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REAL PROJECTIVE STRUCTURES ON THE (2,2,2,2)-ORBIFOLD

JINHA JUN

ABSTRACT. The (2,2,2,2)-orbifold is a 2-dimensional orbifold with four order 2 cone points having 2-sphere as an underlying space. The (2,2,2,2)-orbifold admits different geometric structures. The purpose of this paper is to find some real projective structures on the (2,2,2,2)-orbifold.

1. Introduction

When a group Γ acts properly discontinuously but do not necessarily act freely on a space X, the quotient space X/Γ is called *orbifold.* Orbifold was first introduced by I. Satake in the name of *V-manifold.* In section 3, we give the precise definition of the orbifold and discuss its geometric structures. There are many reasons to study the orbifolds. 2-dimensional orbifolds occur naturally in the study of 3-dimensional manifolds, e.g., Seifert fibered spaces. In [T1], Thurston gave a quite complete treatment of the two dimensional case, and raised many interesting questions.

2. (X,G)-manifolds

Let X be a manifold and G a Lie group acting (transitively) on X. Let M be a manifold of the same dimension as X. An (X, G)-*atlas* on M is a pair $(\mathcal{U}, \Phi)$ where $\mathcal{U}$ is an open covering of M and $\Phi = \{\phi_\alpha : U_\alpha \to X\}_{U_\alpha \in \mathcal{U}}$ is a collection of coordinate charts such that for each pair

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$(U_\alpha, U_\beta) \in \mathcal{U} \times \mathcal{U}$ and connected components C of $U_\alpha \cap U_\beta$ there exists $g_{C,\alpha,\beta} \in G$ such that $g_{C,\alpha,\beta} \circ \phi_\alpha = \phi_\beta$. An (X, G)-*structure* on M is a maximal (X, G)-atlas and an (X, G)-*manifold* is a manifold together with an (X, G)-structure on it. Suppose that M and N are two (X, G)-manifolds and $f : M \to N$ is a map. Then f is an $(X, G)-$*map* if for each pair of charts $\phi_\alpha : U_\alpha \to X$ and $\psi_\beta : V_\beta \to X$ (for M and N respectively) and a component C of $U_\alpha \cap f^{-1}(V_\beta)$ there exists $g = g(C, \alpha, \beta) \in G$ such that the restriction of f to C equals $\psi_\beta^{-1} \circ g \circ \phi_\alpha$. There is a useful globalization of the coordinate charts of a geometric structure in terms of the universal covering space and the fundamental group. The proof of the following basic result can be found in Goldman [G2].

DEVELOPMENT THEOREM. *Let M be an (X, G)-manifold with universal covering space $p : \tilde{M} \to M$ and group of deck transformation $\pi = \pi_1(M)$. Then there exists a pair $(\mathbf{dev}, h)$ such that $\mathbf{dev} : \tilde{M} \to X$ is an immersion and $h : \pi \to G$ is a homomorphism such that, for each $\gamma \in \pi$,*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X \end{array}$$

commutes. Furthermore if $(\mathbf{dev}', h')$ is another such pair, there exists $g \in G$ such that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h'(\gamma) = g h(\gamma) g^{-1}$ for each $\gamma \in \pi$.

We say that such a pair (**dev**,h) is a *development pair*, and **dev** *the developing map* and the homomorphism h *a holonomy representation.*

3. Orbifold

An *n-dimensional orbifold* (without boundary) is defined to be a space equipped with a covering by open sets $\{U_i\}$ closed under finite intersections. To each U_i is associated a finite group Γ_i, an action of Γ_i on an open subset $\tilde{U}_i$ of $\mathbb{R}^n$, a homeomorphism $\phi_i : \tilde{U}_i/\Gamma_i \to U_i$. Whenever $U_i \subset U_j$, there is to be an inclusion $f_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ equivariant with respect to f_{ij} such that the following diagram commutes (see Scott [Sc]):

$$\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\tilde{\phi}_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/\Gamma_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j/f_{ij}\Gamma_i \\
 & & \downarrow \\
\phi_i \downarrow & & \tilde{U}_j/\Gamma_j \\
 & & \downarrow \phi_j \\
U_i & \subset & U_j
\end{array}$$

