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ON AN OPTIMIZATION PROBLEM OF EVASION PARAMETERS IN MINMAX DIFFERENTIAL GAMES

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ABSTRACT. The problem of optimization in choosing of evasion parameters in differential games is considered. Existence of optimal parameters is proved and algorithm of their choice is shown. The example is cited. This work adjoins investigations [1-11].

1. Introduction

We consider quasilinear differential game described by the system of equations

$$\dot{z} = Cz + f(u, v), \tag{1.1}$$

where $z \in R^n, C$ is a constant $(n \times n)$ - matrix, $u \in P \subset R^p, v \in Q \subset R^q$ are controls of the pursuer and evader respectively, P and Q are non-empty compacts, $f(u, v)$ is a given function continuous with respect to $(u, v) \in P \times Q$ such that origin 0 of $\mathbb{R}^n$ lies in $f(P, Q)$.

The game is ended when the state z reaches given terminal set M. We suppose that M is given subspace of R^n. The goal of the pursuer is to terminate the game (1.1) by choosing a suitable measurable control function $u = u(t) \in P, t \geq 0$. The goal of the evader is to avoid from the terminal set M for all $t \geq 0$ the trajectory of (1.1), beginning at the initial point $z_0 \in R^n \setminus M$, by using of suitable mesurable control function $v = v(t) \in Q$ [1]. It is supposed that the evader by construction of the value $v(t) \in Q$ may uses the values $u(s) \in P, 0 \leq s \leq t$, equation (1.1), initial point z_0, and terminal set M. Such games are colled minmax differential games [2].

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Many papers have been devoted to the problem of evasion in linear, quasilinear and nonlinear differential games (see, for example [1-10]), where different sufficient conditions was presented. Usually these sufficient conditions ensuring the possibility of evasion are formulated in terms of some parameters, wich not determined uniquely. So the problems of optimal choice of these parameters is arisen.

In this paper definitions of optimal parameters are introduced, their existence is proved and the algorithm of choice is shown.

2. A structure of orthogonal complement

We introduce now following notations. Let L be the orthogonal complement of M in $R^n, \omega \subset L$ be some nonzero subspase, $S(\omega)$ be the unit sphere of ω with the center in origin of $R^n, \pi(\omega)$ be orthogonal projector of R^n onto $\omega, < b, c >$ be the scalar product of vectors $b \in R^n$ and $c \in R^n$, $\mid b \mid = \sqrt{< b, b >}$, dim A and $[A]$ be respectively the dimension and carrier subspace of a set $A \subset R^n, \{a\}$ be the set with unique point $a \in R^n, \omega_1 \dotplus \omega_2$ be the direct sum of subspaces $\omega_1 \subset L$ and $\omega_2 \subset L$, $B_d(\omega)$ is the ball of radius $d > 0$ with the center in $0 \in R^n$.

In addition, let $G(V)$ be the set of all subspaces of $V \subset L$ with the opening metric [12, p. 222]

$$\theta(\omega_1, \omega_2) = \max\{ \sup_{y \in S(\omega_2)} \inf_{x \in S(\omega_1)} \mid y - x \mid, \sup_{y \in S(\omega_1)} \inf_{y \in S(\omega_1)} \mid y - x \mid\},$$

where $\omega_1 \subset V$ and $\omega_2 \subset V$.

Put for every subspace $\omega \subset L(\omega \neq \{0\})$

$$T_{\omega,i} = \pi(\omega) C^{i-1} f(P, Q),$$

$$k(\omega) = \begin{cases} \min\{i : T_{\omega,i} \not\equiv \{0\}\}, \\ 0, \quad if \quad T_{\omega,i} \equiv \{0\}, i = 1, 2, ... \end{cases}$$

LEMMA 1. *Let $\omega_j, j = 1, 2, ..., p,$ be mutually orthogonal subspaces of $G(L)$ and let $\omega = \omega_1 + \omega_2 + ... + \omega_p$. Then*

$$T_{\omega,i} \subset T_{\omega_1,i} + T_{\omega_2,i} + ... + T_{\omega_p,i}, \quad i = 1, 2, ..., p.$$

The proof follows easily from the relation for orthogonal projectors :

$$\pi(\omega) = \pi(\omega_1) + \pi(\omega_2) + ... + \pi(\omega_p).$$

LEMMA 2. *For arbitrary subspace $\omega \subset L$ satisfying $k(\omega) \geq 1$, the inequality $k(\omega) \geq k(L)$ is fulfilled.*

Proof. The case $k(L) = 0$ is trivial. Let us consider the case $k(L) \geq 1$.

