Bull. Korean Math. Soc. **34** (1997), No. 4, pp. 561–571

# ON SOLUTION AND STABILITY OF FUNCTIONAL EQUATION $\mathbf{f(x+y)^2 = af(x)f(y) + bf(x)^2 + cf(y)^2}$

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ABSTRACT. The general (continuous) solution and the asymptotic behaviors of the unbounded solution of the functional equation $f(x+y)^2 = af(x)f(y) + bf(x)^2 + cf(y)^2$ and the Hyers-Ulam stability of that functional equation for the case when $a = 2$ and $b = c = 1$ shall be investigated.

## 1. Introduction

In the last decades the explicit solutions of several functional equations have been extensively investigated (cf. [1], [3] and [4]). It is well-known that every continuous solution $f : \mathbf{R} \to \mathbf{R}$ of the Cauchy equation $f(x+y) = f(x) + f(y)$ have the form of $f(x) = \alpha x$, where $\alpha$ is a real constant. Based on this fact we can easily presume that each continuous solution of the following functional equation

$$f(x+y)^2 = f(x)^2 + f(y)^2, \quad \text{for all } x, y \geq 0,$$

has the form

$$f(x) = \beta\sqrt{x} \ \text{ or } \ f(x) = -\beta\sqrt{x}\,,$$

for any $x \geq 0$, where $\beta$ is a real constant.

Throughout the paper, let $E$ be a real normed space. The following functional equation

$$(1) \qquad f(x+y)^2 = af(x)f(y) + bf(x)^2 + cf(y)^2, \quad \text{for all } x, y \in E,$$

where $a, b, c$ are complex numbers, can be considered as a generalization of the above functional equation. In section 2, the bounded solutions

Received October 9, 1996.
1991 Mathematics Subject Classification: Primary 39B32, 39B72.
Key words and phrases: Functional equation, Hyers-Ulam stability.

where $a, b, c$ are complex numbers, can be considered as a generalization of the above functional equation. In section 2, the bounded solutions of the functional equation (1) shall be determined. Further, the general continuous solution of the functional equation (1) with $a = 2$ and $b = c = 1$ ($E = \mathbf{R}$) shall be investigated in Theorem 1. We also investigate in Theorem 3 the asymptotic behaviors of the unbounded complex-valued solutions of the functional equation (1). Furthermore, the Hyers-Ulam stability for the functional equation $f(x+y)^2 = (f(x) + f(y))^2$ shall be proved in Theorem 4 and Theorem 5.

## 2. The case of b $\neq$ 1 or c $\neq$ 1

The complex-valued solutions of the functional equation (1) in the case when $b \neq 1$ or $c \neq 1$ shall be investigated.

By setting $y = 0$ in (1) when $b \neq 1$ or by setting $x = 0$ and $y = x$ in (1) when $c \neq 1$, we can transform the functional equation (1) into

$$\begin{cases} (1-b)f(x)^2 - af(0)f(x) - cf(0)^2 = 0 & (\text{for } b \neq 1) \\ \text{or} & \\ (1-c)f(x)^2 - af(0)f(x) - bf(0)^2 = 0 & (\text{for } c \neq 1), \end{cases}$$

and their solutions are

$$(2) \qquad f(x) = \begin{cases} \frac{a+\sqrt{a^2+4(1-b)c}}{2(1-b)} f(0) & \text{for all } x \in E \quad (\text{for } b \neq 1) \\ \text{or} & \\ \frac{a-\sqrt{a^2+4(1-b)c}}{2(1-b)} f(0) & \text{for all } x \in E \quad (\text{for } b \neq 1) \\ \text{or} & \\ \frac{a+\sqrt{a^2+4(1-c)b}}{2(1-c)} f(0) & \text{for all } x \in E \quad (\text{for } c \neq 1) \\ \text{or} & \\ \frac{a-\sqrt{a^2+4(1-c)b}}{2(1-c)} f(0) & \text{for all } x \in E \quad (\text{for } c \neq 1). \end{cases}$$

First, we assume that $a + b + c \neq 1$. Then, by applying (2) to (1), we obtain $f(x) = 0$ for all $x \in E$. Hence, $f(x) = 0$ is the unique solution of the functional equation (1) when $a+b+c \neq 1$ and $b \neq 1$ or $a+b+c \neq 1$ and $c \neq 1$. Now, let's assume that $a+b+c = 1$. By putting $a = 1-b-c$ in (2) we get $f(x) = f(0)$ for all $x \in E$. Therefore, $f(x) = \alpha$, where $\alpha$ is a complex number, is the general solution of the functional equation (1) when $a+b+c=1$ and $b \neq 1$ or $a+b+c=1$ and $c \neq 1$.

