Bull. Korean Math. Soc. **34** (1997), No. 4, pp. 525–533

SPECTRAL SUBSPACES FOR COMPACT GROUP ACTIONS ON C^*-ALGEBRAS

SUN YOUNG JANG

ABSTRACT. We analysis the spectral subspaces of C^*-algebra for a compacr group action. And we prove the condition that the fixed point algebra of the product action is the tensor product of the fixed point algebras.

1. Introduction

In the study of C^*-dynamical systems one of important tasks is the analysis of the structure of C^*-crossed products by a continuous group G. But the known facts on this problem are very limited (See [3], [4], and [6]). When G is a compact group or an abelian group, the spectral theory of group automorphisms plays a some role to analysis the structures of C^*-crossed products. In this paper we try to add a little more informations on the spectral theory of a C^*-dynamical system (A, G, α) when G is a compact group. First we introduce the spectrum of the action when the group G is a locally compact abelian group. Let G be a locally compact abelian group with Haar measure dg and $\widehat{G}$ be the dual group of G, i.e. the set of all unitary characters with the dual group. A triple (A, G, α) is a C^*-dynamical system where A is a C^*-algebra and $\alpha : G \to Aut(A)$ is a strongly continuous action.

For each $f \in L^1(G)$ we define a map α_f from A to A as

$$\alpha_f(x) = \int_G f(g)\alpha_g(x)dg \qquad x \in G.$$

Received June 17, 1996.

1991 Mathematics Subject Classification: 46L55, 46L05.

Key words and phrases: C^*-dynamical system, spectral subspaces, spectrums, fixed point algebras.

Partially supported by the Basic Science Research Institute Program, MOE, 1994. Partially supported by GARC-KOSEF, 1995.

For a subset Y of A we put

$$I_Y^\alpha = \{f \in L^1(G) |\ \alpha_f(x) = 0, \quad x \in Y\}.$$

Then I_Y^α is an ideal of $L^1(G)$. The α-spectrum of Y, $Spec^\alpha(Y)$ is defined by

$$Spec^\alpha(Y) = \{\gamma \in \widehat{G} \mid \hat{f}(\gamma) = 0, \quad f \in I_Y^\alpha\}$$

where $\hat{f}(\gamma) = \int_G \gamma(g) f(g) dg$. The Arveson spectrum of α, $Sp(\alpha)$ is defined by

$$Sp(\alpha) = \{\gamma \in \widehat{G} \mid \hat{f}(\gamma) = 0, \quad f \in I_A^\alpha\}.$$

For a subset of E of $\widehat{G}$, the spectral subspace $A^\alpha(E)$ is defined by

$$A^\alpha(E) = \text{the norm closure of } \{x \in A \mid Spec^\alpha(x) \subset E\}.$$

The set

$$A_F^\alpha = \{x \in A |\ Spec^\alpha(x) \text{ is compact in } \widehat{G}\}.$$

is called the algebra of G-finite elements. Next we consider a compact group G with the normalized Haar measure dg. For each $\gamma \in \widehat{G}$, the space of equivalence classes of irreducible unitary representations of G, we denote by H_γ the finite dimensional Hilbert space which γ acts on. We put $d(\gamma) =$ the dimension of H_γ and fix a matrix representative

$$\gamma(g) = [\gamma_{ij}(g)]_{i,j=1}^{d(\gamma)}.$$

For each $\gamma \in \widehat{G}$, define the linear map $P_\gamma : A \rightarrow A$ by

$$P_\gamma(x) = \int_G d(\gamma) \overline{Tr((\gamma(g))} \alpha_g(x) dg \qquad x \in A.$$

Then P_γ is a projection, and the range $A^\alpha(\gamma)$ of P_γ, i.e., $\{a \in A \mid P_\gamma(x) = x\}$, is called the spectral subspace of A associated with γ. Especially if γ is trivial, P_γ is denoted by P_0 which becomes the conditional expectation from A onto the fixed point algebra A^α. We put $A_F^\alpha =$ the linear span of $\{x \in A^\alpha(\gamma) \ |\gamma \in \widehat{G}\}$ the algebra of G-finite elements, and call elements of A_F^α G-finite elements of A.

