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A SIMPLE PROOF OF ANALYTIC CHARACTERIZATION THEOREM FOR OPERATOR SYMBOLS

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ABSTRACT. In this paper we first give a simple proof of the analytic characterization theorems of the operator symbols by using the characterization theorem for white noise functionals. We next give a criterion for the convergence of operators on white noise functionals in terms of their symbols and then use this result to give a proof for the Fock expansion theorem of operators on white noise functionals.

1. Introduction

The white noise analysis, initiated by Hida [1] in 1975, has been established as an infinite dimensional distribution theory on a Gaussian space with successful applications to quantum physics, infinite dimensional harmonic analysis, stochastic analysis, see e.g. [1, 4, 7].

In the recent years the theory of operators in white noise analysis has been considerably developed (see [6, 7]). The symbol of operators on white noise functionals was introduced by Obata [6] as a generalization of the S-transform of white noise functionals. In [6], Obata obtained two remarkable results which play an important role for the study of operator theory in white noise analysis. One is the analytic characterization theorem of symbols of operators on white noise functionals which is an operator version of the characterization theorem for white noise

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functionals. The other is the Fock expansion theorem which shows that an operator on white noise functional can be represented as an infinite series of integral kernel operators. The main purpose of this paper is to give rather simple proofs for two theorems mentioned above.

In this paper we first give a simple proof of the analytic characterization theorems of the operator symbols by using the characterization theorem for white noise functionals. We next give a criterion for the convergence of operators on white noise functionals in terms of their symbols and then use this result to give a proof for the Fock expansion theorem of operators on white noise functionals.

2. Preliminaries

Let A be a positive self-adjoint operator on a real separable Hilbert space H with eigenvectors $\{e_j\}$ corresponding to eigenvalues $\{\lambda_j\}$ such that $\rho \equiv \|A^{-1}\|_{OP} < 1$ and $\|A^{-1}\|_{HS} < \infty$. With this A, a Gel'fand triple $E \subset H \subset E^*$ is constructed in the standard manner (see [7]). Recall that E is a nuclear Frechet space with the topology generated by a family of Hilbertian norms: $|\xi|_p = |A^p\xi|_0$, $\xi \in H, p \geq 0$.

Let μ be the standard Gaussian measure on E^*, i.e., its characteristic function is given by $e^{-\frac{1}{2}|\xi|_0^2}$, $\xi \in E$. Let (L^2) be the space of $\mathbb{C}$-valued μ-square integrable functions on E^*. Then by the Wiener-Itô decomposition theorem, each $\phi \in (L^2)$ admits a unique expansion:

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle, \quad x \in E^*, f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n},$$

where $: x^{\otimes n} :$ is a Wick ordering of $x^{\otimes n}$ (see [2, 4]) and $H_{\mathbb{C}}^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of the complexification of H. In this case, we simply write $\phi \sim (f_n)$.

Let (E) be the space of $\phi \sim (f_n) \in (L^2)$ such that $f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$ for all n, and $\|\phi\|_p^2 \equiv \sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty$ for all $p \geq 0$. Then we have a Gel'fand triple: $(E) \subset (L^2) \subset (E)^*$, where $(E)^*$ is the strong dual space of (E). Moreover, it is known that for each $\Phi \in (E)^*$ there exists a unique sequence $\{F_n\}_{n=0}^{\infty}$ with $F_n \in (E_{\mathbb{C}}^{\otimes n})_{\mathrm{sym}}^*$ such that

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \quad \phi \sim (f_n) \in (E),$$

and $\|\Phi\|_{-p}^2 \equiv \sum_{n=0}^{\infty} n! |F_n|_{-p}^2 < \infty$ for some $p \geq 0$. In this case we also write $\Phi \sim (F_n)$. These elements $\phi \in (E)$ and $\Phi \in (E)^*$ are called a *test (white noise) functional* and a *generalized (white noise) functional*, respectively.