A *covering orbifold* of an orbifold $\mathcal{O}$ is an orbifold $\tilde{\mathcal{O}}$ with a projection $p : X_{\tilde{\mathcal{O}}} \to X_{\mathcal{O}}$ between the underlying spaces, such that p is a *local covering* , that is, each point $x \in X_{\tilde{\mathcal{O}}}$ in the domain has a neighborhood $U = \tilde{U}/\Gamma$ (where $\tilde{U}$ is an open subset of $\mathbb{R}^n$) such that p restricted to U is isomorphic to a map $\tilde{U}/\Gamma \to \tilde{U}/\Gamma'(\Gamma \subset \Gamma')$ and p is an *even covering*, that is, each point $x' \in \mathcal{O}$ in the range has a neighborhood $V = \tilde{V}/\Gamma$ for which each component U_i of $p^{-1}(V)$ is isomorphic to $\tilde{V}/\Gamma_i$, where $\Gamma_i \subset \Gamma$ is some subgroup. The isomorphism must respect the projections.

$$\begin{array}{ccc}
U & \xleftarrow{\cong} & \tilde{U}/\Gamma \\
p\downarrow & & \downarrow q \\
U' & \xleftarrow[\cong]{} & \tilde{U}'/\Gamma'
\end{array}
\qquad\qquad
\begin{array}{ccc}
U_i & \xleftarrow{\cong} & \tilde{V}/\Gamma_i \\
p\downarrow & & \downarrow q \\
V & \xleftarrow[\cong]{} & \tilde{V}/\Gamma
\end{array}$$

Similarly to (X, G)-structures on manifolds, we can define locally homogeneous geometries on orbifolds by using in the definition of orbifolds all the mappings and group actions related to (X, G)-category. In that sense, we can speak about $(X,G\ )$*-orbifold.*

The *cone point of order* n of a 2-dimensional orbifold means a point whose neighborhood is modeled on $\mathbb{R}^2/\mathbb{Z}_n$ with $\mathbb{Z}_n$ acting by rotation of order n. The (2,2,2,2)-*orbifold* is a 2-dimensional orbifold with four order two cone points. We will denote it by $S^2(2,2,2,2)$. As defined in Scott [Sc], the Euler number of $S^2(2,2,2,2)$ is zero. Then it is known that our orbifold has a Euclidean structure and the Euclidean plane E^2

is the universal covering space (see Thurston [T1]). For convenience, we will write $\mathrm{M} = \mathrm{S}^2(2,2,2,2)$ and $\widetilde{\mathrm{M}} = \mathrm{E}^2$. Every Euclidean structure is a similarity structure which induces an obvious affine structure. Similarly every affine structure determines a projective structure, using embedding $(\mathbb{R}^n, \mathrm{Aff}(\mathbb{R}^n)) \rightarrow (\mathbb{RP}^n, \mathrm{PGL}(n+1,\mathbb{R}))$. According to the Development Theorem, we can deduce that there exists a developing map $\mathbf{dev}\colon \widetilde{\mathrm{M}} \rightarrow \mathbb{RP}^2$.

To express the developing image clearly, we lift $\mathbf{dev}\colon \widetilde{\mathrm{M}} \rightarrow \mathbb{RP}^2$ to the universal covering $\widetilde{\mathbf{dev}} : \widetilde{\mathrm{M}} \rightarrow \mathrm{S}^2$. The universal covering space S^2 of $\mathbb{RP}^2$ is realized geometrically as the *sphere of directions* in $\mathbb{R}^3$. Furthermore the group of lifts of $\mathrm{PGL}(3,\mathbb{R})$ to S^2 equals the quotient

$$\mathrm{GL}(3,\mathbb{R})/\mathbb{R}^+ \cong \mathrm{SL}_\pm(3,\mathbb{R}) = \{\mathrm{A} \in \mathrm{GL}(3,\mathbb{R}) | \det(\mathrm{A}) = \pm 1\}.$$

Hence there exists a lift of the holonomy map $\mathrm{h} : \pi_1(\mathrm{M}) \rightarrow \mathrm{PGL}(3,\mathbb{R})$ to $\tilde{\mathrm{h}} : \pi_1(\mathrm{M}) \rightarrow \mathrm{SL}(3,\mathbb{R})$.