Landstad [5] and Peligrad [6] observed another spectral subspace

$$A_2^\alpha(\gamma) = \{x \in A \otimes B(H_\gamma) |\ x(I_A \otimes \gamma_g) = (\alpha_g \otimes id)(x), \qquad g \in G\}$$

for an element $\gamma \in \widehat{g}$. These spectral subspaces are more useful for studying the properties and ideal structures of the crossed product algebra. If G is abelian, $A_2^\alpha(\gamma)$ is equal to $A^\alpha(\gamma)$.

Gootman, Lazar, and Peligard [2] defined the spectrum of α as follows;

$$Sp(\alpha) = \{\gamma \in \widehat{G} | \overline{A_2^\alpha(\gamma)^* A_2^\alpha(\gamma)} \text{is an essential ideal in} (A \otimes B(H_\gamma))^{\alpha \otimes ad_\gamma}\},$$

$$\widetilde{Sp(\alpha)} = \{\gamma \in \widehat{G} | \overline{A_2^\alpha(\gamma)^* A_2^\alpha(\gamma)} = (A \otimes B(H_\gamma))^{\alpha \otimes ad_\gamma}\}$$

where $\overline{(\ )}$ means the closure of ( ).

2. Main Result

Let A be a C^*-algebra and ϕ be a faithful state on A. Then we can define an inner product $\langle\ ,\ \rangle_\phi$ on A by letting

$$\langle a, b \rangle_\phi = \phi(b^* a)$$

for all $a, b \in A$. Let H_ϕ denote the completion of A in this inner product. Regard A as a subspace imbedded in the Hilbert space H_ϕ.

Let G be a compact group, and γ and σ be irreducible matricial unitary representations of compact group G. Let $\gamma_{ij}(g)$ and $\sigma_{ij}(g)$ be the (i,j)-element of the matrices γ_g and σ_g respectively. Then the inner products in $L^2(G)$ between matricial elements are given by

$$\langle \sigma_{ij}, \gamma_{kl} \rangle = \begin{cases} 0, & \text{if } \sigma \text{ is inequivalent to } \gamma, \\ d(\sigma)^{-1} \delta_{ik} \delta_{jl}, & \text{if } \sigma \simeq \gamma. \end{cases}$$

PROPOSITION 2.1. *Let G be a compact group and (A, G, α) be a C^*-dynamical system. Then*

(1) *The spectral subspace $A_2^\alpha(\gamma)$ is invariant under $\alpha \otimes ad_\gamma$.*
(2) *If $\gamma \in Sp(\alpha)$, then $A^\alpha(\gamma) \neq 0$*

Proof. For $V \in A \otimes B(H_\gamma)$ V can be expressed as $V = \sum_{i,j=1}^{d(\gamma)} v_{ij} \otimes E_{ij}$, where $\{E_{ij} | i,j = 1, \ldots, d(\gamma)\}$ is the cannonical matrix unit of $B(H_\gamma)$. We have for each g and $t \in G$,

$$(\alpha_t \otimes id)(\alpha_g \otimes ad_\gamma)(\sum_{i,j=1}^{d(\gamma)} v_{ij} \otimes E_{ij})$$

$$= (\alpha_t \otimes id)(I_A \otimes \gamma_g)(\sum_{i,j=1}^{d(\gamma)} v_{ij} \otimes E_{ij})$$

$$= (I_A \otimes \gamma_g)(\sum_{i,j=1}^{d(\gamma)} \alpha_t(v_{ij} \otimes E_{ij}))$$

$$= (I_A \otimes \gamma_g)(\sum_{i,j=1}^{d(\gamma)} \alpha_g(v_{ij} \otimes E_{ij}))(I \otimes \gamma_g^*)(I_A \otimes \gamma_t)$$

$$= (\alpha_g \otimes ad_\gamma)(\sum_{i,j=1}^{d(\gamma)} v_{ij} \otimes E_{ij})(I_A \otimes \gamma_t).$$