## 3. The case of $b = c = 1$

Suppose that $b = c = 1$ in the functional equation (1). By putting $x = y = 0$ in (1), we get

$$(a+1)f(0)^2 = 0.$$

First, assume that $a = -1$. By putting $y = 0$ in (1), we obtain

$$f(0)\left(f(0) - f(x)\right) = 0.$$

Hence, if $f(0) \neq 0$ then $f(x) = f(0) \neq 0$. Otherwise, by substituting $-x$ for $y$ in (1),

$$\begin{aligned} 0 = f(0)^2 &= -f(x)f(-x) + f(x)^2 + f(-x)^2 \\ &= \left(f(-x) - \tfrac{1}{2}f(x)\right)^2 + \tfrac{3}{4}f(x)^2. \end{aligned}$$

Hence, it follows that $f(x) = 0$ for any $x \in E$. Therefore, $f(x) = \alpha$ ($\alpha$ is a complex number), for all $x \in E$, is the general solution of the functional equation

$$f(x+y)^2 = -f(x)f(y) + f(x)^2 + f(y)^2, \quad \text{for all } x, y \in E.$$

Now, let $a \neq -1$. We then have $f(0) = 0$. For this case, it follows from the functional equation (1) that

$$f(-x) = \left(-\frac{a}{2} + \frac{1}{2}\sqrt{a^2-4}\right) f(x)$$

or

$$f(-x) = \left(-\frac{a}{2} - \frac{1}{2}\sqrt{a^2-4}\right) f(x)$$

by putting $y = -x$ in (1) and by using $f(0) = 0$. Further, by substituting $-x$ for $x$ in the above equations, we get

$$f(x) = \left(-\frac{a}{2} + \frac{1}{2}\sqrt{a^2-4}\right)^2 f(x)$$

or

$$f(x) = \left(-\frac{a}{2} - \frac{1}{2}\sqrt{a^2-4}\right)^2 f(x)$$

for all $x \in E$. Thus, we obtain $a = -2$ or $a = 2$ if there exists at least one $x \in E$ such that $f(x) \neq 0$. If $a = -2$ then it follows from (1) that $f(2x)^2 = 0$, i.e., $f(x) = 0$, for any $x \in E$, by putting $y = x$ in (1). Hence, $f(x) = 0$ (for $x \in E$) is the unique solution of (1) with $a \notin \{-1, 2\}$ and $b = c = 1$.

Now let $a = 2$, $b = c = 1$ and $f(0) = 0$. For this case the functional equation (1) can be transformed into

$$(3) \qquad f(x+y)^2 = \left(f(x) + f(y)\right)^2, \quad \text{for all } x, y \in E,$$

which is equivalent to

$$(4) \qquad f(x+y) = f(x) + f(y) \quad \text{or} \quad f(x+y) = -f(x) - f(y), \quad \text{for all } x, y \in E.$$

THEOREM 1. *Let $f$ be a complex-valued mapping, defined on $\mathbf{R}$, which is continuous at 0. The mapping $f$ is a solution of the functional equation (3) or (4) with $E = \mathbf{R}$ if and only if $f$ is linear, that is, there exists a complex number $\alpha$ such that $f(x) = \alpha x$ for all $x \in \mathbf{R}$.*

*Proof.* It follows from (3) that $f(0) = 0$ (putting $x = y = 0$). By putting $y = -x$ in (3) and by using $f(0) = 0$ we can show the oddness of $f$. Now, we shall prove

$$(5) \qquad f(nx) = nf(x)$$

for all integers $n$ and any $x \in \mathbf{R}$. On account of the oddness of $f$ we may prove (5) only for integers $n \geq 2$. By putting $y = x$ in (3) we get

$$(6) \qquad f(2x) = 2f(x) \ \text{ or } \ f(2x) = -2f(x).$$

Replacing $y$ in (3) by $2x$ yields

$$(7) \qquad \begin{aligned} f(3x)^2 &= \left(f(x) + f(2x)\right)^2 \\ &= \begin{cases} 9f(x)^2 & \text{if } f(2x) = 2f(x), \\ f(x)^2 & \text{if } f(2x) = -2f(x). \end{cases} \end{aligned}$$