4. Main Computation

Now we find some examples of $\mathbb{RP}^2$-structures on $\mathrm{S}^2(2,2,2,2)$. Let rectangle Q be the fundamental domain of our orbifold in E^2. For computational ease we will assume the developing image of Q in S^2 has vertices at [0,0,1], [1,0,1], [1,1,1], [0,1,1] in homogeneous coordinate; i.e., v is equivalent to w if and only if $v = \lambda w$ for some $\lambda > 0$ for $v, w \in \mathbb{R}^3$. Let p_i be the midpoints of each sides in Q and R_i the order two deck transformation in S^2 fixing p_i for i=1,2,3,4. If Γ is the deck transformation group of $\widetilde{\mathrm{M}}$ with generators R_i's , then Γ admits the presentation

$$\Gamma = < \mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3, \mathrm{R}_4 \mid \mathrm{R}_1^2 = \mathrm{R}_2^2 = \mathrm{R}_3^2 = \mathrm{R}_4^2 = \mathrm{I},\ \mathrm{R}_1\mathrm{R}_2\mathrm{R}_3\mathrm{R}_4 = \mathrm{I} > .$$

We want to find A,B,C,D in $\mathrm{SL}(3,\mathbb{R})$ acting on S^2 satisfying

$$A^2 = B^2 = C^2 = D^2 = I \tag{1}$$

$$ABCD = I \tag{2}$$

$$A[0,0,1] = [1,0,1] \tag{3}$$

$$B[1,0,1] = [1,1,1] \tag{4}$$

$$C[1,1,1] = [0,1,1] \tag{5}$$

$$D[0,1,1] = [0,0,1] \tag{6}$$

The possible $A, B, C, D \in SL(3,\mathbb{R})$ satisfying the conditions (1) and (3) $\sim$ (6) are easily computed. They turn out to be

$$A = \left[\begin{pmatrix} -1 & a_1 & 1 \\ 0 & a_2 & 0 \\ a_2^2 - 1 & a_1(1-a_2) & 1 \end{pmatrix}\right]$$

with fixed points $[1, 0, 1 - a_2]$,

$$B = \left[\begin{pmatrix} -b_1 - b_2 - b_1 b_2 & b_2^2 - 1 & (1+b_1)(1+b_2) \\ -b_1 & -1 & 1 + b_1 \\ -b_1(1+b_2) & b_2^2 - 1 & 1 + b_1 + b_1 b_2 \end{pmatrix}\right]$$

with fixed points $[1 + b_2, 1, 1 + b_2]$,

$$C = \left[\begin{pmatrix} -1 & -c_1 & 1 + c_1 \\ c_2^2 - 1 & -c_1 - c_2 - c_1 c_2 & (1+c_1)(1+c_2) \\ c_2^2 - 1 & -c_1(1+c_2) & 1 + c_1 + c_1 c_2 \end{pmatrix}\right]$$

with fixed points $[1, 1 + c_2, 1 + c_2]$,

$$D = \left[\begin{pmatrix} -d_1 & 0 & 0 \\ -d_2 & -1 & 1 \\ -d_2(1+d_1) & d_1^2 - 1 & 1 \end{pmatrix}\right]$$

with fixed points $[0, 1, 1 + d_1]$.