It follows that $\alpha \otimes ad_\gamma(A_2^\alpha(\gamma)) \subset A_2^\alpha(\gamma)$. If $\gamma \in Sp(\alpha)$, then $A_2^\alpha(\gamma) \neq \{0\}$. For each $V = [v_{ij}] \in A_2^\alpha(\gamma)$

$$\begin{aligned} P_\gamma(v_{ij}) &= \int_G d(\gamma)\overline{Tr(\gamma(g))}\alpha_g(v_{ij})dg \\ &= \int_G d(\gamma)\overline{Tr(\gamma(g))} \sum_{k=1} v_{ik}\gamma_{kj} dg \\ &= v_{ij}. \end{aligned}$$

Since every entry of $V = [v_{ij}]$ is contained in $A^\alpha(\gamma)$. □

THEOREM 2.2. *Let A be a unital C^*-algebra and (A, G, α) be a C^*-dynamical system. Let G be a compact group and ϕ be a faithful α-invariant state on A.*

(1) *If γ and σ are not inequivalent, then the spectral subspace $A^\alpha(\gamma)$ and $A^\alpha(\sigma)$ are mutually orthogonal with respect to the inner product $\langle\ ,\ \rangle_\phi$.*

(2) *For each $x \in A$, x can be converged by the elements whose orbits are finite dimensional and mutually orthogonal with respect to $\langle\ ,\ \rangle_\phi$.*

(3) *$A^\alpha(\gamma)$ doesn't contain non-zero positive element for a non-trivial representation γ.*

Proof. For each $x \in A^\alpha(\gamma)$ there exists a family of irreducible subspaces $V_1(\gamma), \dots, V_{n_x}(\gamma)$ of A such that $\dim(V_i(\gamma)) = d(\gamma)$, $x \in \sum \oplus V_i$ (γ) and $\alpha|_{V_i(\gamma)}$, which means that $(\alpha|_{V_i(\gamma)})_g = \alpha_g|_{V_i(\gamma)}$ for all $g \in G$, is equivalent to γ for each $\gamma \in \widehat{G}$. We can choose $x_{11}, \dots, x_{1d(\gamma)}$ in $V_1(\gamma)$ such that they form an orthonormal basis for V_1 with respect to $\langle\ ,\ \rangle_\phi$. P_{V_1} be a projection from A onto $V_1(\gamma)$ defined by

$$P_{V_1}(x) = \sum_{i=1}^{d(\gamma)} \langle x, x_{1i} \rangle x_{1i} \qquad x \in A.$$

Since $V_1(\gamma)$ is α-invariant subspace of A, $(id - P_{V_1})(A^\alpha(\gamma))$ is closed α -invariant subspace of A orthogonal to $V_1(\gamma)$. We can choose a orthonomal basis $x_{21}, \dots, x_{2d(\gamma)}$ of V_2. We define a projection P_{V_2} from A onto V_2 as above

$$P_{V_2}(x) = \sum_{i=1}^{d(\gamma)} \langle x, x_{2i} \rangle x_{2i} \qquad x \in A.$$

We consoder $(id - (P_{V_1} + P_{V_2}))(A^\alpha(\gamma)$ and proceed the same way as above. So $V_1, \dots, V_{d(\gamma)}$ is mutually orthogonal with respect to $\langle\ ,\ \rangle_\phi$. For inequivalent unitary representations γ and σ in $\widehat{G}$, choose any elements x and y in $A^\alpha(\gamma)$ and $A^\alpha(\sigma)$ respectively. We may assume that

$$\alpha_g(x) = \sum_{i=1}^{n_x} \sum_{p,j=1}^{d(\gamma)} c_{ij} \gamma_{pj}(g) x_{ip},$$

$$\alpha_g(y) = \sum_{r=1}^{n_y} \sum_{q,s=1}^{d(\sigma)} d_{rs} \sigma_{qs}(g) y_{rq},$$

where $x \in \sum \oplus V_r(\sigma)$, $\{x_{i1}, x_{i2}, \ldots, x_{id(\gamma)}\}$ is an orthonormal basis of $V_i(\gamma)$, $y \in \sum \oplus V_r(\sigma)$ and $\{y_{r1}, \ldots, y_{rd(\sigma)}\}$ is an orthonormal basis for $V_r(\sigma)$. Since ϕ is α-invariant, we have for all $x \in A$

$$\phi \circ P_0(x) = \int_G \phi(\alpha_g(x))dg = \phi(x).$$

Hence we get by the orthogonality relations,

$$\begin{aligned}\langle x, y\rangle_\phi &= \int_G \phi(\alpha_g(y^*x))dg \\ &= \phi\Big(\sum \int_G c_{ij} d_{rs} \sigma_{qs}^{-1}(g)\gamma_{pj}(g) y_{rq}^* x_{ip} dg\Big) = 0.\end{aligned}$$