In view of (3) and (6), it follows

$$(8) \qquad f(4x)^2 = \left(f(2x) + f(2x)\right)^2 = 16f(x)^2.$$

On the other hand, by using (7), we obtain

$$f(4x)^2 = (f(x)+f(3x))^2$$
$$= \begin{cases} 16f(x)^2 & \text{if } f(3x) = 3f(x), \\ 4f(x)^2 & \text{if } f(3x) = -3f(x) \text{ or } f(3x) = f(x), \\ 0 & \text{if } f(3x) = -f(x). \end{cases} \tag{9}$$

By comparing (8) with (9) and taking (7) into consideration, we get $f(3x) = 3f(x)$. Hence, it follows from (6) and (7) that

$$f(2x) = 2f(x).$$

Thus, (5) holds true for $n = 2$. Assume that (5) holds true for all positive integers $\leq n$ $(n \geq 2)$. Then, by putting $y = nx$ in (3), we obtain

$$f((n+1)x) = (n+1)f(x) \quad \text{or} \quad f((n+1)x) = -(n+1)f(x). \tag{10}$$

Replacing $y$ in (3) by $(n+1)x$ yields

$$f((n+2)x)^2$$
$$= \begin{cases} (n+2)^2 f(x)^2 & \text{if } f((n+1)x) = (n+1)f(x), \\ n^2 f(x)^2 & \text{if } f((n+1)x) = -(n+1)f(x). \end{cases} \tag{11}$$

On the other hand, by substituting $2x$ and $nx$ for $x$ and $y$ in (3), respectively, and by using induction hypothesis, it holds

$$f((n+2)x)^2 = (n+2)^2 f(x)^2. \tag{12}$$

By comparing (11) with (12) and by considering (10) we conclude

$$f((n+1)x) = (n+1)f(x),$$

which completes the proof of (5).

By substituting $x/n$, $n \neq 0$, for $x$ in (5) we get

$$f\left(\frac{x}{n}\right) = \frac{1}{n} f(x). \tag{13}$$

Hence, for every rational number $q$, we have

$$f(qx) = qf(x)$$

by (5) and (13). If we put $x = 1$ in the above equation then

$$f(q) = f(1)q. \tag{14}$$

The continuity of $f$ at 0, together with (3), implies that $f(x)^2$ is continuous at each $x \in \mathbf{R}$. From this fact and (14) it follows

$$(15) \qquad f(r) = f(1)r \ \text{ or } \ f(r) = -f(1)r$$

for all irrational numbers $r$. Assume that $f$ satisfies $f(r) = -f(1)r$ for some irrational number $r$. Then, by (15), it holds

$$(16) \qquad f(q+r) = f(1)(q+r) \ \text{ or } \ f(q+r) = -f(1)(q+r)$$

for any rational number $q \neq 0$. On the other hand, by (3), (14) and the assumption, we get

$$f(q+r)^2 = f(1)^2(q-r)^2.$$

By comparing this equation with (16) we conclude $f(1) = 0$. Hence, if $f(r) = -f(1)r$ holds for some irrational number $r$ then it follows

$$f(x) = 0$$

for all $x \in \mathbf{R}$. Now, assume that $f(r) = f(1)r$ for all irrational numbers $r$. Then this assumption, together with (14), yields

$$f(x) = f(1)x$$

for all $x \in \mathbf{R}$.

Conversely, every complex-valued mapping $f$, defined on $\mathbf{R}$, of the form $f(x) = \alpha x$ with a constant $\alpha$ satisfies the functional equation (3). □

REMARK. If $f$ is a real-valued mapping, the functional equation (3) is equivalent to

$$(17) \qquad |f(x+y)| = |f(x) + f(y)|, \quad \text{for all } x, y \in E.$$

Since $(\mathbf{R}, |\cdot|)$ is a strictly normed space, according to Skof [6], the mapping $f$ is a solution of the functional equation (3) if and only if $f$ is an additive mapping. (When $f$ is complex-valued, the functional equation (3) implies the functional equation (17) but, in general, (17) does not imply (3).)

It is easy to prove the following lemma. Hence, we omit the proof.

LEMMA 2. *Let $f$ be a complex-valued solution of the functional equation (3). Then it holds*

$$|f(2^k x)| = 2^k\,|f(x)|$$

*for all $x \in E$ and all integers $k$.*

The asymptotic behaviors of unbounded complex-valued solutions of the functional equation (3) shall be investigated.