Since $\det A = -a_2^3 > 0, \det B = b_2^3 > 0, \det C = c_2^3 > 0$ and $\det D = d_1^3 > 0$, we see that

(7) $$a_2 < 0, b_2 > 0, c_2 > 0, d_1 > 0 .$$

The fact that all R_i have order two implies (2) is equivalent to

(2′) $$CD = BA.$$

$$CD = \left[\begin{pmatrix} d_1 - d_2 - (1+c_1)d_1 d_2 & -1 + (1+c_1)d_1^2 & 1 \\ d_1(1-c_2^2) - d_2 - (1+c_1)(1+c_2)d_1 d_2 & -1 + (1+c_1)(1+c_2)d_1^2 & 1 \\ d_1(1-c_2^2) - d_2 - d_1 d_2(1+c_1+c_1c_2) & -1 + (1+c_1+c_1c_2)d_1^2 & 1 \end{pmatrix}\right]$$

$$BA = \left[\begin{pmatrix} -1 + (1+b_1)(1+b_2)a_2^2 & a_1 - a_2(1-b_2^2) - a_1a_2(1+b_1)(1+b_2) & 1 \\ -1 + (1+b_1)a_2^2 & a_1 - a_2 - a_1a_2(1+b_1) & 1 \\ -1 + (1+b_1+b_1b_2)a_2^2 & a_1 - a_2(1-b_2^2) - a_1a_2(1+b_1+b_1b_2) & 1 \end{pmatrix}\right]$$

From $(2')$, we get the following 6 equations with 8 unknowns.

$$d_1 - d_2 - (1+c_1)d_1d_2 = -1 + (1+b_1)(1+b_2)a_2^2 \tag{8}$$

$$d_1(1-c_2^2) - d_2 - (1+c_1)(1+c_2)d_1d_2 = -1 + (1+b_1)a_2^2 \tag{9}$$

$$d_1(1-c_2^2) - d_2 - d_1d_2(1+c_1+c_1c_2) = (1+b_1+b_1b_2)a_2^2 - 1 \tag{10}$$

$$-1 + (1+c_1)d_1^2 = a_1 - a_2(1-b_2^2) - a_1a_2(1+b_1)(1+b_2) \tag{11}$$

$$-1 + (1+c_1)(1+c_2)d_1^2 = a_1 - a_2 - a_1a_2(1+b_1) \tag{12}$$

$$-1 + (1+c_1+c_1c_2)d_1^2 = a_1 - a_2(1-b_2^2) - a_1a_2(1+b_1+b_1b_2) \tag{13}$$

To determine A, B, C, D, we only need to solve the above equations with $a_2 < 0$, $b_2 > 0$, $c_2 > 0$, and $d_1 > 0$.
Subtract (11) from (12), (9) from (8), (13) from (12), (9) from (10), (11) from (13), and (10) from (8) respectively.