First we will show that there exists a subspace $\omega \subset L, \omega \neq L$, such that $k(\omega) \geq 1$. Really, in opposite case we have $k(\omega) = 0$ for all $\omega \subset L(\omega \neq L)$. Then for the orthogonal complement $\bar{\omega}$ to ω in L we have also $k(\bar{\omega}) = 0$, hence

$$T_{\omega,k(L)} = T_{\bar{\omega},k(L)} = \{0\}.$$

Last equalities and lemma 1 give us following contradiction

$$\{0\} \not\equiv T_{L,k(L)} \subset T_{\omega,k(L)} + T_{\bar{\omega},k(L)} \equiv \{0\}.$$

Let $\omega \in G(L)$ and $k(\omega) \geq 1$. First of all it should be note that from $T_{L,i} \equiv \{0\}$ follows $T_{\omega,i} \equiv \{0\}$ for all $i = 1, 2, ..., k(L) - 1$, becouse in opposite case there exists a vector $\bar{z} \in C^{i_0-1}f(P,Q)$ for some $i_0, 1 \leq i_0 \leq k(L) - 1$, such that $\pi(\omega)(L)\bar{z} \not\equiv 0$. Then from the representation $L = \omega + \bar{\omega}$ one has contradiction $0 = \pi(L)\bar{z} = \pi(\omega)\bar{z} + \pi(\bar{\omega})\bar{z} \neq 0$. So, $T_{\omega,i} \equiv \{0\}, i = 1, 2, ..., k(L) - 1$, and $T_{L,i} \not\equiv \{0\}$, that's why we have only two cases for $i = k(L)$:

$T_{\omega,i} \equiv \{0\}$ and then $k(\omega) > k(L)$,

or

$T_{\omega,i} \not\equiv \{0\}$ and then $k(\omega) = k(L)$,

both of them implies $k(\omega) \geq k(L)$. □

LEMMA 3. *Let $L_1 = [T_{L,k(L)}]$ be the carrier subspace of the compact set $T_{L,k(L)}$ and $k(L) \geq 1$. Then $k(L) = k(L_1)$. The proof of this lemma is obtained easily from lemmas 1 and 2.*

LEMMA 4. (expansion of orthogonal complement).

Let $k(L) \geq 1$ in the game (1.1). Then there exist mutually orthogonal subspaces $L_i \subset L, i = 1, 2, ..., s + 1$, such that

a) $L = L_1 + L_2 + \cdots + L_{s+1}$,

b) $k(L_1) < k(L_2) < \cdots < k(L_s), \quad k(L_{s+1}) = 0$,

c) $k(\omega) = k(L_1)$ *for every subspace* $\omega \subset L_i, i = 1, 2, ..., s, \omega \neq \{0\}$.

Proof. Put $L_1 = [T_{L,k(L)}]$. If $L_1 = L$, then the lemma's proof is finished $(s = 1, L_1 = \{0\})$. If $L_1 \subset L, L_1 \neq L$, then for orthogonal complement $\Gamma_1 (L = L_1 + \Gamma_1)$ are possible only two cases :

$$k(\Gamma_1) = 0 \qquad \text{or} \qquad k(\Gamma_1) \geq 1.$$

First case give us $L_2 = \Gamma_1$ and the proof of lemma is finished. In the second case putting $L_2 = [T_{\Gamma_1, k(\Gamma_1)}]$ we get the representation

$$\Gamma_1 = L_2 + \Gamma_2. \tag{2.1}$$

Lemmas 2 and 3 and (2.1) implies

$$k(L_2) = k(\Gamma_1) > k(L_1), \tag{2.2}$$

becouse if $k(L_2) = k(L_1)$, then in accordence with definition of number $k(L_2)$ we get

$$\pi(L_2)C^{k(L_1)-1} f(P,Q) = \pi(L_2)C^{k(L_2)-1} f(P,Q) \not\equiv \{0\}. \tag{2.3}$$

On other side L_2 is orthogonal to $L_1 = [T_{L,k(L)}]$, so from lemma 3 one has

$$\pi(L_2)C^{k(L_1)-1} f(P,Q) \equiv \{0\},$$

that contradicts with (2.3). □

Repeating all arguments for Γ_2 (see (2.1) and (2.2)) and using certain induction, we get the fulfilment of lemma's assertions a) and b).

