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A NECESSARY AND SUFFICIENT CONDITION FOR THE GENERALIZED REGULAR RELATION TO BE TRANSITIVE

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ABSTRACT. In this paper we define the generalized regular relations with respect to homomorphisms and find some properties of those in transformation groups. And we also investigate the equivalent conditions for the generalized regular relations to be transitive.

1. Introduction

The regular relations, which are the generalizations of proximal relations in transformation groups introduced by Yu [12]. The regular relation is reflexive, symmetric and invariant, but is not in general transitive or closed. The proximalities and regularities play an important role to characterize the transformation groups. Proximal transformation groups and regular minimal sets are closely connected with the proximal relation and regular relation, respectively. In this paper we define the generalized regular relations with respect to homomorphisms and find some properties of those in transformation groups. And we also investigate the equivalent conditions for the generalized regular relations to be transitive.

2. Preliminaries

Let (X, T, h) be a transformation group. Then X is called the *phase space* and T is called the *phase group.* If $x \in X, t \in T$, then we shall write xt instead of $h(x, t)$, when there is no danger of ambiguity. Instead of (X, T, h), we shall write (X, T).

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Let $t \in T$. A map $h^t : X \to X$ is called *t-shift* if $h^t(x) = h(x,t)$ for all $x \in X$. Let $G = \{h^t | t \in T\}$. A map $h_x : T \to X$ defined by $h_x(t) = h(x,t) = xt$ is called the *motion* of the point $x \in X$.

DEFINITION 2.1. Let (X,T) be a transformation group. A subset $A \subset X$ is called *invariant* if $AT \subset A$. A non-empty closed invariant subset $M \subset X$ is called a *minimal subset* if M does not contain a proper closed invariant subset. If X is itself minimal, X is called a *minimal transformation group* or *minimal* set.

Let $x \in X$. The set xT is called the *orbit* of the point x. It is clear that the orbit xT is the smallest invariant subset of X. If $y \in xT$, then $xT = yT$. The set $\overline{xT}$ is the smallest closed invariant set containing x, and a set M is minimal if and only if $M = \overline{xT}$ for every point $x \in M$.

Let $\{(X_i, T, h_i) | i \in I\}$ be a family of transformation groups indexed by a set I and let $X = \prod\{X_i | i \in I\}$. Let us define a map $h : X \times T \to X$ as follows. If $x =< x_i | i \in I >$ and $t \in T$, then $h(x,t) =< h_i^t(x_i) | i \in I >$. It is easy to verify that (X,T,h) is a topological transformation group.

DEFINITION 2.2. Let (X,T,h) and (Y,T,k) be topological transformation groups. A continuous map $\pi : (X,T) \to (Y,T)$(or simply, $\pi : X \to Y$) is said to be a *homomorphism* if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ {\scriptstyle h^t}\downarrow & & \downarrow{\scriptstyle k^t} \\ X & \xrightarrow{\pi} & Y \end{array} \tag{1}$$

is commutative, i.e., $\pi(xt) = \pi(x)t \quad (x \in X, t \in T)$.

If Y is minimal, the homomorphism π is always onto. Especially, if π is onto, π is called an *epimorphism.* If the homomorphism π is a homeomorphism, π is called an *isomorphism.* A homomorphism π from (X,T) onto itself (not necessarily onto) is called an *endomorphism* of (X,T). and an isomorphism $\pi : (X,T) \to (X,T)$ is called an automorphism of (X,T). The set of automorphisms of (X,T) is denoted by $A(X)$.

Let (X, T, h) be a transformation group and let R be a relation in X. Then the relation R is called *invariant* if $(x, y) \in R$ implies $(h^t(x), h^t(y)) \in R$ for all $t \in T$.

Let (X, T, h) be a transformation group with compact Hausdorff phase space X. Let X^X denote the set of all functions from X to X, provided with the topology of pointwise convergence. Then for $t \in T$, $h^t : X \to X$ is a map of X into X, hence an element of the compact Hausdorff space X^X. The *enveloping semigroup* $E(X)$, or E of (X, T, h) is the closure of $\{h^t | t \in T\}$ in X^X. The space X^X is naturally provided with a semigroup structure, if $p, q \in X^X$ then $pq : X \to X$ is such that $x(pq) = (xp)q$ $(x \in X)$.

Throughout this paper it will be assumed that the phase spaces of all transformation groups are compact Hausdorff space.