$$c_2(1+c_1)d_1^2 = a_1a_2b_2(1+b_1) - a_2b_2^2 \tag{14}$$

$$d_1c_2^2 + c_2(1+c_1)d_1d_2 = b_2(1+b_1)a_2^2 \tag{15}$$

$$c_2d_1^2 = -a_2b_2^2 + a_1a_2b_1b_2 \tag{16}$$

$$c_2d_1d_2 = b_1b_2a_2^2 \tag{17}$$

$$c_1c_2d_1^2 = a_1a_2b_2 \tag{18}$$

$$c_2^2d_1 + c_1c_2d_1d_2 = b_2a_2^2 \tag{19}$$

Because $c_2d_1 \neq 0$ by (7),

$$c_1 = \frac{a_1a_2b_2}{c_2d_1^2} \quad \text{by (18)}, \tag{18'}$$

$$d_2 = \frac{b_1b_2a_2^2}{c_2d_1} \quad \text{by (17)}. \tag{17'}$$

Firstly, assume

(assumption 1) $$b_1 = 0$$

which implies

$$d_2 = 0 \quad \text{by } (17'), \tag{20}$$

$$c_2d_1^2 = -a_2b_2^2 \quad \text{by (16)},$$

$$d_1c_2^2 = b_2a_2^2 \quad \text{by (15) and (20)}. \tag{21}$$

Multiplying the above two equations gives

$$(22) \qquad c_2 d_1 = -a_2 b_2 \ .$$

Substituting the above into (21) gives

$$(23) \qquad c_2 = -a_2 \qquad \text{by (7)}$$

which yields

$$(24) \qquad d_1 = b_2 \qquad \text{by (22)}.$$

Thus

$$(25) \qquad c_1 = -\frac{a_1}{b_2} \qquad \text{by (18}'\text{)}.$$

Finally, put (24) and (25) into (11), we obtain

$$-1 + (1 + (-\frac{a_1}{b_2}))b_2^2 = a_1 - a_2(1 - b_2^2) - a_1 a_2(1 + b_2)$$

which becomes

$$(a_2 - 1)(b_2 + 1)(b_2 - (1 + a_1)) = 0 \ .$$

Since $a_2 < 0$ and $b_2 > 0$ by (7),

$$(26) \qquad b_2 = 1 + a_1 > 0$$

which yields in turn

$$(27) \qquad d_1 = 1 + a_1 \qquad \text{by (24)},$$

$$(28) \qquad c_1 = -\frac{a_1}{1 + a_1} \qquad \text{by (25)}.$$

Putting (20), (26), (27) and (assumption 1) into (8) gives

$$(2 + a_1) = (2 + a_1)a_2^2.$$

Because $1 + a_1 = b_2 > 0$ and $a_2 < 0$ by (7) and (26),

$$(29) \qquad a_2 = -1$$

which implies

$$(30) \qquad c_2 = 1 \qquad \text{by (23)} \ .$$

In summary, we get the solution of our 6 equations under $b_1 = 0$ as follows.

Solution (1)

$$\begin{cases} a_1 & \text{independent variable greater than -1} & \text{by (26)} \\ a_2 = -1 & & \text{by (29)} \\ b_1 = 0 & & \text{by (assumption1)} \\ b_2 = 1 + a_1 & & \text{by (26)} \\ c_1 = -\frac{a_1}{1+a_1} & & \text{by (28)} \\ c_2 = 1 & & \text{by (30)} \\ d_1 = 1 + a_1 & & \text{by (27)} \\ d_2 = 0 & & \text{by (20)} \end{cases}$$

Secondly, assume

(assumption 2) $\qquad b_1 \neq 0 .$

Note that $d_1 \neq 0$ by (7). Then (16) becomes

$$\begin{aligned} a_1 &= \frac{c_2 d_1^2}{a_2 b_1 b_2} + \frac{a_2 b_2^2}{a_2 b_1 b_2} \\ &= \frac{d_1 b_1 b_2 a_2^2}{d_2 a_2 b_1 b_2} + \frac{b_2}{b_1} \qquad \text{by (17)} \\ &= \frac{d_1 a_2}{d_2} + \frac{b_2}{b_1} . \end{aligned} \tag{31}$$

Substitute the above into (18), then we get

$$\begin{aligned} c_1 &= \frac{a_1 a_2 b_2}{c_2 d_1^2} = (\frac{d_1 a_2}{d_2} + \frac{b_2}{b_1})\frac{a_2 b_2}{c_2 d_1^2} \\ &= \frac{a_2^2 b_2}{c_2 d_1 d_2} + \frac{b_2^2 a_2}{b_1 c_2 d_1^2} = \frac{a_2^2 b_2}{b_1 b_2 a_2^2} + \frac{b_2^2 a_2}{b_1 c_2 d_1^2} \qquad \text{by (17)} \\ &= \frac{1}{b_1} + \frac{b_2^2 a_2}{b_1 c_2 d_1^2} . \end{aligned} \tag{32}$$

Putting (17) and (32) into (19) gives

$$c_2^2 d_1 + (\frac{1}{b_1} + \frac{b_2^2 a_2}{b_1 c_2 d_1^2}) b_1 b_2 a_2^2 = b_2 a_2^2 .$$