Let us proof the assertion c).

Let ω be a subspace of L and $\omega \neq L, \omega \neq \{0\}$. First we show that $k(\omega) \geq 1$. If $k(\omega) = 0$, then from $L_i = \omega + \bar{\omega}$ we have for all $u \in P$ and $v \in Q$

$$\begin{aligned} \pi(L_i)C^{k_1-1} f(u,v) &= \pi(\omega)C^{k_1-1} f(u,v) + \pi(\bar{\omega})C^{k_1-1} f(u,v) \\ &= \pi(\bar{\omega})C^{k_i-1} f(u,v), \end{aligned} \tag{2.4}$$

where $k_i = k(L_i), i \in J = \{1, 2, ..., s\}, \bar{\omega}$ is orthogonal complement of ω in L.

Similarly (2.2) and inductively we can get the equality

$$k(L_i + \Gamma_i) = k(L_i), \quad i \in J, \tag{2.5}$$

which with (2.4) implies

$$T_{\Gamma_{i-1}, k(\Gamma_{i-1})} = \bigcup_{u \in P, v \in Q} \pi(\Gamma_{i-1}) C^{k(\Gamma_{i-1})-1} f(u,v)$$

$$(2.6) \qquad = \bigcup_{u\in P, v\in Q} [\pi(L_i)C^{k_i-1}f(u,v) + \pi(\Gamma_i)C^{k_i-1}f(u,v)]$$

$$= \bigcup_{u\in P, v\in Q} \pi(L_i)C^{k_i-1}f(u,v) = \pi(\bar{\omega})C^{k_i-1}f(P,Q)$$

that is in contradiction with the definition of L_i $(L_i = [T_{\Gamma_{i-1}}, k(\Gamma_{i-1})]$ and dim $\bar{\omega} <$ dim $L_i)$.

Thus $k(\omega) \geq 1$ and according to the lemma 2 we get $k(\omega) \geq k_i$. Further, if $k(\omega) > k_i$, then $\pi(\omega)C^{k_i-1}f(P,Q) = \{0\}$ in (2.4) and we get one more (2.6) and the contradiction with the definition of L_i. Lemma 4 is proved.

LEMMA 5. (uniqueness of the expansion of orthogonal complement). *The expansion a) of orthogonal complement L satisfying conditions b) and c) of lemma 4 is unique.*

Proof. From the proof of the lemma 4 we have

$$(2.7) \qquad L_i = [T_{\Gamma_{i-1}}, k(\Gamma_{i-1})], \Gamma_{i-1} = L_i + \Gamma_i, \quad i \in J, \quad \Gamma_0 = L.$$

Let there exists some different expansion of L :

$$(2.8) \qquad L = L_1' + L_2' + \ldots + L_m' + L_{m+1}'$$

satisfying conditions like b) and c) of lemma 4.

We shall show that

$$(2.9) \qquad L_i = L_i', \quad k(L_i) = k_i = k(L_i'), \quad s = m, \quad i \in J.$$

Let $i = 1$ and let us prove that $L_1 = L_1', k(L_1) = k(L_1')$.

Put $L = L_1 + \Gamma_1, l = L_1' + \Gamma_1'$, then from (2.5) we get

$$(2.10) \qquad k_1 = k(L_1) = k(L_1 + \Gamma_1) = k(L_1' + \Gamma_1') = k(L_1') = k_1'.$$

Let $L_1 \subset L_1'$ and dim $L_1 <$ dim L_1', then

$$(2.11) \qquad L_1' = L_1 + L_1^*, \quad L_1^* \not\equiv \{0\},$$

and by property c) of lemma 4 and (2.10) one has

$$(2.12) \qquad k(L_1^*) = k(L_1') = k(L_1) = k_1.$$

On the other hand from (2.11) by using the fact $L_1 = [T_{L,k(L)}]$ we have

$$\pi(L_1^*)C^{k_1-1}f(P,Q) = \{0\},$$

so $k(L_1^*) > k(L_1) = k_1$ that give us the contradiction with (2.12).