For $\xi \in E_{\mathbb{C}}$ an *exponential vector* φ_ξ is defined by $\varphi_\xi \sim (\frac{1}{n!}\xi^{\otimes n})$. The *S-transform* of $\Phi \in (E)^*$ is a $\mathbb{C}$-valued function on $E_{\mathbb{C}}$ defined by

$$S\Phi(\xi) = \langle\!\langle \Phi, \varphi_\xi \rangle\!\rangle, \quad \xi \in E_{\mathbb{C}}.$$

Since $\{\varphi_\xi \; ; \; \xi \in E_{\mathbb{C}}\}$ spans a dense linear subspace of (E), $\Phi \in (E)^*$ is uniquely determined by its S-transform.

We need the following characterization theorems for generalized and test functionals in terms of their S-transforms.

THEOREM 2.1. [3, 4, 8] *The S-transform $F = S\Phi$ of $\Phi \in (E)^*$ satisfies the following conditions:*

- (F1) *For any $\xi, \eta \in E_{\mathbb{C}}$ the map $z \mapsto F(z\xi + \eta)$ is an entire function on $\mathbb{C}$.*
- (F2) *There exist $K > 0, a > 0$ and $p \geq 0$ such that*

$$|F(\xi)| \leq K e^{a|\xi|_p^2}, \qquad \xi \in E_{\mathbb{C}}.$$

Conversely, assume that a $\mathbb{C}$-valued function F defined on $E_{\mathbb{C}}$ satisfies the above two conditions. Then there exists a unique $\Phi \in (E)^*$ such that $F = S\Phi$. Moreover, for any $q > 0$ with $2ae^2\|A^{-q}\|_{\mathrm{HS}}^2 < 1$, we have the following norm estimate:

$$\|\Phi\|_{-(p+q)} \leq K\big(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\big)^{-1/2}.$$

THEOREM 2.2. [3, 4, 5] *The S-transform $F = S\phi$ of $\phi \in (E)$ satisfies the following conditions:*

(F1′) *For any $\xi, \eta \in E_{\mathbb{C}}$ the map $z \mapsto F(z\xi + \eta)$ is an entire function on $\mathbb{C}$.*

(F2′) *For any $p \geq 0$ and $a > 0$ there exists a constant $K > 0$ such that*

$$|F(\xi)| \leq Ke^{a|\xi|_{-p}^2}, \qquad \xi \in E_{\mathbb{C}}.$$

Conversely, assume that a $\mathbb{C}$-valued function F defined on $E_{\mathbb{C}}$ satisfies the above two conditions. Then there exists a unique $\phi \in (E)$ such that $F = S\phi$. Moreover, if a is a constant in (F2′) with $0 < a < (2e^2\|A^{-1}\|_{\mathrm{HS}}^2)^{-1}$ then there exists $K > 0$ such that

$$\|\phi\|_{p-1} \leq K\left(1 - 2ae^2\|A^{-1}\|_{\mathrm{HS}}^2\right)^{-1/2}.$$

3. Analytic characterization theorem for operator symbols

Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ denote the space of all continuous operators from a locally convex space $\mathfrak{X}$ into a locally convex space $\mathfrak{Y}$, given with the topology of uniform convergence on bounded subsets of $\mathfrak{X}$.

For any $\Xi \in \mathcal{L}((E), (E)^*)$, the *symbol* $\widehat{\Xi}$ of Ξ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\varphi_\xi, \varphi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

We here give a simple proof for analytic characterization theorem for operator symbols due to Obata [6].

THEOREM 3.1. *The symbol $F = \widehat{\Xi}$ of $\Xi \in \mathcal{L}((E), (E)^*)$ satisfies the following conditions:*

(S1) *For any ξ, ξ', η and η' in $E_{\mathbb{C}}$, the map $(z, w) \mapsto F(z\xi + \xi', w\eta + \eta')$ is an entire function on $\mathbb{C} \times \mathbb{C}$.*

(S2) *There exist $p \geq 0, a > 0$ and $K > 0$ such that*

$$|F(\xi, \eta)| \leq Ke^{a(|\xi|_p^2 + |\eta|_p^2)}, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Conversely, assume that a $\mathbb{C}$-valued function F on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ satisfies the above two conditions. Then there exists a unique $\Xi \in \mathcal{L}((E),(E)^*)$ such that $F = \widehat{\Xi}$. Moreover, for any $q > 0$ with $2ae^2\|A^{-q}\|_{\mathrm{HS}}^2 < 1$, we have

$$\|\Xi\phi\|_{-(p+q)} \le K\big(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\big)^{-1}\|\phi\|_{p+q}, \quad \phi \in (E).$$