Simplifying the above, we have

$$a_2 b_2 = -d_1 c_2 \; . \tag{33}$$

Then (31) and (32) become

$$a_1 = \frac{d_1}{d_2}\left(-\frac{c_2 d_1}{b_2}\right) + \frac{b_2}{b_1} = \frac{b_2}{b_1} - \frac{d_1^2 c_2}{d_2 b_2} \; , \tag{34}$$

$$c_1 = \frac{1}{b_1} - \frac{b_2}{b_1 d_1} \; . \tag{35}$$

Substituting the above two equations and (33) into (18) gives

$$\left(\frac{1}{b_1} - \frac{b_2}{b_1 d_1}\right) c_2 d_1^2 = \left(\frac{b_2}{b_1} - \frac{c_2 d_1^2}{b_2 d_2}\right)(-d_1 c_2) \; ,$$

which simplified as

$$b_2 d_2 = b_1 c_2 d_1 \; . \tag{36}$$

Thus

$$a_1 = \frac{b_2}{b_1} - \frac{d_1}{b_1} \qquad \text{by (34)} \; . \tag{37}$$

Putting (35) and (36) into (8) gives

$$\begin{aligned}
& b_1 c_2 d_1 (c_2 d_1 + b_2 c_2 d_1 + b_2 + b_2 d_1) \\
& \qquad - ((c_2 d_1 + d_1 + 1) b_2^2 - (d_1 + c_2 d_1) c_2 d_1 b_2 - c_2^2 d_1^2) \\
& = b_1 c_2 d_1 (c_2 d_1 + b_2 c_2 d_1 + b_2 + b_2 d_1) \\
& \qquad - ((c_2 d_1 + d_1 + 1) b_2 + c_2 d_1)(b_2 - c_2 d_1) \\
& = (c_2 d_1 + b_2 c_2 d_1 + b_2 + b_2 d_1)(b_1 c_2 d_1 - b_2 + c_2 d_1) = 0 \; .
\end{aligned}$$

Therefore $c_2 d_1 + b_2 c_2 d_1 + b_2 + b_2 d_1 = 0$, or $b_1 c_2 d_1 - b_2 + c_2 d_1 = 0$. By (7), only the second is true. Hence

$$b_2 = c_2 d_1 (1 + b_1) \neq 0 \; . \tag{38}$$

which yields in turn

$$a_1 = \frac{c_2 d_1(1+b_1) - d_1}{b_1} = \frac{d_1}{b_1}(c_2 + c_2 b_1 - 1) \quad \text{by (37)}, \tag{39}$$

$$c_1 = \frac{d_1 - (1+b_1)c_2 d_1}{b_1 d_1} = -\frac{c_2 + b_1 c_2 - 1}{b_1} \quad \text{by (35)}, \tag{40}$$

$$a_2 = -\frac{c_2 d_1}{b_2} = -\frac{1}{1+b_1} \quad \text{by (33)}, \tag{41}$$

$$d_2 = \frac{b_1 c_2 d_1}{c_2 d_1(1+b_1)} = \frac{b_1}{1+b_1} \quad \text{by (36)}. \tag{42}$$

Then (12) becomes

$$\begin{aligned}(b_1(1+b_1) - (1+b_1)(c_2 + b_1 c_2 - 1))(1+c_2)d_1^2 \\ - 2(1+b_1)(c_2 + b_1 c_2 - 1)d_1 - b_1(1+b_1) - b_1\end{aligned}$$

Since $c_2, d_1 > 0$ and $1+b_1 > 0$ by (41) and (7), $(1+c_2)(1+b_1)d_1+b_1+2 \neq 0$. Hence

$$(1-c_2)(1+b_1)d_1 - b_1 = 0.$$

Note that $c_2 = 1$ implies $b_1 = 0$ which is contradicted to (assumption 2). Moreover $1 + b_1 > 0$ by (41) and (7). Therefore

$$d_1 = \frac{b_1}{(1-c_2)(1+b_1)}. \tag{43}$$

Putting the above into (38) and (39) gives

$$b_2 = \frac{b_1 c_2}{1-c_2}, \tag{44}$$

$$a_1 = \frac{c_2 + c_2 b_1 - 1}{(1-c_2)(1+b_1)}, \tag{45}$$

respectively. Under $b_1 \neq 0$, we get the other solution as follows.