Let us assume that $L_1' \subset L_1$ and $L_1' \neq L_1$, then $L_1 = L_1' + \tilde{L}_1$ and from the facts $\tilde{L}_1 \subset L_1$ and $\tilde{L}_1 \not\equiv \{0\}$ by condition c) of lemma 4 we get $k(\tilde{L}_1) = k_1$. From here and (2.5) and obvious relations

$$k(\tilde{L}_1) = k_1 < k_2 = k(\Gamma_1)$$

we get

$$k_2' = k(\Gamma_1') = k(\tilde{L}_1 + \Gamma_1) = k(\tilde{L}_1) = k_1 = k_1'$$

that contradicts with the inequality $k_2' > k_1'$ (see b) of lemma 4). Thus, we proved that $L_1 = L_1'$ and $k_1 = k_1'$, hence (2.9) is true for $i = 1$. Let now

$$L_j = L_j', \quad k_j = k_j' \quad for \quad j = 2, 3, \ldots, p-1 \tag{2.13}$$

and let for the definiteness $p \leq s \leq m$.

We will prove now that

$$L_p = L_p' \quad \text{and} \quad k_p = k_p'. \tag{2.14}$$

From (2.5) we get equalities

$$k(L_p + \ldots + L_{s+1}) = K(L_p) = k_p,$$

$$k(L_p' + \ldots + L_{m+1}') = k(L_p)' = k_p', \tag{2.15}$$

and according to (2.13) we have

$$L_p + \ldots + L_{s+1} = L_p' + \ldots + L_{m+1}',$$

hence (see (2.15)) $k_p = k_p'$. The equality $L_p = L_p'$ is obtained easily by full repeating the process for $p = 1$.

For the complete proof of lemma we must show that $s = m$. Let us assume $s < m$ (the case $s > m$ is considered analogously) respectively with (2.13), then from (2.14) we get $L_{s+1} = L_{s+1}' + \ldots + L_{m+1}'$, from here and (2.5) we obtain the contradiction

$$0 = k(L_{s+1}) = k(L_{s+1}') = k_{s+1}' \geq 1,$$

wich yields $s = m$. Lemma 5 is proved. □

LEMMA 6. (on a structure of orthogonal complement).

Let in the game (1.1) there exist subspaces $L_i \subset L$, $i \in J = \{1, 2, \ldots, s\}$, satisfying the conditions

1) $L_i = [T_{\Gamma_{i-1}, k(\Gamma_{i-1})}]$, $\quad \Gamma_{i-1} = L_i + \Gamma_i$, $\quad i \in J$, $\quad \Gamma_0 = L$.

2) $1 \leq k(L_1) < k(L_2) < \cdots < k(L_s)$, $\quad k(\Gamma_s) = 0$.

Then

$$k(\omega) = k(L_i) \tag{2.16}$$

for arbitrary nonzero subspace $\omega \subset L$ *such that* $\omega \not\subset \Gamma_{i-1}$ *and* $\omega \subset \Gamma_i, i \in J$.

Proof. From condition 1) and lemma 4 using to pair Γ_{i-1} and $L_i, i \in J$, we get

$$L = L_1 + L_2 + \cdots + L_s + \Gamma_s,$$

$$T_{\Gamma_{i-1},k(\Gamma_{i-1})} = T_{L_i,k(L_i)}, \quad k(\Gamma_{i-1}) = k(L_i). \tag{2.17}$$

Let $i \in J$ be fixed and let $\omega \not\subset \Gamma_{i-1}$ and $\omega \subset \Gamma_i$.

For the fulfillment (2.16) enough to show the true of the inequality

$$\pi(\omega)C^{k_i-1}f(P,Q) \not\equiv \{0\}. \tag{2.18}$$

Really, (2.18) and lemma 2 implies

$$k(\omega) \geq k(\Gamma_{i-1}) = k(L_i) = k_i$$

and $1 \leq k(\omega) \leq k_i$, from here we obtain $k(\omega) = k_i$.

Let us proof (2.18). Let assume that left side of (2.18) is zero. Then from representations

$$\Gamma_{i-1} = L_i + \Gamma_i = \omega + \bar{\omega} \tag{2.19}$$

and formula (2.5) we have

$$\pi(L_i)C^{k_i-1}f(P,Q) = \pi(\bar{\omega})C^{k_i-1}f(P,Q),$$

and from here we get the strong inclusions ($\omega \not\subset \Gamma_{i-1}, \omega \subset \Gamma_i$)

$$\bar{\omega} \subset L_i \quad \text{or} \quad \bar{\omega} \supset L_i.$$

If $\bar{\omega} \subset L_i$, then from (2.19) we have $L_i \subset \omega$ and $\dim \bar{\omega} < \dim L_i$ ($\omega \not\subset \Gamma_i$!), that contradicts with the definition of L_i as the carrier subspace.

