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FOURIER TRANSFORMATIONS OF W^{Φ}-SPACES

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ABSTRACT. The W^{Φ}-spaces generalizing the W^p-spaces due to Pathak and Upadhyay are investigated and the Fourier transformations on these spaces are continuous.

0. Introduction

Let μ be a real valued function defined on $[0,\infty)$ with the properties:

(1) $\mu(0)=0$, $\mu(t)>0$ if $t>0$, $\lim_{t\to\infty}\mu(t)=\infty$.
(2) μ is nondecreasing.
(3) μ is right continuous.

Then the real valued function M defined on $(-\infty,\infty)$ by

$$M(x)=\int_0^{|x|}\mu(t)dt$$

is called an *N-function.* We know M is continuous, convex and

$$\lim_{|x|\to 0}\frac{M(x)}{x}=0.$$

We define $v(s)=\sup_{\mu(t)\le s} t,\ \ s\ge 0$ and $\Omega(y)=\int_0^{|y|}v(s)ds$. Then Ω is an N-function and $\Omega(y)=\sup_x\{x|y|-M(x)\}$. We call (M,Ω) is a *complementary pair of N-functions.* In the sequel, let M, Ω and Φ be N-functions.

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Now the class K_M^Φ is defined as the set of all infinitely differentiable functions $\varphi(x)$ satisfying

$$\|\varphi\|_{M,q}^\Phi = \inf\{\lambda \geq 0 \mid \int_{-\infty}^{\infty} \Phi(\frac{1}{\lambda}|e^{M(ax)}\varphi^{(q)}(x)|)dx \leq 1\} < \infty$$

for each nonnegative integer q where the positive constant a depends upon the function φ. In general, K_M^Φ is not a vector space.

The space W_M^Φ is defined to be the linear convex hull of the class K_M^Φ. W_M^Φ is a Banach space with respect to the norm $\|\cdot\|_{M,q}^\Phi$. The space W_M^Φ can be regarded as the union of countably normed spaces $W_{M,a}^\Phi$ of all infinitely differentiable functions φ, which for any $\delta > 0$ satisfy

$$\|\varphi\|_{M,q,a}^\Phi = \inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi(\frac{1}{\lambda}|e^{M[(a-\delta)x]}\varphi^{(q)}(x)|)dx \leq 1\}, \quad q = 0, 1, 2, \dots .$$

The class $K^{\Omega,\Phi}$ is defined to be the set of all entire functions $\varphi(z)$, $z = x + iy$ satisfying

$$\|\varphi\|^{\Omega,k,\Phi} = \sup_y \inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi(\frac{1}{\lambda}|e^{-\Omega(by)} z^k \varphi(z)|)dx \leq 1\} < \infty, \quad k = 0, 1, 2, \dots .$$

The space $W^{\Omega,\Phi}$ is defined to be the linear convex hull of the class $K^{\Omega,\Phi}$ with the norm $\|\cdot\|^{\Omega,k,\Phi}$. The space $W^{\Omega,b,\Phi}$ is the set of all function φ in $W^{\Omega,\Phi}$ with the norm

$$\|\varphi\|^{\Omega,k,b,\Phi} = \sup_y \inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi(\frac{1}{\lambda}|e^{-\Omega[(b+\rho)y]} z^k \varphi(z)|)dx \leq 1\}, \quad k = 0, 1, 2, \dots .$$

By the monotone convergence theorem, we have $\int_{-\infty}^{\infty} \Phi(e^{M[(a-\delta)x]}|\varphi^{(q)}(x)|/\|\varphi\|_{M,q,a}^\Phi)dx \leq 1$ and $\int_{-\infty}^{\infty} \Phi(e^{-\Omega[(b+\rho)y]}|z^k\varphi(z)|/\|\varphi\|^{\Omega,k,b,\Phi})dx \leq 1$.