Proof. The first assertion has been proved in [2, 4, 7]. To prove the second assertion, let $\eta \in E_{\mathbb{C}}$ be fixed and define a $\mathbb{C}$-valued function F_η on $E_{\mathbb{C}}$ by $F_\eta(\xi) = F(\xi, \eta)$, $\xi \in E_{\mathbb{C}}$. Then the function F_η satisfies (F1) and (F2) in Theorem 2.1: In fact, the function $z \mapsto F_\eta(z\xi + \xi') = F(z\xi + \xi', \eta)$ is entire on $\mathbb{C}$ and we have

$$|F_\eta(\xi)| = |F(\xi,\eta)| \le (Ke^{a|\eta|_p^2})e^{a|\xi|_p^2}.$$

Hence there exists a unique $\Phi_\eta \in (E)^*$ such that $S\Phi_\eta = F_\eta$, i.e.,

$$\langle\langle \Phi_\eta, \varphi_\xi \rangle\rangle = F(\xi,\eta), \quad \xi,\eta \in E_{\mathbb{C}}.$$

Let $q > 0$ be such that $2ae^2\|A^{-q}\|_{\mathrm{HS}}^2 < 1$. Then Theorem 2.1 gives the norm estimate:

$$\text{(3.2)} \qquad \|\Phi_\eta\|_{-(p+q)} \le Ke^{a|\eta|_p^2}\big(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\big)^{-1/2}.$$

Now fix $\phi \in (E)$, and define a $\mathbb{C}$-valued function G_ϕ on $E_{\mathbb{C}}$ by

$$G_\phi(\eta) = \langle\langle \Phi_\eta, \phi \rangle\rangle, \quad \eta \in E_{\mathbb{C}}.$$

Then in view of (3.2), we have

$$\text{(3.3)} \qquad \begin{aligned} |G_\phi(\eta)| &\le \|\Phi_\eta\|_{-(p+q)}\|\phi\|_{p+q} \\ &\le K\|\phi\|_{p+q}\big(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\big)^{-1/2}e^{a|\eta|_p^2}. \end{aligned}$$

Hence G_ϕ satisfies (F2) in Theorem 2.1. Note that G_ψ satisfies (F1) for all $\psi \in V$, where V is the linear span of $\{\varphi_\xi \; ; \; \xi \in E_{\mathbb{C}}\}$. Since V is dense in (E), we can choose a sequence $\{\phi_k\}$ in V such that $\phi_k \to \phi$ in (E). Fix $\eta, \eta' \in E_{\mathbb{C}}$ then by (3.3), we obtain

$$\begin{aligned}|G_\phi(z\eta+\eta') - G_{\phi_k}(z\eta+\eta')| &= |\langle\langle \Phi_{z\eta+\eta'}, \phi - \phi_k \rangle\rangle| \\ &\le K\big(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\big)^{-1/2} e^{a|z\eta+\eta'|_p^2} \|\phi - \phi_k\|_{p+q}.\end{aligned}$$

So, the function $G_{\phi_k}(z\eta+\eta')$ of $z \in \mathbb{C}$ converges to $G_\phi(z\eta+\eta')$ uniformly on every compact subset of $\mathbb{C}$ and hence the function $z \mapsto G_\phi(z\eta+\eta')$ is entire on $\mathbb{C}$. Therefore G_ϕ satisfies (F1) in Theorem 2.1. By using Theorem 2.1 again, there exists a unique $\Psi_\phi \in (E)^*$ such that

$$\langle\langle \Psi_\phi, \varphi_\eta \rangle\rangle = G_\phi(\eta) = \langle\langle \Phi_\eta, \phi \rangle\rangle, \quad \eta \in E_{\mathbb{C}}. \tag{3.4}$$

Moreover, we have

$$\|\Psi_\phi\|_{-(p+q)} \le K\big(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\big)^{-1}\|\phi\|_{p+q}, \quad \phi \in (E). \tag{3.5}$$