In the case $\bar{\omega} \supset L_i$ (2.19) implies the conclusion $\omega \subset \Gamma_i$ that contradicts with a choice of $\omega(\omega \subset \Gamma_i!)$.

Thus (2.18) is proved and so, as we note above, (2.16) is true. Lemma 6 is completely proved. □

COROLLARY 1. *The function $k(\omega), \omega \in G(L)$, has the structure*

$$k(\omega) = \begin{cases} k_i, & if \ \ \omega \subset \Gamma_{i-1}, \omega \not\subset \Gamma_i, i \in J. \\ k_{i+1}, & if \ \ \omega \subset \Gamma_i, i \in J \setminus \{s\}, \\ 0, & if \ \ \omega \subset \Gamma_s, \end{cases}$$

i.e., it is discontinuous function.

3. Optimal evasion parameters

In this part on the base of orthogonal complement's structure advantage's functions are defined and their properties are considered. Then by using of function $k(\omega)$ and advantage's functions optimal evasion parameters are introduced.

Let $X = G(L) \times R^n \times R^n \times P \times P \times Q, \quad \alpha = (\omega, l, \psi, u, v) \in X$,
Put

$$N_i(\alpha) = \langle \psi, \pi(\omega) C^{i-1} f(u,v) - l \rangle, \quad \alpha \in X,$$

$$\lambda_i(\omega) = \max_{l \in D_i} \min_{\psi \in S(\omega)} \max_{v \in Q} N_i(\alpha),$$

$$\lambda(\omega) = \lambda_{k(\omega)}(\omega), \quad k(\omega) \in I,$$

where $\omega \in G(L), I \equiv \{1, 2, ..., n\}, D_i = C^{i-1} f(P, Q), i \in I$.

DEFINITION 1. We will call the functions $\lambda_i(\omega), i \in I$, the advantage's functions.

LEMMA 7. *Functions $N_i(\alpha), i \in I$, are continuous with respect to $\alpha \in X$ and advantage's functions achieve their least upper and greatest lower bounds.*

Proof. The continuity of $N_i(\alpha)$ at the point $\alpha_0 \in X$ follows from the inequality [9]

$$| N_i(\alpha) - N_i(\alpha_0) | \leq | \psi - \psi_0 | (| l | + d) + (| \psi_0 | + d)[| l - l_0 | +$$

$$\Theta(\omega, \omega_0) + | f_i(u,v) - f_i(u_0, v_0) |],$$

where $f_i(u,v) = C^{i-1} f(u,v), d > 0$ is a constant.

Let us consider one of the advantage's functions, for example, $\lambda_{i_0}(\omega), i_0 \in I$. Since the operations of minimum and maximum on compacts save the continuity property of functions, $\lambda_{i_0}(\omega)$ is continuous at the set $G(L)$.

Due to compactness of the space $G(L)$ [12, p. 226] and established continuity we have the accessibility of bounds on $G(L)$ for the function $\lambda_{i_0}(\omega)$. □

Continuity and bounds accessibility for other advantage's functions are proved analogously.

Now we may introduce optimal evasion parameters for quasilinear game (1.1).

Put for every $i \in I$

$$F_i = \{\omega \subset L : k(\omega) = i\},$$

$$H_i = \{\omega \in G(F_i) : \dim \omega \geq i+1, \lambda_i(\omega) > 0\},$$

and $H = \bigcup\limits_{i \in I} H_i$.

DEFINITION 2. We will call a positive integer number k_0, subspace $\omega_0 \subset L$ and number $N_0 > 0$ **optimal evasion parameters** in quasilinear game (1.1) if

$$k_0 = \min_{i \in I}\{i : H_i \neq \emptyset\},$$

$$N_0 = \max_{\omega \in H_0} \lambda(\omega),$$

$$\dim \omega_0 = \min\{\dim \omega : \omega \in \quad Arg \quad \max_{\omega \in H_0} \lambda(\omega)\},$$

where $\lambda(\omega) = \lambda_{k_0}(\omega), H_0 = H_{k_0}$.