Hence $A^\alpha(\gamma)$ and $A^\alpha(\sigma)$ are mutually orthogonal. Since A_F^α is a dense subspace, it follows from the above that 1) and 2) hold. Now let x be a non-zero positive element in $A^\alpha(\gamma)$ for a non-trivial representation γ in $\widehat{G}$ and y be I_A. Since y exists in A^α, by the above computation, we have $\phi(x) = 0$. Since ϕ is faithful, $x = 0$. Thus the spectral subspace $A^\alpha(\gamma)$ has no non-zero positive element. □

REMARK 2.3. We have the similar result when G is a locally compact abelian group. Let A be a unital C^*-algebra. If ϕ is a faithful α-invariant state on A, then the spectral subspace $A^\alpha(\gamma)$ and $A^\alpha(\sigma)$ are mutually orthogonal with respect to the inner product $\langle\ ,\ \rangle_\phi$ for inequivalent unitary representations γ and σ in $\widehat{G}$. For by the Tauberian theorem, we have for any $x \in A^\alpha(\gamma)$ and $y \in A^\alpha(\sigma)$

$$\phi(x^*y) = \phi(\alpha_g(x^*y)) = \overline{\gamma(g)}\sigma(g)\phi(x^*y).$$

COROLLARY 2.4. *Let (A, G, α) be topologically transitive and G be a compact group. Then there exists a faithful α-invariant state ϕ on A, the spectral subspace $A^\alpha(\gamma)$ has no non-zero positive element for each non-trivial element γ in $\widehat{G}$, and the spectral subspaces $A^\alpha(\gamma)$ and $A^\alpha(\sigma)$ are orthogonal with respect to the inner product $\langle\ ,\ \rangle_\phi$ for inequivalent elements γ and σ in $\widehat{G}$.*

Proof. By Corollary 2.3 of [7] ϕ is a unique α-invariant state. Hence A^α has only one state, say, $\widetilde{\phi}$. Since P_0 is faithful and $\widetilde{\phi}$ is faithful on A^α, $\phi = \widetilde{\phi} \circ P_0$ is also faithful. Then the result follows from Theorem 2.2. □

Next we are going to consider the spectral subspace of the product actions. For two C^*-algebras A and B, $A \otimes B$ denotes the C^*-tensor product of A and B with respect to some C^*-cross norm.

THEOREM 2.5. *Let (A, G, α) and (B, G, β) be C^*-dynamical systems and G be a compact group. Then the fixed point algebra $(A \otimes B)^{\alpha\otimes\beta}$ of $A \otimes B$ under the product action $\alpha \otimes \beta$ of G is the closed linear span of $P_0^{\alpha\otimes\beta}(A^\alpha(\bar{\gamma}) \otimes B^\beta(\gamma))$.*

Proof. Since A_F^α and B_F^β are dense in A and B respectively and $P_0^{\alpha\otimes\beta}$ is of norm 1, the result follows from the following computation. Choose any elements x and y in $A^\alpha(\gamma)$ and $B^\beta(\sigma)$ respectively. As in the proof of Theorem 2.2 we may assume that

$$x = \sum_{i=1}^{n_x} \sum_{j=1}^{d(\gamma)} c_{ij} x_{ij}, \qquad \alpha_g(x) = \sum_{i=1}^{n_x} \sum_{j,p=1}^{d(\gamma)} c_{ij} \gamma_{pj}(g) x_{ip}$$

and

$$y = \sum_{r=1}^{n_y} \sum_{s=1}^{d(\sigma)} d_{rs} y_{rs}, \qquad \beta_g(y) = \sum_{r=1}^{n_y} \sum_{s,q=1}^{d(\sigma)} d_{rs} \sigma_{qs}(g) y_{rq}.$$

We have

$$P_0^{\alpha\otimes\beta}(x \otimes y) = \sum_{i=1}^{n_x} \sum_{r=1}^{n_y} \sum_{j,s,p,q=1} \int_G c_{ij} d_{rs} \gamma_{pj}(g) \sigma_{qs}(g) x_{ip} \otimes y_{rq} dg$$