Now define an operator Ξ from (E) into $(E)^*$ by $\Xi\phi = \Psi_\phi$, $\phi \in (E)$. Then $\Xi \in \mathcal{L}((E),(E)^*)$ by (3.4) and (3.5). Since F is clearly the symbol of Ξ, we complete the proof. □

THEOREM 3.2. *The symbol $F = \widehat{\Xi}$ of $\Xi \in \mathcal{L}((E),(E))$ satisfies the following conditions:*

(S1′) *For any $\xi, \xi', \eta, \eta' \in E_{\mathbb{C}}$, the map $(z,w) \mapsto F(z\xi+\xi', w\eta+\eta')$ is an entire function on $\mathbb{C}\times\mathbb{C}$.*

(S2′) *For any $p \ge 0, a > 0$, there exist $q \ge 0$ and $K > 0$ such that*

$$|F(\xi,\eta)| \le K e^{a(|\xi|_{p+q}^2 + |\eta|_{-p}^2)}, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Conversely, assume that a $\mathbb{C}$-valued function F on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ satisfies the above two conditions. Then there exists a unique $\Xi \in \mathcal{L}((E),(E))$ such that $F = \widehat{\Xi}$. Moreover, if a is a constant in (S2′) with $0 < a < (2e^2\|A^{-1}\|_{\mathrm{H}S}^2)^{-1}$ then there exist $q \ge 0$ and $K > 0$ such that

$$\|\Xi\phi\|_{p-1} \le K(1 - 2ae^2\|A^{-1}\|_{\mathrm{H}S}^2)^{-1}\|\phi\|_{p+q+1}.$$

Proof. The first assertion has been proved in [2, 4, 7]. We shall prove the second assertion. Since condition (S2′) in Theorem 3.2 is stronger than condition (S2) in Theorem 3.1, there exists a unique continuous linear operator $\Xi \in \mathcal{L}((E),(E)^*)$ such that F is the symbol of Ξ. Now we shall show that $\Xi \in \mathcal{L}((E),(E))$. We first recall from the proof of Theorem 3.1 that for $\eta \in E_{\mathbb{C}}$ there exists a unique $\Phi_\eta \in (E)^*$ such that

$$\langle\langle \Phi_\eta, \varphi_\xi \rangle\rangle = F(\xi, \eta), \quad \xi \in E_{\mathbb{C}} \tag{3.6}$$

and that for any fixed $\phi \in (E)$,

$$S(\Xi\phi)(\eta) = \langle\langle \Xi\phi, \varphi_\eta \rangle\rangle = \langle\langle \Phi_\eta, \phi \rangle\rangle, \quad \eta \in E_{\mathbb{C}}.$$

To show that $S(\Xi\phi)$ satisfies (F2′) in Theorem 2.2, choose arbitrary $p \geq 0$ and $a > 0$. Then by (S2′), there exist $q \geq 0$ and $K > 0$ such that

$$|F(\xi,\eta)| \leq (K e^{a|\eta|_{-p}^2}) e^{a|\xi|_{p+q}^2}, \quad \xi, \eta \in E_{\mathbb{C}}.$$

If r is a positive number such that $2ae^2\|A^{-r}\|_{\mathrm{HS}}^2 < 1$, then by Theorem 2.1, we have

$$\|\Phi_\eta\|_{-(p+q+r)} \leq K\big(1 - 2ae^2\|A^{-r}\|_{\mathrm{HS}}^2\big)^{-1/2} e^{a|\eta|_{-p}^2}, \quad \eta \in E_{\mathbb{C}}.$$

Hence for any $\phi \in (E)$ we obtain

$$\begin{aligned} |S(\Xi\phi)(\eta)| &\leq \|\Phi_\eta\|_{-(p+q+r)} \|\phi\|_{p+q+r} \\ &\leq K\big(1 - 2ae^2\|A^{-r}\|_{\mathrm{HS}}^2\big)^{-1/2} \|\phi\|_{p+q+r} e^{a|\eta|_{-p}^2}. \end{aligned}$$

So, $S(\Xi\phi)$ satisfies (F2′) in Theorem 2.2. Thus by Theorem 2.2, if a is a constant with $0 < a < (2e^2\|A^{-1}\|_{\mathrm{H}S}^2)^{-1}$ then there exist $q \geq 0$ and $K > 0$ such that

$$\|\Xi\phi\|_{p-1} \leq K(1 - 2ae^2\|A^{-1}\|_{\mathrm{H}S}^2)^{-1}\|\phi\|_{p+q+1}.$$

We complete the proof. □

4. Convergence of operators on white noise functionals

In this section we will find a criterion for the convergence of operators on white noise functionals in terms of their symbols.