4. Main result and example

THEOREM. Let for the game (1.1) the set $H \neq \emptyset$. Then there exist optimal evasion parameters ensuring the possibility of evasion from every initial position $z_0 \in R^n \setminus M$ for all $t \geq 0$. In addition, the distances $\xi(t)$ and $\eta(t)$ from a point $z(t)$ to M and L respectively satisfy the estimate

$$\xi(t) > \begin{cases} c\xi_0^k[1+\eta(t)]^{-m} & if \;\; \xi_0 < \delta_1, \\ c\delta_1^m[1+\eta(t)]^{-m} & if \;\; \xi_0 \geq \delta_1, \end{cases} \tag{4.1}$$

here c, δ_1 and m are positive constants.

Proof. Existence of optimal evasion parameters k_0, N_0 and ω_0 follows from nonemptity of H and lemmas 4-7. Denote $\pi_0 = \pi(\omega_0), S_0 = S(\omega_0), \omega_0 \in H_0$.

Let the evasion game begins at the moment $t = 0$ from the point $z_0 \in R^n \setminus M$ and the pursuer choosed admissible control $u(t) \in P, 0 \leq t \leq 1$.

Let $l_0 \in D_0 \equiv C^{k_0-1} f(P, Q)$ be a vector such that

$$N_0 = \min_{\psi \in S_0} \min_{u \in P} \max_{v \in Q} \langle \psi, \pi_0 [C^{k_0-1} fu, v) - l_0] \rangle$$

and let $\psi = \psi(z_0) \in S_0$ be a "vector -direction", for wich $(\dim \omega_0 \geq k_0 + 1, \omega_0 \in H_0!)$

$$(4.2) \qquad < \psi(z_0), g(z_0, t) > \geq 0, t \in [0, 1],$$

where

$$g(z_0, t) = \sum_{\nu=0}^{k_0} \frac{t^\nu}{\nu!} \pi_0 C^\nu z_0 + \frac{t^{k_0}}{k_0!} \pi_0 l_0.$$

As follows from [7] the set

$$\bigcup_{s \in [0,1]} Arg \quad \max_{v \in Q} \langle \psi(z_0), \pi_0 [C^{k_0-1} f(u(s), v) - l_0] \rangle$$

contains mesurable function $\tilde{v}(s) \in Q, s \in [0, 1]$, wich may be choosed by lexicographical way.

It is clear that for all $s \in [0, 1]$ we have

$$(4.3) \qquad \langle \psi(z_0), \pi_0 [C^{k_0-1} f(u(s), \tilde{v}(s)) - l_0] \rangle \geq N_0.$$

Let the evader uses control function $\tilde{v}(s), s \geq 0$, and $z(t)$ is the solution of (1.1), corresponding $u(t)$ and $\tilde{v}(t)$ with initial condition $z(0) = z_0$. By Couchy's formula

$$z(t) = e^{tC} z_0 + \int_0^t e^{(t-s)C} f(u(s), \tilde{v}(s)) ds$$

and properties of $\omega_0, k_0 (\pi_0 C^j f(P, Q) \equiv \{0\}, j = 0, 1, ..., k_0 - 2)$

one has

(4.4)

$$\pi_0 z(t) = g(z_0, t) + \int_0^t \frac{(t-s)^{k_0-1}}{(k_0-1)!} [\pi_0 C^{k_0-1} f(u(s), \tilde{v}(s)) - \pi_0 l_0] ds + h_0(t, z_0),$$

here $h_0(z_0, t)$ satisfies inequality

$$| h_0(z_0, t) | \leq d_1(1 + \xi_0 + \eta_0) T^{k_0+1}, t \in [0, 1],$$

and $\xi_0 = \xi(0), \eta_0 = \eta(0), d_1 > 0$ is a constant.

Relations (4.2) - (4.4) imply (if $\xi_0 \leq 1$) the inequality

$$\langle \pi_0 z(t), \psi(z_0) \rangle \geq \frac{N_0 t^{k_0}}{2k_0!} \tag{4.5}$$

for all $t \in [0, \Theta_1], \Theta_1 = \Theta_1(z_0) = d_2 N_0 [1 + \eta_0]^{-1}$.

The inequality (4.5) is main one in differential evasion games, becouse it permits to proof easy avoiding of trajectory $z(t)$ from terminal set M on the interval $[0, \Theta_1]$. Really, (4.5) implies inequalities

$$| \pi_0 z(t) | \geq \langle \pi_0 z(t), \psi(z_0) \rangle \quad > 0, \quad t \in (0, \Theta_1],$$

wich mean that $z(t) \notin M$ for all $t \in [0, \Theta_1]$, i.e., local evasion process is possible.