REMARK. In particular, if $\Phi(x) = x^p$, $1 \le p < \infty$, we have

$$W_M^{\Phi} = W_M^p,\ W_{M,a}^{\Phi} = W_{M,a}^p,\ W^{\Omega,\Phi} = W^{\Omega,p} \text{ and } W^{\Omega,b,\Phi} = W^{\Omega,b,p} \text{ [cf. 2]}$$

LEMMA 1. *Let Φ_i, $i = 1,2,3$ be N-functions such that $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \le \Phi_3^{-1}(x)$, $f_1 \in W^{\Omega,b_0,\Phi_1}$ and $f_2 \in W^{\Omega,b,\Phi_2}$. Then $f_1 f_2 \in W^{\Omega,b_0+b,\Phi_3}$ and we have*

$$\|f_1 f_2\|^{\Omega,k,b_0+b,\Phi_3} \le 2\|f_1\|^{\Omega,0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2}.$$

Proof. We may assume $\|f_1\|^{\Omega,0,b_0,\Phi_1} = \|f_2\|^{\Omega,k,b,\Phi_2} = 1$. By the convexity of Φ_3 and $\Phi_3(xy) \le \Phi_1(x) + \Phi_2(y)$ in [3], we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Phi_3\left(\frac{1}{2}|e^{-\Omega[(b_0+b+\rho)y]} z^k f_1(z) f_2(z)|\right)dx \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} \Phi_3\left(|e^{-\Omega[(b_0+\frac{\varrho}{2})y]} f_1(z)| \cdot |e^{-\Omega[(b+\frac{\varrho}{2})y]} z^k f_2(z)|\right)dx \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} \Phi_1\left(|e^{-\Omega[(b_0+\frac{\varrho}{2})y]} f_1(z)|\right) + \frac{1}{2}\int_{-\infty}^{\infty} \Phi_2\left(|e^{-\Omega[(b+\frac{\varrho}{2})y]} z^k f_2(z)|\right)dx \\
&\le \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

Thus $\|f_1 f_2\|^{\Omega,k,b+b_0,\Phi_3} \le 2\|f_1\|^{\Omega,0,b_0,\Phi_1}\|f_2\|^{\Omega,k,b,\Phi_2}$. □

THEOREM 2. *Let Φ_i, $i = 1,2,3$ be N-functions such that*

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x) \le \Phi_3^{-1}(x)$$

and $f(z)$ be an entire function satisfying

$$\|(1 + |x|^h)^{-1} f(z)\|^{\Omega,0,b_0,\Phi_1} = D_{\Phi_1} < \infty.$$

Then $\varphi f \in W^{\Omega,k,b_0+b,\Phi_3}$ for all $\varphi \in W^{\Omega,k,b,\Phi_2}$.

Proof. By the Lemma 1, we have

$$\begin{aligned}\|\varphi f\|^{\Omega,k,b_0+b,\Phi_3} &\le 2\|(1+|x|^h)^{-1}f(z)\|^{\Omega,0,b_0,\Phi_1}\|(1+|x|^h)\varphi(z)\|^{\Omega,k,b,\Omega_2} \\ &\le 2D_{\Phi_1}(\|\varphi\|^{\Omega,k,b,\Phi_2}+\|\varphi\|^{\Omega,k+h,b,\Phi_2})<\infty.\end{aligned}$$

□

We denote by $K_M^{\Omega,\Phi}$ the set of all entire functions $\varphi(z)$, $z=x+iy$ with the norm

$$\|\varphi\|_M^{\Omega,\Phi}=\sup_y \inf\{\lambda \mid \int_{-\infty}^{\infty}\Phi(\frac{1}{\lambda}|e^{[M(ax)-\Omega(by)]}\varphi(z)|)dx\le 1\}<\infty.$$

The space $W_M^{\Omega,\Phi}$ is the convex hull of $K_M^{\Omega,\Phi}$ with the norm $\|\cdot\|_M^{\Omega,\Phi}$. The space $W_M^{\Omega,\Phi}$ can also be represented as a union of countably normed linear spaces. We denote by $W_{M,a}^{\Omega,b,\Phi}$ the set of all functions belonging to the space $W_M^{\Omega,\Phi}$ with the norm

$$\|\varphi\|_{M,a}^{\Omega,b,\Phi}=\sup_y \inf\{\lambda \mid \int_{-\infty}^{\infty}\Phi(\frac{1}{\lambda}|e^{M[(a-\delta)x]-\Omega[(b+\rho)y]}\varphi(z)|)dx\le 1\}.$$

REMARK. In particular, if $\Phi(x)=x^p$, $1\le p<\infty$, we have

$$W_M^{\Omega,\Phi}=W_M^{\Omega,p} \text{ and } W_{M,a}^{\Omega,b,\Phi}=W_{M,a}^{\Omega,b,p} \quad \text{[cf. 2]}.$$