A simple application of Banach-Steinhaus theorem gives the following lemma:

LEMMA 4.1. (1) *Let* $\{\Xi_n\}_{n\geq 1}$ *and* Ξ *be in* $\mathcal{L}((E),(E)^*)$. *Then* Ξ_n *converges to* Ξ *in* $\mathcal{L}((E),(E)^*)$ *if and only if* $\Xi_n\phi$ *converges to* $\Xi\phi$ *in* $(E)^*$ *for all* $\phi \in (E)$.

(2) *Let* $\{\Xi_n\}_{n\geq 1}$ *and* Ξ *be in* $\mathcal{L}((E),(E))$. *Then* Ξ_n *converges to* Ξ *in* $\mathcal{L}((E),(E))$ *if and only if* $\Xi_n\phi$ *converges to* $\Xi\phi$ *in* (E) *for all* $\phi \in (E)$.

THEOREM 4.2. *Let* $\{\Xi_n\}_{n=1}^{\infty}$ *and* Ξ *be in* $\mathcal{L}((E),(E)^*)$. *Then* Ξ_n *converges to* Ξ *in* $\mathcal{L}((E),(E)^*)$ *if and only if the following conditions hold:*

(U1) $\widehat{\Xi}_n(\xi,\eta)$ *converges to* $\widehat{\Xi}(\xi,\eta)$ *for each* $\xi,\eta \in E_{\mathbb{C}}$.

(U2) *There exist* $p \geq 0, a > 0$ *and* $K > 0$ *such that*

$$|\widehat{\Xi}_n(\xi,\eta)| \leq K e^{a(|\xi|_p^2+|\eta|_p^2)}, \qquad \xi,\eta \in E_{\mathbb{C}}, n \in \mathbb{N}.$$

Proof. Suppose that Ξ_n converges to Ξ in $\mathcal{L}((E),(E)^*)$. Then by using the fact that $\mathcal{L}((E),(E)^*) \cong ((E)\otimes(E))^*$ and a simple application of the kernel theorem, there exists $p \geq 0$ such that Ξ_n and Ξ can be regarded as continuous linear operators from (E_p) into $(E_p)^*$ and $\lim_{n\to\infty} \|\Xi_n\phi - \Xi\phi\|_{-p} = 0$ for all $\phi \in (E)$. So, (U1) is easily satisfied. Since $\{\|\Xi_n\phi\|_{-p}\ ;\ n \in \mathbb{N}\}$ is bounded for any $\phi \in (E_p)$, it follows from the uniform boundedness theorem that $K \equiv \sup\{\|\Xi_n\|_{\mathcal{L}((E_p),(E_p)^*)}\ ;\ n \in \mathbb{N}\}$ is finite. Hence we obtain

$$|\widehat{\Xi}_n(\xi,\eta)| \leq K e^{\frac{1}{2}(|\xi|_p^2+|\eta|_p^2)}, \quad \xi,\eta \in E_{\mathbb{C}}, n \in \mathbb{N}.$$

So, (U2) is satisfied.