THEOREM 3. *Let $f$ be any complex-valued solution of the functional equation (3) or (4). We further assume that there exist some $M_1, M_2 > 0$ and an integer $n_0$ such that $M_1 < |f(x)| < M_2$ for any $x \in E$ with $2^{n_0} \le \|x\| \le 2^{n_0+1}$. If $p > 0$ is given, then*
(a) $\|x\|^{1-p} = o(|f(x)|)$ *and* $|f(x)| = o(\|x\|^{1+p})$, *as* $\|x\| \to \infty$,
(b) $|f(x)| = o(\|x\|^{1-p})$ *and* $\|x\|^{1+p} = o(|f(x)|)$, *as* $\|x\| \to 0$.

*Proof.* It follows from the assumptions for $|f(x)|$ that there are some $m_1, m_2 > 0$ for which

$$m_1\|x\| \le |f(x)| \le m_2\|x\| \tag{18}$$

holds for all $x \in E$ with $2^{n_0} \le \|x\| \le 2^{n_0+1}$. Now, let $x \in E$, $x \ne 0$, be given arbitrarily. We may choose an integer $n$ such that $2^{n_0} \le \|2^n x\| \le 2^{n_0+1}$. From Lemma 2 and (18), we get

(19)
$$m_1\|x\| = \frac{1}{2^n}m_1\|2^n x\| \le \frac{1}{2^n}\,|f(2^n x)| = |f(x)| \le \frac{1}{2^n}m_2\|2^n x\| = m_2\|x\|.$$

By using (19) we prove the part (a):

$$\frac{\|x\|^{1-p}}{|f(x)|} \le \frac{1}{m_1}\frac{1}{\|x\|^p} \to 0 \quad\text{and}\quad \frac{|f(x)|}{\|x\|^{1+p}} \le m_2\frac{1}{\|x\|^p} \to 0$$

as $\|x\| \to \infty$. Analogously, we can easily prove the part (b) by using (19). □

## 4. The stability of $\mathbf{f(x+y)^2 = (f(x)+f(y))^2}$

Using ideas from the papers of Hyers [2] and Rassias [5], we shall investigate the Hyers-Ulam stability problem for the functional equation (3).

Suppose $f : E \to \mathbf{R}$ to be a mapping which fulfils

$$|f(x+y)^2 - (f(x)+f(y))^2| \leq \delta \tag{20}$$

for some $\delta \geq 0$ and any $x, y \in E$.

THEOREM 4. *There exists an additive mapping $T : E \to \mathbf{R}$ which satisfies*

$$|T(x)^2 - f(x)^2| \leq \frac{\delta}{3}, \quad \text{for all } x \in E. \tag{21}$$

*Moreover, if $U : E \to \mathbf{R}$ is another additive mapping which fulfils (21) then*

$$T(x)^2 = U(x)^2 \tag{22}$$

*for any $x \in E$.*

*Proof.* By using induction on $n$ we first prove that

$$|f(2^n x)^2 - (2^n f(x))^2| \leq \delta \sum_{i=0}^{n-1} 2^{2i} \tag{23}$$

for $n \in \mathbf{N}$. For $n = 1$, it is trivial by (20). Assume that (23) holds true for some $n$. Then, by substituting $2^n x$ for $x$ and $y$ in (20) and by using (23), we show

$$\begin{aligned}
|f(2^{n+1}x)^2 - (2^{n+1}f(x))^2| &\leq \\
&\leq |f(2^{n+1}x)^2 - (2f(2^n x))^2| + 2^2|f(2^n x)^2 - (2^n f(x))^2| \\
&\leq \delta + 2^2\delta \sum_{i=0}^{n-1} 2^{2i} \\
&\leq \delta \sum_{i=0}^{n} 2^{2i},
\end{aligned}$$

which completes the proof of (23). Dividing the both sides in (23) by $2^{2n}$ yields

$$|f(2^n x)^2/2^{2n} - f(x)^2| \leq \frac{\delta}{3} \tag{24}$$

for all $x \in E$ and $n \in \mathbf{N}$. It follows from (24) that, for $n \geq m > 0$,

$$\begin{aligned} & \left| (f(2^n x)/2^n)^2 - (f(2^m x)/2^m)^2 \right| \\ (25) \qquad = & \ \frac{1}{2^{2m}} \left| \left(f(2^{n-m} 2^m x)/2^{n-m}\right)^2 - f(2^m x)^2 \right| \\ \leq & \ \frac{1}{2^{2m}} \frac{\delta}{3} \\ \longrightarrow & \ 0, \quad \text{as } m \to \infty. \end{aligned}$$

For each $x \in E$ we define

$$I_x^+ = \{n \in \mathbf{N} \mid f(2^n x) \geq 0\} \quad \text{and} \quad I_x^- = \{n \in \mathbf{N} \mid f(2^n x) < 0\}.$$