THEOREM 2.1. *Let $\pi : (X, T) \to (Y, T)$ be an epimorphism.*

(1) *There exists a unique epimorphism $\tilde{\pi}(E(X), T) \to (E(Y), T)$ such that the diagram*

$$\begin{array}{ccc} E(X) & \xrightarrow{\tilde{\pi}} & E(Y) \\ \theta_x \downarrow & & \downarrow \theta_{\pi(x)} \\ X & \xrightarrow{\pi} & Y \end{array} \tag{2}$$

commutes $(x \in X)$. Moreover $\tilde{\pi}$ is also a semigroup homomorphism.

(2) *If (Y, T) coincides with (X, T), then the map $\tilde{\pi}$ in (1) is the identity map.*

(3) *If K is a right ideal of $E(Y)$, then $\tilde{\pi}^{-1}(K)$ is a right ideal of $E(X)$. Moreover, if K is a minimal right ideal of $E(Y)$, then there exists a minimal right ideal I of $E(X)$ such that $\tilde{\pi}(I) = K$.*

(4) *If K is a minimal right ideal of $E(Y)$ and $v \in K$ is an idempotent, then there is an idempotent $u \in E(X)$ belonging to some minimal right ideal of $E(X)$ such that $\tilde{\pi}(u) = v$.*

Proof. See ([5], Theorem 1.4.21, p.30). □

A non-empty subset $I \subset E(X)$ is called a *right ideal* of $E(X)$ if $IE(X) \subset I$. A right ideal $I \subset E(X)$ is said to be a *minimal right ideal* if it does not contain proper subsets which are right ideals of $E(X)$.

Let u and v be two idempotents of $E(X)$. We say that u and v are *equivalent idempotents*, writing $u \sim v$ if $uv = u$ and $vu = v$. If $u \sim v$ and $v \sim w$, then $uw = uvw = uv = u$ and $wu = wvu = wv = w$. Thus $u \sim w$. Hence, above relation is indeed an equivalence relation.

LEMMA 2.1 [5]. *Let S be a compact space provided with a semigroup structure such that the left multiplication by each element is continuous. Then S contains an idempotent u.*

The compact Hausdorff space X carries a natural uniformity $\mathcal{U}[X]$ whose indices are all the neighborhoods of the diagonal in $X \times X$.

DEFINITION 2.3. Let (X,T) be a transformation group. Two points x and y of X are called *proximal* if for every index $\alpha \in \mathcal{U}[X]$ there exists an element $t \in T$ such that $(xt, yt) \in \alpha$. Two points, which are not proximal are called *distal*. The set of all proximal pairs of points is called the proximal relation and is denoted by $P(X,T)$ or, simply $P(X)$. (X,T) is said to be a *proximal* (resp. *distal*) *transformation group* if the proximal relation equals $X \times X$(resp. $\Delta(X)$ the diagonal of X).

LEMMA 2.2 [6]. *Let (X,T) be a transformation group, and $x, y \in X$. Then the following statements are pairwise equivalent:*

(1) $(x,y) \in P(X,T)$.
(2) *There exists $p \in E(X)$ with $xp = yp$.*
(3) *There exists a minimal right ideal I in E(X) such that $xq = yq (q \in I)$.*

Let (X,T) be a transformation group, and $x \in X$. Then x is called an almost periodic point if $\overline{xT}$ is a compact minimal subset of X.

LEMMA 2.3 [6]. *Let (X,T) be a transformation group, E its enveloping semigroup, I a minimal ideal in E, and $x \in X$. Then the* *following statements are equivalent:*

(1) *x is an almost periodic point of (X, T).*
(2) *There exists an idempotent $u \in I$ with $xu = x$.*

LEMMA 2.4 [6]. *Let $(x, y) \in p(x)$ and let (x, y) be an almost periodic point of $(X \times X, T)$. Then $x = y$.*

THEOREM 2.2. *Let (X, T) and (Y, T) be transformation groups. $\pi : (X, T) \to (Y, T)$ be an epimorphism and let $\breve{\pi} : X \times X \to Y \times Y$ be the map defined by $\breve{\pi}(x_1, x_2) = (\pi(x_1), \pi(x_2))$. Then the following statements hold:*

(1) *If $P(X, T)$ is an equivalence relation, so is $P(Y, T)$.*
(2) *$\breve{\pi} P(X, T) \subset P(Y, T)$.*
(3) *If (Y, T) is pointwise almost periodic, then $\breve{\pi} P(X, T) = P(Y, T)$.*

Proof. See ([6], Proposition 5.25, p.41). □

A minimal transformation group is said to be *regular minimal* if it is isomorphic to a minimal right ideal in the enveloping semigroup.