THEOREM 3. *Let Φ_i, $i=1,2,3$ be N-functions such that*

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x)\le\Phi_3^{-1}(x)$$

and $f(z)$ be an entire function satisfying

$$\sup_y \inf\{\lambda \mid \int_{-\infty}^{\infty}\Phi_1(\frac{1}{\lambda}|e^{-M(a_0x)-\Omega(b_0y)}f(z)|)dx\le 1\}=D_{\Phi_1}<\infty.$$

Then $\varphi f\in W_{M,a-a_0}^{\Omega,b+b_0,\Phi_3}$ for all $\varphi\in W_{M,a}^{\Omega,b,\Phi_2}$.

Proof. By the similar argument as the proof of Lemma 1, we have

$$\|\varphi f\|_{M,a-a_0}^{\Omega,b+b_0,\Phi_3} \leq 2D_{\Phi_1}\|\varphi\|_{M,a}^{\Omega,b,\Phi_2} < \infty.$$

Let $\wedge$ denote the Fourier transform in the sequel. □

THEOREM 4. *Let (M,Ω) be a complementary pair of N-functions. Then $\widehat{W}_{M,a}^{\Phi_1} \subset W^{\Omega,\frac{1}{a},\Phi_2}$ for any N-functions Φ_1 and Φ_2.*

Proof. Let $\varphi \in W_{M,a}^{\Phi_1}$. Then the Fourier transform $\hat{\varphi}$ of φ exists in the L^1-sense and $\hat{\varphi}$ is an entire function of $s = \sigma + i\tau$ [1]. We have

$$\begin{aligned}
&\|\hat{\varphi}\|^{\Omega,k,\frac{1}{a},\Phi_2} \\
&= \sup_{\tau}\inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda}|e^{-\Omega[(\frac{1}{a}+\rho)\tau]}s^k\hat{\varphi}(s)|)d\sigma \leq 1\} \\
&\leq \sup_{\tau}\inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda(\sigma^2+1)}|e^{-\Omega[(\frac{1}{a}+\rho)\tau]}(|s|^{k+2}+|s|^k)\hat{\varphi}(s)|)d\sigma \leq 1\} \\
&\leq \sup_{\tau}\inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda(\sigma^2+1)}|e^{-\Omega[(\frac{1}{a}+\rho)\tau]}\int_{-\infty}^{\infty}|\varphi^{(k+2)}(x)|e^{|x||\tau|}dx)d\sigma \leq 1\} \\
&\quad + \sup_{\tau}\inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda(\sigma^2+1)}|e^{-\Omega[(\frac{1}{a}+\rho)\tau]}\int_{-\infty}^{\infty}|\varphi^{(k)}(x)|e^{|x||\tau|}dx)d\sigma \leq 1\} \\
&= \inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda(\sigma^2+1)}\int_{-\infty}^{\infty} e^{M(\frac{x}{1/a+\rho})}|\varphi^{(k+2)}(x)|dx)d\sigma \leq 1\} \\
&\quad + \inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda(\sigma^2+1)}\int_{-\infty}^{\infty} e^{M(\frac{x}{1/a+\rho})}|\varphi^{(k)}(x)|dx)d\sigma \leq 1\}.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{M(\frac{x}{1/a+\rho})}|\varphi^{(k)}(x)|dx \leq \int_{-\infty}^{\infty} \Phi_1(e^{M(\frac{x}{1/a+\rho})}|\varphi^{(k)}(x)|)dx \\
&= \int_{-\infty}^{\infty} \Phi_1(e^{M[(a-\delta)x]}|\varphi^{(k)}(x)|)dx = c < \infty \quad (\frac{1}{a}+\rho = \frac{1}{a-\delta}),
\end{aligned}$$

$$\begin{aligned}
\|\hat{\varphi}\|^{\Omega,k,\frac{1}{a},\Phi_2} &\leq \inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{c_1}{\lambda(\sigma^2+1)})d\sigma \leq 1\} \\
&\quad + \inf\{\lambda \mid \int_{-\infty}^{\infty} \Phi_2(\frac{c_2}{\lambda(\sigma^2+1)})d\sigma \leq 1\} < \infty
\end{aligned}$$

because $\int_{-\infty}^{\infty} \Phi_2(\frac{c}{\sigma^2+1})d\sigma < \infty$ by an easy calculation.