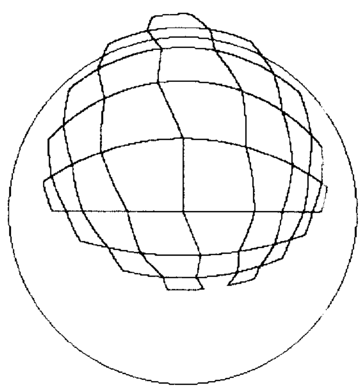
Solution (2)

$$\begin{cases} a_1 = \dfrac{c_2 + c_2 b_1 - 1}{(1-c_2)(1+b_1)} & \text{by (45)} \\ a_2 = \dfrac{-1}{1+b_1} & \text{by (41)} \\ b_1 \quad \text{independent variable greater than -1 not equal to 0} & \\ b_2 = \dfrac{b_1 c_2}{1-c_2} & \text{by (44)} \\ c_1 = -\dfrac{c_2 + b_1 c_2 - 1}{b_1} & \text{by (40)} \\ c_2 \quad \text{positive independent variable not equal to 1} & \\ d_1 = \dfrac{b_1}{(1-c_2)(1+b_1)} & \text{by (43)} \\ d_2 = \dfrac{b_1}{1+b_1} & \text{by (42)} \end{cases}$$

Note that $d_1 > 0$ implies either $-1 < b_1 < 0, c_2 > 1$ or $b_1 > 0, 0 < c_2 < 1$.

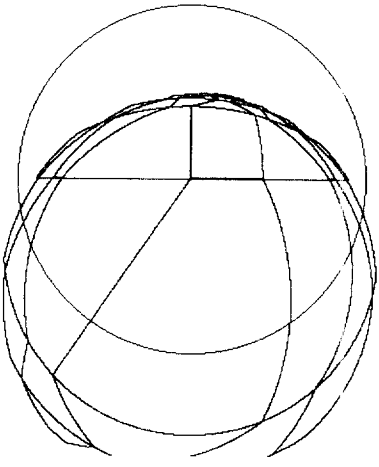
Pictured below are the developing images in E^2 using the stereographic projection from (0, 0, -1) with various choice of value of the each parameters. The equator in S^2 is drawn as the circles in the pictures.

Solution(1)
a_1 = 0.4



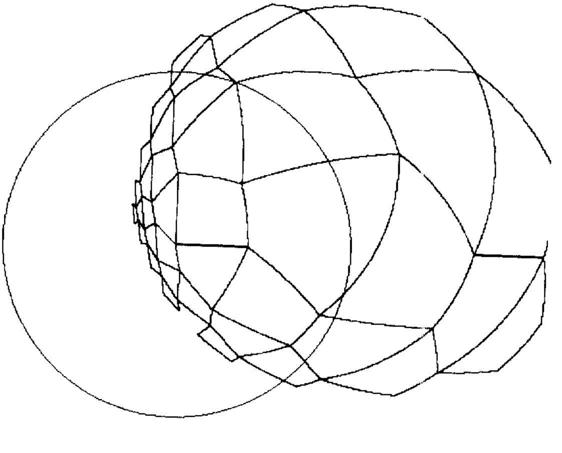
Solution(1)

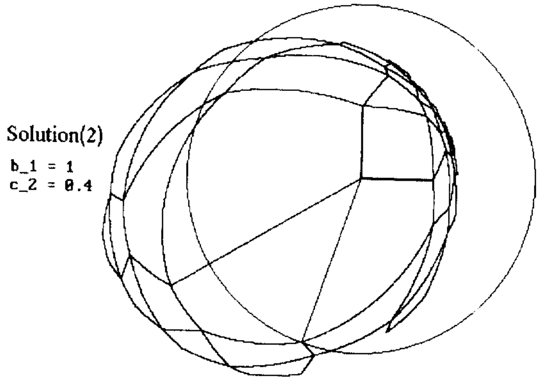
a_1 = -0.7



Solution(2)

b_1 = -.6
c_2 = 2





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DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA
E-mail: jhjun@math.snu.ac.kr