By orthogonality relations

$$P_0^{\alpha\otimes\beta}(x \otimes y) = \begin{cases} o & \text{if } \sigma \text{ is inequivalent to } \bar{\gamma} \\ d(\gamma) \sum_{i,r,p} x_{ip} \otimes y_{rp} & \text{if } \sigma \simeq \bar{\gamma}. \end{cases}$$

□

THEOREM 2.6. *Let (A, G, α) be C^*-dynamical system. If G is a compact abelian group, then $(A \otimes B)^{\alpha\otimes\beta}$ is the closed linear span of $A^\alpha(\gamma) \otimes B^\alpha(-\gamma)$ for $\gamma \in \widehat{G}$.*

Proof. If G is a compact abelian group, $A^\alpha(\gamma) = \{x \in A \mid \alpha_g(x) = \gamma(g)x\}$ for each $\gamma \in \widehat{G}$. So for each $x \in A^\alpha(\gamma)$ and $y \in B^\beta(\sigma)$

$$P_0^{\alpha\otimes\beta}(x \otimes y) = \int_G \gamma(g)\sigma(g)x \otimes y dg = \begin{cases} 0, & \text{if } \sigma \text{ is inequivalent to } \bar{\gamma} \\ x \otimes y & \text{if } \sigma \simeq \bar{\gamma}. \end{cases}$$

□

THEOREM 2.7. *Let (A, G, α) and (B, G, β) be C^*-dynamical systems of a compact abelian group G. $\mathrm{Sp}(\alpha) \cap \mathrm{Sp}(\beta) = \{$identity of $\widehat{G}\}$ if and only if the fixed point algebra $(A \otimes B)^{\alpha\otimes\beta}$ is $A^\alpha \otimes B^\beta$.*

Proof. Since G is compact, the dual group $\widehat{G}$ is discrete. So $\gamma \in \mathrm{Sp}(\alpha)$ if and only if $A^\alpha(\gamma) \neq \{0\}$. Let P_0 be a conditional expectation onto the fixed point algebra. Since P_0 is faithful, the fixed point algebra is not $\{0\}$. So the identity of $\widehat{G}$ is contained in the spectrum of the action. If $\mathrm{Sp}(\alpha) \cap \mathrm{Sp}(\beta) \neq \{$identity of $\widehat{G}\}$, then $(A \otimes B)^{(\alpha\otimes\beta)} \neq A^\alpha \otimes B^\beta$. The converse is trivial. □

COROLLARY 2.8. *Let C^*-dynamical systems (A, G, α) and (B, G, β) be ergodic and G be a compact abelian group. If $\mathrm{Sp}(\alpha) \cap \mathrm{Sp}(\beta) = \{0\}$, then the C^*-dynamical system $(A \otimes B, G, \alpha \otimes \beta)$ is also ergodic.*

References

[1] O. Bratteli, *Derivations, dissipations and group actions on C^*-algebras*, Springer-Verlag, Berlin, 1986.

[2] E. C. Gootman, A. J. Lazar, and C. Peligrad, *Spectra for compact group actions*, to appear in J. Operator Theory.

[3] A. Kishimoto and M. Takai, *On the invariant $\Gamma(\alpha)$ in C^*-dynamical systems*, Tohoku. Math. Jour. **30** (1978), 83-94.

[4] D. Olesen, G. K. Pedersen,and E. Størmer, *Compact abelian groups of automorphisms of simple C^*-algebras*, Invent. Math. **39** (1977), 53-64.

[5] M. B. Landstad, *Operator algebras and compact groups*, Proceedings, International Conference held in Neptun, Romania, 1980 **II** (1981), Pitnam,Boston, London/Melbourne, 33–47.
[6] C. Peligrad, *Locally compact group actions on C^*-algebras and compact subgroups*, J. Func. Anal. **76** (1988), 126–139.
[7] S. Y. Jang and S. G. Lee, *Topological transitive of compact actions on C^*-algebras*, Proc. A. M. S **110** (1990), 741–744.

Department of Mathematics, University of Ulsan, Ulsan 680–749, Korea