In view of (25), we know that if $I_x^+$ or $I_x^-$ is an infinite set then the sequence $(f(2^n x)/2^n)_{n \in I_x^+}$ or $(f(2^n x)/2^n)_{n \in I_x^-}$ is a Cauchy sequence, respectively. Now, let us define

$$T(x) = \begin{cases} \lim\limits_{\substack{n \to \infty \\ n \in I_x^+}} \frac{1}{2^n} f(2^n x) & \text{if } I_x^+ \text{ is infinite,} \\ \lim\limits_{\substack{n \to \infty \\ n \in I_x^-}} \frac{1}{2^n} f(2^n x) & \text{otherwise.} \end{cases}$$

If both $I_x^+$ and $I_x^-$ are infinite sets then

$$(26) \qquad T(x) = - \lim_{\substack{n \to \infty \\ n \in I_x^-}} \frac{1}{2^n} f(2^n x).$$

The definition of $T$ and (24) imply the validity of (21). Let $x, y \in E$ be given arbitrarily. It is not difficult to prove that there is at least one infinite set among the sets $I_x^+ \cap I_y^+ \cap I_{x+y}^+$, $I_x^+ \cap I_y^+ \cap I_{x+y}^-$, $\cdots$, $I_x^- \cap I_y^- \cap I_{x+y}^-$. We may choose such an infinite set and denote this set by $I$. Let $n \in I$ be given. Replacing $x$ and $y$ in (20) by $2^n x$ and $2^n y$, respectively, and then dividing the both sides in (20) by $2^{2n}$, we obtain

$$(27) \qquad \left| (f(2^n(x+y))/2^n)^2 - (f(2^n x)/2^n + f(2^n y)/2^n)^2 \right| \leq \frac{\delta}{2^{2n}}.$$

By letting $n \to \infty$ through $I$ in (27) and taking (26) into consideration, we immediately get

$$(28) \qquad \begin{aligned} T(x+y)^2 \ = \ (T(x)+T(y))^2 \ \text{ or } \ T(x+y)^2 = (T(x)-T(y))^2, \\ \text{for all } x, y \in E. \end{aligned}$$

The second equality in (28) can take place, e.g., when both $I_y^+$ and $I_y^-$ are infinite sets and $I = I_x^+ \cap I_y^- \cap I_{x+y}^+$ for some $x, y \in E$, because (27) and $I = I_x^+ \cap I_y^- \cap I_{x+y}^+$ lead to

$$\lim_{\substack{n \to \infty \\ n \in I}} \frac{1}{2^n} f(2^n(x+y)) = \lim_{\substack{n \to \infty \\ n \in I_{x+y}^+}} \frac{1}{2^n} f(2^n(x+y)) = T(x+y),$$

$$\lim_{\substack{n \to \infty \\ n \in I}} \frac{1}{2^n} f(2^n x) = \lim_{\substack{n \to \infty \\ n \in I_x^+}} \frac{1}{2^n} f(2^n x) = T(x)$$

and

$$\lim_{\substack{n \to \infty \\ n \in I}} \frac{1}{2^n} f(2^n y) = \lim_{\substack{n \to \infty \\ n \in I_y^-}} \frac{1}{2^n} f(2^n y) = -T(y).$$

$T(x) = 0$, for al $x \in E$, is not only the unique solution of the second functional equation in (28) but also a solution of the first equation in (28). Hence, it holds

$$T(x+y)^2 = (T(x)+T(y))^2 \quad \text{or} \quad |T(x+y)| = |T(x)+T(y)|, \quad \text{for all } x, y \in E.$$

According to Remark in section 3, $T$ is an additive mapping.

Now, suppose $U : E \to \mathbf{R}$ to be another additive mapping which satisfies (21). Since $T$ and $U$ are additive mappings, it is clear that

$$T(nx) = nT(x) \text{ and } U(nx) = nU(x)$$

for all $n \in \mathbf{N}$ and any $x \in E$. Hence, by (21), we get

$$\begin{aligned} |T(x)^2 - U(x)^2| &= \frac{1}{n^2}|T(nx)^2 - U(nx)^2| \\ &\leq \frac{1}{n^2}\left(|T(nx)^2 - f(nx)^2| + |f(nx)^2 - U(nx)^2|\right) \\ &\leq \frac{2}{n^2}\frac{\delta}{3} \\ &\to 0, \text{ as } n \to \infty, \end{aligned}$$

which completes the proof of (22). □

ACKNOWLEDGMENT. The author would like to thank Professor Themistocles M. Rassias for his kind advice.

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