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LIE DERIVATIVES ON HOMOGENEOUS REAL HYPERSURFACES OF TYPE A IN COMPLEX SPACE FORMS

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ABSTRACT. The purpose of this paper is to give some characterizations of homogeneous real hypersurfaces of type A in complex space forms $M_n(c)$, $c \neq 0$, in terms of Lie derivatives.

1. Introduction

A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. The complete and simply connected complex space form is isometric to a complex projective space $P_n(C)$, a complex Euclidean space C^n, or a complex hyperbolic space $H_n(C)$ according as $c > 0$, $c = 0$ or $c < 0$ respectively. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

Now, there exist many studies about real hypersurfaces of $M_n(c)$, $c \neq 0$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space $P_n(C)$ by Takagi [13], who showed that these hypersurfaces of $P_n(C)$ could be divided into six types which are said to be of type A_1, A_2, B, C, D and E, and in [3] Cecil-Ryan and in [8] Kimura proved that they are realized as the tubes of constant radius over Hermitian symmetric spaces of compact type of rank 1 or rank 2. Also Berndt [1], [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex

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hyperbolic space $H_n(C)$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal.

On the other hand, Okumura [12] and Montiel and Romero [11] proved the followings respectively.

THEOREM A. *Let M be a real hypersurface of $P_n(C)$, $n\geq 2$. If it satisfies*

$$A\phi - \phi A = 0, \tag{1.1}$$

then M is locally congruent to a tube of radius r over one of the following Kaehler submanifolds:

(A_1) *a hyperplane $P_{n-1}(C)$, where $0 < r < \frac{\pi}{2}$,*
(A_2) *a totally geodesic $P_k(C)$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.*

THEOREM B. *Let M be a real hypersurface of $H_n(C)$, $n\geq 2$. If it satisfies* (1.1), *then M is locally congruent to one of the following hypersurfaces:*

(A_0) *a horosphere in $H_n(C)$, i.e., a Montiel tube,*
(A_1) *a tube of a totally geodesic hyperplane $H_k(C)$ $(k = 0$ or $n-1)$,*
(A_2) *a tube of a totally geodesic $H_k(C)$ $(1 \leq k \leq n-2)$.*

Now hereafter, unless otherwise stated, the above kind of real hypersurfaces in Theorem A or in Theorem B are said to be of *real hypersurfaces of type A*.

From two decades ago there have been so many investigations for real hypersurfaces of type A in $M_n(c)$, $c\neq 0$ and several characterizations of this type have been obtained by many differential geometers (See [1], [3], [7], [11] and [12]). But until now in terms of Lie derivatives only a few characterizations are known to us. From this point of view we have paid our attention to the works of Okumura [12] and Montiel and Romero [11] as in Theorem A and in Theorem B respectively. They showed that a real hypersurface M in $P_n(C)$ or in $H_n(C)$ is locally congruent to a real hypersurface of type A if and only if the structure vector ξ is an infinetesimal isometry, that is $\mathcal{L}_\xi g = 0$, which is equivalent to (1.1), where $\mathcal{L}_\xi$ denotes the Lie derivative along the structure vector ξ.

Being motivated by these results Ki, Kim and Lee [4] proved that the Lie derivatives $\mathcal{L}_\xi g = 0$, $\mathcal{L}_\xi \phi = 0$ or $\mathcal{L}_\xi A = 0$ are equivalent to each other, where A denotes the second fundamental tensor of M in $M_n(c)$.

In this paper we want to generalize these results and to investigate further properties of real hypersurfaces of type A in terms of the tensorial formulas concerned with the Lie derivatives along the structure vector field ξ as follows:

THEOREM. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$. Assume that the structure vector ξ of M satisfies one of the followings :*

(1) $\mathcal{L}_\xi g = fg$ *for the induced Rimannian metric g,*
(2) $\mathcal{L}_\xi \phi = f\phi$ *for the structure tensor ϕ,*
(3) $\mathcal{L}_\xi \phi = fA$ *for the second fundamental tensor A,*
(4) $\mathcal{L}_\xi \phi = fA\phi$ *for the certain tensor $A\phi$ of type (1,1) or,*
(5) $\mathcal{L}_\xi \phi = f\phi A$ *for the certain tensor ϕA of type (1,1),*

where f denotes any differentiable function defined on M. Then M is locally congruent to a real hypersurface of type A.

In section 2 the theory of real hypersurfaces in complex space forms is recalled and in section 3 we will prove the first part of the Theorem when ξ becomes an infinitesimal conformal transformation. In section 4 we will give the complete proof of the latter parts of the Theorem in above. Namely, some characterizations of real hypersurfaces in $M_n(c)$ will be given in terms of the tensorial formulas concerned with the Lie derivatives $\mathcal{L}_\xi \phi$.