J. Auslander [2] proved that a minimal transformation group (X, T) is regular minimal if and only if $(x, y) \in X \times X$ implies there is an endomorphism ϕ of (X, T) such that $\phi(x)$ and y are proximal. In regular minimal transformation group, every endomorphism is , in fact, an automorphism [2].

DEFINITION 2.4 [12]. Let (X, T) be a transformation group and $x, y \in X$. Then x and y are regular if there exists a ϕ in $A(X)$ such that $(\phi(x), y) \in P(X, T)$. The set of all regular pairs in X is called the regular relation and is denoted by $R(X, T)$ or, simply $R(X)$.

Note that a minimal transfortion group (X, T) is regular minimal if and only if $R(X, T) = X \times X$.

Yu [12] showed that if $P(X, T)$ is transitive, then so is $R(X, T)$, but the converse is not always true.

THEOREM 2.3 [12]. *Let (X, T) be a transformation group. Then the following statements are equivalent:*

(1) *$R(X)$ is an equivalence relation.*

(2) *Let u and v be the equivalent idempotents in any two minimal right ideals of $E(X)$. Then $(xu, xv) \in R(X)$ for all $x \in X$.*

We will generalize the above theorem in Theorem 3.6 with respect to an epimorphism $\pi : (X,T) \to (Y,T)$.

3. The generalized regular relations

In this section, we define generalized regular relations in transformation groups.

DEFINITION 3.1. Let (X,T) and (Y,T) be transformation groups and let $\pi : (X,T) \to (Y,T)$ be a homomorphism. Two points x_1 and x_2 are said to be the regular with respect to π if $\pi(x_1)$ and $\pi(x_2)$ are regular in Y, i.e., $(\pi(x_1), \pi(x_2)) \in R(Y,T)$. The set of all regular pair with respect to π is called the regular relation with respect to π and is denoted by $R_\pi(X,T)$, or more briefly $R_\pi(X)$, that is,

$$\begin{aligned} R_\pi(X,T) &= \{(x_1,x_2) \in X \times X | (\pi(x_1), \pi(x_2)) \in R(Y,T)\} \\ &= \{(x_1,x_2) \in X \times X | (\phi\pi(x_1), \pi(x_2)) \in P(Y,T) \\ &\qquad \text{for some } \phi \in A(Y)\} \end{aligned}$$

Similarly, we also define the proximal relation with respect to π as follows:

$$P_\pi(X,T) = \{(x_1,x_2) \in X \times X | (\pi(x_1), \pi(x_2)) \in P(Y,T)\}$$

We also denote $P_\pi(X,T)$ as $P_\pi(X)$, simply. Clearly $P_\pi(X) \subset R_\pi(X)$.

REMARK 3.1. If we take $X = Y$ and $\pi = 1_X$, the identity map of X, then $R_\pi(X,T)$ coincides with $R(X,T)$.

REMARK 3.2. Given a homomorphism $\pi : (X,T) \to (Y,T)$, define

$$S = \{(x_1,x_2) \in X \times X | \pi(x_1) = \pi(x_2)\}.$$

Then

(1) S is a closed invariant equivalence relation and obviously $S \subseteq R_\pi(X)$.
(2) If (Y,T) is distal transformation group, then $S = R_\pi(X)$.
(3) If (Y,T) is a regular minimal, then $R_\pi(X) = X \times X$.

LEMMA 3.1. *Let $x \in X$, $p \in E(X)$ and φ an endomorphism of (X,T). Then $\varphi(xp) = \varphi(x)p$.*

Proof. See([1], Lemma 1, p.606). □

THEOREM 3.1. *Let (X,T) and (Y,T) be transformation groups and let $\pi : (X,T) \to (Y,T)$ be a homomorphism. The following hold:*

(1) *$R_\pi(X)$ is a reflexive, symmetric and invariant relation.*
(2) *If $E(Y)$ contains just one minimal right ideal, then $R_\pi(X)$ is an equivalence relation.*