Let $(\Phi, \tilde{\Phi})$ be a complementary pair of N-functions. Then a generalized version of Hölder's inequality is $\|uv\|_{L_1} \le 2\|u\|_{\Phi}\|v\|_{\tilde{\Phi}}$ where $\|u\|_{\Phi} = \inf\{\lambda | \int_{-\infty}^{\infty} \Phi(\frac{1}{\lambda}|u(x)|)dx \le 1\}$. □

THEOREM 5. *Let (M, Ω) be a complementary pair of N-functions. Then $\widehat{W}^{\Omega,b,\Phi_1} \subset W^{\Phi_2}_{M,\frac{1}{b}}$ for any N-functions Φ_1 and Φ_2.*

Proof.

$$
\begin{aligned}
&\|\hat{\varphi}\|^{\Phi_2}_{M,q,\frac{1}{b}} \\
&= \inf\{\lambda | \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda}|e^{M[(\frac{1}{b}-\delta)\sigma]}\hat{\varphi}^{(q)}(\sigma)|)d\sigma \le 1\} \\
&\le \inf\{\lambda | \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda}|e^{M[(\frac{1}{b}-\delta)\sigma]-\sigma y} \int_{-\infty}^{\infty} \frac{(|z|^{q+2} + |z|^{q})}{x^2+1}|\varphi(z)|dx|)d\sigma \le 1\} \\
&\le \inf\{\lambda | \int_{-\infty}^{\infty} \Phi_2[\frac{1}{\lambda}|e^{M[(\frac{1}{b}-\delta)\sigma]-\sigma y+\Omega[(b+\rho)y]} \cdot 2\|\frac{1}{x^2+1}\|_{\tilde{\Phi}_1} \times \\
&\qquad (\|\varphi\|^{\Omega,k+2,b,\Phi_1} + \|\varphi\|^{\Omega,k,b,\Phi_1})]d\sigma \le 1\} \\
&= \inf\{\lambda | \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda}|ce^{-M(\frac{\rho^2}{b^3}\sigma)}|)d\sigma \le 1\} < \infty
\end{aligned}
$$

because $\int_{-\infty}^{\infty} \Phi_2(|ce^{-M(\frac{\rho^2}{b^3}\sigma)}|)d\sigma \le \int_{-\infty}^{\infty} \Phi_2(ce^{-\frac{\rho^2}{b^3}|\sigma|})d\sigma < \infty$ by an easy calculation. □

COROLLARY. $\widehat{W}^{\Phi}_{M,a} = W^{\Omega,\frac{1}{a},\Phi}$, $\widehat{W}^{\Omega,b,\Phi} = W^{\Phi}_{M,\frac{1}{b}}$.

THEOREM 6. *Let (M, Ω_1) and (M_1, Ω) be two complementary pairs of N-functions. Then*

$$\widehat{W}^{\Omega,b,\Phi_1}_{M,a} \subset W^{\Omega_1,\frac{1}{a},\Phi_2}_{M_1,\frac{1}{b}}$$

for any N-functions Φ_1 and Φ_2.

Proof. Let $\varphi \in W_{M,a}^{\Omega,b,\Phi_1}$. Then we have

$$\begin{aligned}
&\|\hat{\varphi}\|_{M_1,\frac{1}{b}}^{\Omega,\frac{1}{a},\Phi_2} \\
&= \sup_{\tau} \inf\{\lambda| \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda}|e^{M_1[(\frac{1}{b}-\delta)\sigma]-\Omega_1[(\frac{1}{a}+\rho)\tau]}\hat{\varphi}(\sigma+i\tau)|)d\sigma \le 1\} \\
&\le \inf\{\lambda| \int_{-\infty}^{\infty} \Phi_2(\frac{1}{\lambda}|ce^{-M_1(\frac{\rho^2}{b^3}\sigma)}|)d\sigma \le 1\} < \infty. \qquad \square
\end{aligned}$$

COROLLARY. $\widehat{W}_{M,a}^{\Omega,b,\Phi} = W_{M_1,\frac{1}{b}}^{\Omega_1,\frac{1}{a},\Phi}$.

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