For the moments $t \geq \Theta_1$, setting $z_1 = z(\Theta_1)$ and $\psi = \psi(z(\Theta_1))$, we can organize evasion process by using of control function $\tilde{v}(t) \in Q$ on the length $[0, \Theta_2]$, where $\Theta_2 = d_3 N_0 [1 + \eta(\Theta_1)]^{-1}$ and so on.

It is proved [11] that series $\sum_{p=1}^{+\infty} \Theta_p$ is divergence one, therefore evasion is possible for all $t \geq 0$.

The estimate (4.1) is obtained by standard way [1, 4, 5]. □

EXAMPLE. (L. S. Pontryagin's control example "with soft landing").

The behaviours of the pursuer x and the evader y are described by the equations

$$\ddot{x} + \alpha \dot{x} = 2\rho u, \qquad \ddot{y} + \beta \dot{y} = 2\sigma v, \tag{4.2}$$

where x, y, u, v belong to R^ν, $| u | \leq 1$, $\alpha \geq 0, \beta \geq 0, \rho$ and σ are positive numbers.

The game is considered to be finished if $x = y$ and $\dot{x} = \dot{y}$ (conditions of "soft landing").

After conversation to reduced coordinates

$$z_1 = \frac{1}{2}(y - x), \quad z_2 = \frac{1}{2}(\dot{y} - \dot{x}), \quad z_3 = \frac{1}{2}(\dot{y} + \dot{x}),$$

the equation (4.2) are described by the system

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -\tilde{\alpha} z_2 + \tilde{\beta} z_3 + \sigma v - \rho u \equiv Cz + f(u, v), \\ \dot{z}_3 &= \tilde{\beta} z_2 - \alpha z_3 + \sigma v + \rho u, \end{aligned} \tag{4.3}$$

where

$$z = (z_1, z_2, z_3)^T \in R^{3\nu}, \qquad \tilde{\alpha} = \frac{\alpha + \beta}{2}, \qquad \tilde{\beta} = \frac{\alpha + \beta}{2},$$

T denotes the transposition,

C is $(3\nu \times 3\nu)$ - matrix.

Terminal set $M = M_1 \bigcap M_2$, where $M_1 = \{z : z_1 = 0\}, M_2 = \{z : z_2 = 0\}$ and their orthogonal complements $L = \{z : z_3 = 0\}, L_1 = \{z : z_1 = z_3 = 0\}$ and $L_2 = \{z : z_2 = z_3 = 0\}$.

Simple calculations show that (see lemmas 4.5)

$$k(L) = k(L_1) = 1, \quad k(L_2) = 2, \quad L = L_1 + L_2,$$

$\lambda_i(\omega) \leq \sigma - \rho$ for every subspace $\omega \subset L_i, i = 1, 2$.

Further,

$$F_1 = \{\omega \in G(L) : \omega \subset L, \omega \not\subset L_2\},$$

$$F_2 = \{\omega \in G(L) : \omega \subset L_2, \omega \neq \{0\}\},$$

$$H_1 = \{\omega \in G(F_1) : \dim \omega \geq 2, \lambda_1(\omega) > 0\},$$

$$H_2 = \{\omega \in G(F_2) : \dim \omega \geq 3, \lambda_2(\omega) > 0\},$$

hence it is easy to show that $H_1 \neq \emptyset$ if $\nu \geq 2, \sigma > \rho$ and $H_2 \neq \emptyset$ if $\nu \geq 3, \sigma > \rho$.

Thus, conditions of the teorem are fulfilled if $\nu \geq 2$ and $\sigma > \rho$ and optimal evasion parameters are equal :

$k_0 = 1, N_0 = \sigma - \rho$, ω_0 is arbitrary two - dimensional subspace of L_1.

REMARK 1. According to previous papers' results on evasion differential games (see for example [4-5]) the game (4.3) is evadable in one of the following cases :

$$a)\ \nu \geq 2,\ \sigma > \rho; \qquad b)\ \nu \geq 3,\ \sigma > \rho.$$

Our theorem of evasion rejects from consideration the case b) and points to the algorithm of choice of optimal evasion parameters (case a)).

REMARK 2. The obtained results could be generalize on the l - evasion problem [11] and nonlinear differential games [5-7].

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