Proof. (1): Obvious.
(2): By (1), we only show that $R_\pi(X)$ is transitive. Let I be the only minimal right ideal in $E(Y)$, and let $(x,y) \in R_\pi(X)$ and $(y,z) \in R_\pi(X)$. Then $(h\pi(x), \pi(y)) \in P(Y)$ and $(k\pi(y), \pi(z)) \in P(Y)$ for some automorphisms h and k of Y. Since $E(Y)$ contains just one minimal right ideal I, we have, from Lemma2.2, that

$$h\pi(x)p = \pi(y)p \text{ and } k\pi(y)p = \pi(z)p \tag{3}$$

for all $p \in I$. It follows that

$$hk\pi(x)p = k\pi(y)p = \pi(z)p \tag{4}$$

for all $p \in I$. Since $hk \in A(Y)$, we have $(\pi(x), \pi(z)) \in R(Y)$. Therefore $(x,z) \in R_\pi(X)$. □

THEOREM 3.2. *$R(Y)$ is an equivalence relation if and only if $R_\pi(X)$ is an equivalence relation for all homomorphisms $\pi : (X,T) \to (Y,T)$.*

Proof. Necessity. $(x,y) \in R_\pi(X)$ and $(y,z) \in R_\pi(X)$. Then $(\pi(x), \pi(y)) \in R(Y)$ and $(\pi(y), \pi(z)) \in R(Y)$. Since $R(Y)$ is an equivalence relation, $(\pi(x), \pi(z)) \in R(Y)$. This shows that $(x,z) \in R_\pi(X)$.

Sufficiency. If we take $X = Y$ and $\pi = 1_Y : Y \to Y$, then $R_\pi(X) = R(Y)$. □

LEMMA 3.2. *Let $(x,y) \in R(Y)$ and $h \in A(Y)$, then $(h(x), h(y)) \in R(Y)$.*

Proof. There exists $k \in A(Y)$ such that $(k(x), y) \in P(Y)$. Hence $k(x)p = yp$ for all p in a minimal right ideal I. We also have

$$hk(x)p = h(k(x)p) = h(yp) = h(y)p, \tag{5}$$

that is,

$$(hk(x), h(y)) = ((hkh^{-1})h(x), h(y)) \in P(Y). \tag{6}$$

Since $hkh^{-1} \in A(Y)$, we obtain $(h(x), h(y)) \in R(Y)$. □

THEOREM 3.3. *Let $\pi : (X,T) \to (Y,T)$ be an epimorphism, and let u and v be the idempotents in $E(X)$. If $R_\pi(X)$ is an equivalence relation, then $(xu, xv) \in R_\pi(X)$ for all $x \in X$.*

Proof. Let $\tilde{\pi} : E(X) \to E(Y)$ be the homomorphism induced by π. Let $\tilde{\pi}(u) = u'$ and $\tilde{\pi}(v) = v'$. Then u' and v' are idempotents in $E(Y)$. Let $x \in X$, then we have, from Lemma 2.2, that

$$(\pi(xu), \pi(x)) = (\pi(x)u', \pi(x)) \in P(Y) \subset R(Y), \tag{7}$$

and

$$(\pi(x), \pi(xv)) = (\pi(x), \pi(x)v') \in P(Y) \subset R(Y), \tag{8}$$

that is,

$$(xu, x) = R_\pi(X) \text{ and } (x, xv) = R_\pi(X). \tag{9}$$

Since $R_\pi(X)$ is an equivalence relation, we have $(xu, xv) \in R_\pi(X)$. □

Let $\pi : (X,T) \to (Y,T)$ be an epimorphism and $\tilde{\pi} : E(X) \to E(Y)$ be the induced map by π. Let u be an idempotent in $E(Y)$. Consider $\tilde{\pi}^{-1}(u)$. Since $\tilde{\pi}^{-1}(u)$ is a closed subset of the compact T_2–space $E(X)$, $\tilde{\pi}^{-1}(u)$ is also compact T_2. Then the restriction $L_p|_{\tilde{\pi}^{-1}(u)}$ of $L_p : E \to E$ is continuous. Therefore, by Lemma 2.1 $\tilde{\pi}^{-1}(u)$ contains an idempotent u^*.