2. Preliminaries

Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$, $c \neq 0$, $n \geqq 3$ and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood x in M, the transformation of X and C under J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmertic transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

where I denotes the identity transformation and X denotes any vector field tangent to M. Accordingly, this set (ϕ, ξ, η, g) defines the *almost contact metric structure* on M. Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{2.2}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M. Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows :

(2.3)
$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\varphi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.4}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Now, in order to get our result, we introduce a lemma which was proved by Ki and Suh [6] as follows:

LEMMA 2.1. *Let M be a real hypersurface of a complex space form $M_n(c)$. If $A\phi + \phi A = 0$, then $c = 0$.*

3. The infinitesimal conformal transformations

Before going to prove our assertion in Case (1), let us introduce a slight weaker condition than an infinitesimal isometry.

A vector field X on a Riemannian manifold is said to be an *infinitesimal conformal transformation* if the metric tensor g satisfies $\mathcal{L}_X g = fg$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X and f denotes a differentiable function defined on M.

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$, whose structure vector ξ is an infinitesimal conformal transformation. Then the metric tensor g on M satisfies

$$\begin{aligned}(\mathcal{L}_\xi g)(X,Y) &= g((\phi A - A\phi)X, Y) \\ &= fg(X,Y),\end{aligned}$$

where X and Y are any vector fields tangent to M. It yields that

$$(\phi A - A\phi)X = fX$$

for any differentiable function f on M. From this, putting $X = \xi$, we have

$$\phi A\xi = f\xi. \tag{3.1}$$

So, from applying the operator ϕ we have

$$A\xi = \alpha\xi, \tag{3.2}$$

where α denotes $g(A\xi, \xi)$. By virtue of the latter two formulas (3.1) and (3.2) we know that f identically vanishes. This means the structure vector ξ becomes an infinitesimal isometric transformation. Thus by Theorems A and B in the introduction, we have completed the proof of our Theorem in Case (1).

4. Some characterizations of real hypersurfaces in terms of $\mathcal{L}_\xi\phi$

In this section let us prove the latter part of our main Theorem. Namely, we will give some characterizations of real hypersurfaces of type A in terms of the Lie derivatives of the structure tensor ϕ along the structure vector ξ.

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$ whose structure vector ξ on M satisfies

$$\mathcal{L}_\xi \phi = fT,$$

where f is a differentiable function and T is a tensor field of type $(1,1)$ defined on M. By the definition of the Lie derivative and (2.2) we have

$$\mathcal{L}_\xi \phi = \phi^2 A - \phi A \phi + A\xi \otimes \eta - \xi \otimes \eta(A) = fT, \tag{4.1}$$

from which together with (2.1), it follows that

$$A - A\xi \otimes \eta + \phi A \phi = -fT. \tag{4.2}$$

Operating the linear transformation (4.2) to the structure vector ξ and taking account of (2.1), we have

$$fT\xi = 0. \tag{4.3}$$

Next, operating ϕ to (4.2) to the left and using (2.1), we have

$$A\phi - \phi A + \phi A\xi \otimes \eta - \xi \otimes \eta(A\phi) = f\phi T. \tag{4.4}$$

Operating ϕ to (4.2) to the right and making use of (2.1), we have

$$\phi A - A\phi - \phi A\xi \otimes \eta = fT\phi. \tag{4.5}$$

Taking the inner product of (4.2) with the structure vector ξ, we have for any X in TM

$$g(AX, \xi) - \alpha\eta(X) + fg(TX, \xi) = 0. \tag{4.6}$$

Then from (4.4) and (4.5) we have

LEMMA 4.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$. Assume that the structure vector ξ satisfies $\mathcal{L}_\xi \phi = fT$, where f is a differentiable function and T is a tensor field of type $(1,1)$. If the structure vector ξ is principal, then it satisfies*

$$(4.7) \qquad f\phi T + fT\phi = 0, \quad 2(A\phi - \phi A) = f(\phi T - T\phi).$$

Case (2): $T = \phi$

Assume that $T = \phi$. In this Case (2) the formula (4.6) yields the structure vector ξ is principal. Then, by Lemma 4.1 we have $A\phi - \phi A = 0$. So by virtue of Theorems A and B, we have our assertion under this case.

Case (3): $T = A$.

