilon} C^{\delta}, i = 1,2,3, \text{ and } \epsilon = 0,1, \text{ and } \delta = 0,1,2, \text{ and } \epsilon + \delta \neq 0.$$

By the straight computation, we can show that any of them is one of the three forms. We compute one case here.

$$T_3ABC^2 = (-1)^m \begin{pmatrix} \alpha & \beta \\ -\gamma & -\delta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = (-1)^m \begin{pmatrix} \beta & 2\beta - \alpha \\ -\delta & -(2\delta - \gamma) \end{pmatrix}$$

So, $T_3ABC^2 = (-1)^m \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}, a,b,c,d \geqslant 0$. Now simple computation show that $c \geqslant a$, $d \geqslant b$, $b \geqslant a$, $d \geqslant c$, $d \pm b \geqslant c \pm a$, and $d \pm c \geqslant b \pm a$. □

From Theorem 2.2, we know that any element in Γ is $(2,1)$ or $(2,2)$-cornered, and hence Γ has no identity matrix. Now by computation, we can see that $B^{\alpha}C^{\beta}A^{\gamma} = I$ if and only if $\alpha = \beta = \gamma = 0$ and $B^{\alpha}C^{\beta}A^{\gamma}$ is not of any form in Theorem 2.2. As we remarked before, any element in $\langle A, B, C\rangle$ can be expressed as

$$B^{\alpha}C^{\beta}(AB^{\epsilon_1}C^{\delta_1}) \cdots (AB^{\epsilon_n}C^{\delta_n})A^{\gamma}.$$

where $\alpha, \gamma, \epsilon_i = 0,1$, and $\beta, \delta_i = 0,1,2$, and $n \geqslant 0$, and $\epsilon_i + \delta_i \neq 0, i = 1, \cdots, n$. Since Γ has no identity matrix, the identity matrix in $\langle A, B, C\rangle$ must be of that form with $n = \alpha = \beta = \gamma = 0$. Hence using von Dyck's theorem [p.51, 4] we can get an isomorphism from $C_2 * S_3$ onto $\langle A, B, C\rangle$.

THEOREM 2.3. $\langle A, B, C\rangle = \langle A\rangle * \langle B, C\rangle$

3. An application

It is well known that S_3 has a torsion free normal complement in $V(ZS_3)$ [3]. Also in [1], Allen and Hobby exhibited a non torsion free normal complement of S_3 in $V(ZS_3)$. We know that $D_\infty \cong \langle x\rangle * \langle y\rangle, x^2 = y^2 = 1$, has such a property. This fact motivates the following theorem :

THEOREM 3.1. *Suppose A is an arbitrary group and B is a non-trivial group isomorphic to a subgroup of A. Suppose B is not torsion free. Then $A * B$ has both a normal complement N, to A which is torsion free and a normal complement M, to A which is not torsion free.*

Proof. To construct N, let $\phi : B \longrightarrow A$ denote the embedding of B in A. Then we have a mapping $\bar{\phi} : A * B \longrightarrow A$ gven by

$$(a_1 b_1 \cdots a_n b_n)^{\bar{\phi}} = a_1 b_1^{\phi} a_2 b_2^{\phi} ... a_n b_n^{\phi} \text{ where } a_i \in A, b_i \in B.$$

The normal form theorem for free products shows $\bar{\phi}$ is a homomorphism. Set $N = ker\ \bar{\phi}$. If $x \in A * B$ and $x^{\bar{\phi}} = a \in A$ then $(a^{-1}x)^{\bar{\phi}} = 1$. So $A * B = NA$. Futhermore $N \cap A = N \cap B = 1$ so by the Kuroš subgroup theorem N is a free group, and hence is torsion free.

To construct M define $\theta : A * B \longrightarrow A$ by

$$(a_1 b_1 a_2 b_2 \cdots a_n b_n)^{\theta} = a_1 a_2 \cdots a_n.$$

Then θ is a homomorphism. We set $ker\ \theta = M$. As above $A * B = MA$ and $M \cap A = 1$. However $B \leq M$, so M is not torsion free. □

From Theorem 3.1 and Theorem 2.3 we immediately see that S_3 has torsion free and non torsion free normal complements in $V(ZS_3)$. It is also worth remarking that there is a normal complement of S_3 which is actually free, according to the proof of Theorem 3.1. Also note that we can get some of the results in [2] as corollaries of Theorem 2.3.

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