THEOREM 3.4. *Let $\pi : (X,T) \to (Y,T)$ be an epimorphism and let u and v be idempotents in $E(Y)$. If $R_\pi(X)$ is an equivalence relation, then $(\pi(x)u, \pi(x)v) \in R(Y)$ for all $x \in X$.*

Proof. By the above consideration, we can find idempotents u^* and v^* in $\tilde{\pi}^{-1}(u)$ and $\tilde{\pi}^{-1}(v)$, respectively. Suppose that $R_\pi(X)$ is an equivalence relation. By Theorem 3.3, we have $(xu^*, xv^*) \in R_\pi(X)$ for all $x \in X$, that is

$$(\pi(x)u, \pi(x)v) = (\pi(xu^*), \pi(xv^*)) \in R(Y) \tag{10}$$

for all $x \in X$. □

Let I and K be minimal right ideals of $E(Y)$, and let v be an idempotent of K. Let $L_v : I \to K$ defined by $L_v(p) = vp$. Then L_v is an isomorphism.

For each $y \in Y$, the map $\theta_y^I : p \mapsto yp$ of I into Y is a homomorphism.

THEOREM 3.5. *Let $\pi : (X,T) \to (Y,T)$ be an epimorphism, and let u and v be the equivalent idempotents in any two minimal right ideals of $E(Y)$. If $(\pi(x)u, \pi(x)v) \in R(Y)$ for all $x \in X$, then $R_\pi(X)$ is an equivalence relation.*

Proof. Let $(x,y) \in R_\pi(X)$ and $(y,z) \in R_\pi(X)$. There exist automorphisms k_1 and k_2 of (Y,T) such that $(k_1\pi(x), \pi(y)) \in P(Y)$ and $(k_2\pi(y), \pi(z)) \in P(Y)$. Therefore, there exist minimal right ideals I and K of $E(Y)$ such that

$$k_1\pi(x)p = \pi(y)p \text{ and } k_2\pi(y)q = \pi(z)q \tag{11}$$

for all $p \in I$ and $q \in K$.

Since L_v is an isomorphism, for $q \in K$ there exists $p_0 \in I$ such that $L_v(p_0) = q$. Therefore,

$$k_2\pi(y)vp_0 = k_2\pi(y)q = \pi(z)q = \pi(z)vp_0. \tag{12}$$

By the hypothesis,

$$(k_2\pi(y)u, k_2\pi(y)v) \in R(Y) \text{ and } (\pi(z)u, \pi(z)v) \in R(Y). \tag{13}$$

So, there exist automorphisms h_1 and h_2 of (Y, T) such that

$$(h_1 k_2 \pi(y)u, k_2\pi(y)v) \in P(Y), \ (h_2\pi(z)u, \pi(z)v) \in P(Y). \tag{14}$$

Since

$$(h_1 k_2 \pi(y)u, k_2\pi(y)v)u = (h_1 k_2 \pi(y)u, k_2\pi(y)v) \tag{15}$$

and

$$(h_2\pi(z)u, \pi(z)v)u = (h_2\pi(z)u, \pi(z)v), \tag{16}$$

we know, from Lemma2.3, that $(h_1 k_2\pi(y)u, k_2\pi(y)v)$ and $(h_2\pi(z)u, \pi(z)v)$ are almost periodic points of $(Y \times Y, T)$. Hence we have, from Lemma2.4, that

$$h_1 k_2 \pi(y)u = k_2\pi(y)v \text{ and } h_2\pi(z)u = \pi(z)v. \tag{17}$$

Therefore

$$\begin{aligned} h_1 k_2 \pi(y)p_0 &= h_1 k_2\pi(y)up_0 = k_2\pi(y)vp_0 \\ &= \pi(z)vp_0 = h_2\pi(z)up_0 = h_2\pi(z)p_0. \end{aligned} \tag{18}$$

From (3.10) and (3.17), it follows that

$$h_2^{-1} h_1 k_2 \pi(y)p_0 = h_2^{-1} h_1 k_2 k_1 \pi(x)p_0 = \pi(z)p_0, \tag{19}$$

which implies

$$(h_2^{-1} h_1 k_2 k_1 \pi(x), \pi(z)) \in P(Y). \tag{20}$$

This shows that $(x, z) \in R_\pi(X)$. This completes the proof. □

From Theorem 3.4 and Theorem 3.5, we obtain the following.