We assume that $T = A$. By (4.4) and (4.5), we have

$$(4.8) \qquad A\phi - (1+f)\phi A + \phi A\xi \otimes \eta - \xi \otimes \eta(A\phi) = 0,$$

$$(4.9) \qquad \phi A - (1+f)A\phi - \phi A\xi \otimes \eta = 0.$$

Acting the structure vector ξ to the linear transformation (4.8), we get

$$(4.10) \qquad f\phi A\xi = 0.$$

Taking an inner product (4.9) with the structure vector ξ, we have

$$(4.11) \qquad (1+f)\phi A\xi = 0.$$

From (4.10) and (4.11) we have

$$\phi A\xi = 0,$$

that is, ξ is the principal curvature vector with principal curvature α. Then by Lemma 4.1 we have

$$(4.12) \qquad f(A\phi + \phi A) = 0,$$

$$(4.13) \qquad (2+f)(A\phi - \phi A) = 0.$$

Let us denote by M_1 a subset of M consisting of points at which $f(x) \neq 0$. Now let us assume M_1 is not empty. Then, by (4.12), we see that $A\phi + \phi A = 0$ on M_1, and hence $c = 0$ on M_1 by Lemma 2.1. This makes a contradiction. So M_1 is empty. Therefore the function f vanishes identically on M. Then (4.13) together with Theorems A and B we have our assertion in Case (3).

Case (4): $T = A\phi$

Next, we assume that $T = A\phi$. Then, by (4.6), we have

$$A\xi - \alpha\xi = -f\phi A\xi. \tag{4.14}$$

Applying ϕ to (4.14) and using (2.1) and (4.14), we have $(1+f^2)\phi A\xi = 0$, that is, ξ is the principal curvature vector with principal curvature α. From this and (4.5) we have

$$\phi A - A\phi + f(A - \alpha\eta \otimes \xi) = 0. \tag{4.15}$$

Operating ϕ to (4.15) to the right and using (2.1) and the fact ξ is principal, we get

$$\phi A\phi + fA\phi + (A - \alpha\eta \otimes \xi) = 0 \tag{4.16}$$

from which together with (4.15), it follows

$$\phi A - (1 + f^2)A\phi - f\phi A\phi = 0. \tag{4.17}$$

Next, operating ϕ to (4.16) to the left and using (2.1), we get

$$\phi A - A\phi + f\phi A\phi = 0. \tag{4.18}$$

From (4.15) and (4.18), we find

$$f\phi A\phi - f(A - \alpha\eta \otimes \xi) = 0. \tag{4.19}$$

From this, operating ϕ to the left and using (2.1) and the fact ξ is principal, we have

$$f(A\phi + \phi A) = 0. \tag{4.20}$$

Let M_1 be an open set consisting of points x in M such that $f(x) \neq 0$. If M_1 is not empty, then, by (4.20), we see that $A\phi + \phi A = 0$ on M_1, and hence $c = 0$ on M_1 by Lemma 2.1. This makes a contradiction. Hence M_1 is empty. Therefore the function f vanishes identically on M. From this, together with (4.15), we have $\phi A = A\phi$. So by Theorems A and B, we have our assertion in this case.

Case (5): $T = \phi A$

Finally, we assume that $T = \phi A$. Then, by (4.6), the structure vector ξ is principal curvature vector with principal curvature α. From this together with (4.5) we have

$$\phi A - A\phi = f\phi A\phi. \tag{4.21}$$

From this, applying ϕ to the left and using (2.1) and ξ is principal, we get

$$\phi A\phi + (A - \alpha\eta \otimes \xi) = fA\phi. \tag{4.22}$$

Next, operating ϕ to (4.22) to the right and using (2.1), we find

$$A\phi - \phi A + f(A - \alpha\eta \otimes \xi) = 0, \tag{4.23}$$

from which together with (4.21) and (4.22) it follows

$$2(A\phi - \phi A) + f^2 A\phi = 0. \tag{4.24}$$

Operating ϕ to (4.23) to the left and using (2.1) and also the fact ξ is principal, we have

$$\phi A\phi + f\phi A + (A - \alpha\eta \otimes \xi) = 0,$$

from which together with (4.23) it follows

$$A\phi - \phi A = f\phi A\phi + f^2\phi A. \tag{4.25}$$

From (4.21) and (4.25) we have

$$2(A\phi - \phi A) = f^2\phi A,$$

from which together with (4.24) it follows

$$f^2(A\phi + \phi A) = 0.$$

Let us also denote by M_1 an open set consisting of points x in M such that $f(x) \neq 0$. Then by the same argument as in above, we know that such an open subset M_1 do not exist. So f vanishes identically on M. Thus we also have our assertion in Case (5).

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