THEOREM 3.6. *Let $\pi : (X,T) \to (Y,T)$ be an epimorphism. The following are equivalent:*

(1) *$R_\pi(X)$ is an equivalence relation.*
(2) *Let u and v be the equivalent idempotents in any two minimal right ideals in $E(Y)$. Then $(\pi(x)u, \pi(x)v) \in R(Y)$ for all $x \in X$.*

COROLLARY 3.1. *Let $\pi : (X,T) \to (Y,T)$ be an epimorphism. If $R(Y)$ is a closed subset of $Y \times Y$, then $R_\pi(X)$ is an equivalence relation.*

Proof. Let u and v be the equivalent idempotents in any two minimal right ideals of $E(Y)$. For every $x \in X$, $(\pi(x)u, \pi(x)) \in R(Y)$ and since $R(Y)$ is closed, we obtain $(\pi(x)u, \pi(x))v = (\pi(x)u, \pi(x)v) \in R(Y)$ for all $x \in X$. Therefore, $R_\pi(X)$ is an equivalence relation. □

COROLLARY 3.2. *Let I, K be minimal right ideals in $E(Y)$. For a given epimorphism $\pi : (X,T) \to (Y,T)$, the following are equivalent:*

(1) *$R_\pi(X)$ is an equivalence relation.*

(2) *There is a $\phi \in A(Y)$ such that the following diagram commutes*

$$\begin{array}{ccc} I & \xrightarrow{L_v} & K \\ {\scriptstyle \theta^I_{\pi(x)}}\downarrow & & \downarrow{\scriptstyle \theta^K_{\pi(x)}} \\ Y & \xrightarrow{\phi} & Y \end{array} \tag{21}$$

Proof. (1) implies (2). Suppose that $R_\pi(X)$ is an equivalence relation. Then by Theorem 3.6, $(\pi(x)u, \pi(x)v) \in R(Y)$ for $u \sim v$ in $E(Y)$. This means that there is a $\phi \in A(Y)$ such that $(\phi\pi(x)u, \pi(x)v) \in P(Y)$. Since $(\phi\pi(x)u, \pi(x)v)$ is an almost periodic point, we obtain

$$\phi\pi(x)u = \pi(x)v = \pi(x)vu, \tag{22}$$

which shows that

$$\phi \circ \theta^I_{\pi(x)}(u) = \theta^K_{\pi(x)} \circ L_v(u) \tag{23}$$

and the diagram commutes.

(2) implies (1). Suppose that there exists a $\phi \in A(Y)$ such that the diagram (3.21) commutes.

Let $u \sim v$. Then $\phi \circ \theta^I_{\pi(x)}(u) = \theta^K_{\pi(x)} \circ L_v(u)$, i.e.,

$$\phi\pi(x)u = \pi(x)uv = \pi(x)v, \tag{24}$$

which implies $(\pi(x)u, \pi(x)v) \in R(Y)$. Therefore $R_\pi(X)$ is an equivalence relation. □

THEOREM 3.7. *Let (X,T) and (Y,T) be transformation groups and let $\pi : (X,T) \to (Y,T)$ be an epimorphism. The following are equivalent:*

(1) *$R_\pi(X)$ is an equivalence relation.*
(2) *For $(\pi(x), \pi(y)) \in P(Y)$ and $(\pi(y), \pi(z)) \in P(Y)$, there exists a $\phi \in A(Y)$ such that $(\phi\pi(x), \pi(z)) \in P(Y)$.*

Proof. (1) implies (2). Let $(\pi(x), \pi(y)) \in P(Y)$ and $(\pi(y), \pi(z)) \in P(Y)$. Then $(x, y) \in R_\pi(X)$ and $(y, z) \in R_\pi(X)$, and thus $(x, z) \in R_\pi(X)$, that is, $(\pi(x), \pi(z)) \in R(Y)$. Therefore, there exists a $\phi \in A(Y)$ such that $(\phi\pi(x), \pi(z)) \in P(Y)$.

(2) implies (1). It is sufficient to show that $R_\pi(X)$ is transitive. Let $(x, y) \in R_\pi(X)$ and $(y, z) \in R_\pi(X)$. There exist ϕ_1 and ϕ_2 in $A(Y)$ such that $(\phi_1\pi(x), \pi(y)) \in P(Y)$ and $(\phi_2\pi(y), \pi(z)) \in P(Y)$.

We also have, from Theorem2.2, that $(\phi_2\phi_1\pi(x), \phi_2\pi(y)) \in P(Y)$, and $(\phi_2\pi(y), \pi(z)) \in P(Y)$. By the assumption, there is a ϕ_3 in $A(Y)$ such that

$$(\phi_3\phi_2\phi_1\pi(x), \pi(z)) \in P(Y).$$

Since $\phi_3\phi_2\phi_1$ is in $A(Y)$, it follows that $(\pi(x), \pi(z)) \in R(Y)$ and $(x, z) \in R_\pi(X)$. □

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