Conversely, assume that (U1) and (U2) are satisfied. Let $q > 0$ be such that $2ae^2\|A^{-q}\|_{\mathrm{HS}}^2 < 1$. Then, by Theorem 3.1, we obtain

$$\|(\Xi_n-\Xi)\phi\|_{-(p+q)} \leq 2K\left(1-2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\right)^{-1}\|\phi\|_{p+q}, \quad \phi \in (E), n \in \mathbb{N}.$$

Now we will show that $\Xi_n\varphi_\xi$ converges to $\Xi\varphi_\xi$ in $(E)^*$ for all $\xi \in E_{\mathbb{C}}$. Note that a sequence in $(E)^*$ converges strongly if and only if it converges weakly. Hence it is enough to show that $\lim_{n\to\infty}\langle\langle(\Xi_n - \Xi)\varphi_\xi, \phi\rangle\rangle = 0$ for all $\xi \in E_{\mathbb{C}}$ and $\phi \in (E)$. Let $\xi \in E_{\mathbb{C}}$ be fixed and V be the linear span of the set $\{\varphi_\eta \,;\, \eta \in E_{\mathbb{C}}\}$. Then we can show that $\lim_{n\to\infty}\langle\langle(\Xi_n - \Xi)\varphi_\xi, \psi\rangle\rangle = 0$ for all $\psi \in V$. Let $\phi \in (E)$ be given and consider the inequality:

$$\begin{aligned}|\langle\langle(\Xi_n - \Xi)\varphi_\xi, \phi\rangle\rangle| &\le |\langle\langle(\Xi_n - \Xi)\varphi_\xi, \phi - \psi\rangle\rangle| + |\langle\langle(\Xi_n - \Xi)\varphi_\xi, \psi\rangle\rangle| \\ &\le 2Ke^{\frac{1}{2}|\xi|_{p+q}^2}\left(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\right)^{-1}\|\phi - \psi\|_{p+q} \\ &\quad + |\langle\langle(\Xi_n - \Xi)\varphi_\xi, \psi\rangle\rangle|.\end{aligned}$$

Since we can choose $\psi \in V$ such that $\|\phi - \psi\|_{p+q}$ is sufficiently small and $\lim_{n\to\infty}\langle\langle(\Xi_n - \Xi)\varphi_\xi, \psi\rangle\rangle = 0$, we have $\lim_{n\to\infty}\langle\langle(\Xi_n - \Xi)\varphi_\xi, \phi\rangle\rangle = 0$. So, we have proved that $\Xi_n\varphi_\xi$ converges to $\Xi\varphi_\xi$ in $(E)^*$ and hence we see that $\Xi_n\psi$ converges to $\Xi\psi$ in $(E)^*$ for all $\psi \in V$. For an arbitrary $\phi \in (E)$, observe the inequality:

$$\begin{aligned}\|(\Xi_n - \Xi)\phi\|_{-(p+q)} &\le \|(\Xi_n - \Xi)(\phi - \psi)\|_{-(p+q)} + \|(\Xi_n - \Xi)\psi\|_{-(p+q)} \\ &\le 2K\left(1 - 2ae^2\|A^{-q}\|_{\mathrm{HS}}^2\right)^{-1}\|\phi - \psi\|_{p+q} + \|(\Xi_n - \Xi)\psi\|_{-(p+q)}.\end{aligned}$$

Using similar argument as above, we obtain that $\Xi_n\phi$ converges to $\Xi\phi$ in (E) for all $\phi \in (E)$. By the help of Lemma 4.1, we complete the proof. □

THEOREM 4.3. *Let $\{\Xi_n\}_{n=1}^\infty$ and Ξ be in $\mathcal{L}((E),(E))$. Then Ξ_n converges to Ξ in $\mathcal{L}((E),(E))$ if and only if the following conditions hold:*

(U1′) *$\widehat{\Xi}_n(\xi,\eta)$ converges to $\widehat{\Xi}(\xi,\eta)$ for each ξ, $\eta \in E_{\mathbb{C}}$.*

(U2′) *For any $p \ge 0$ and $a > 0$, there exist $q \ge 0$ and $K > 0$ such that*

$$|\widehat{\Xi}_n(\xi,\eta)| \le Ke^{a(|\xi|_{p+q}^2 + |\eta|_{-p}^2)}, \quad \xi, \eta \in E_{\mathbb{C}}, n \in \mathbb{N}.$$

Proof. Suppose that Ξ_n converges to Ξ in $\mathcal{L}((E),(E))$. Let $p \geq 0$ and $a > 0$ be given. Then there exists $q' \geq 0$ such that Ξ_n converges to Ξ in $\mathcal{L}((E_{p+q'}),(E_p))$. Hence there exists $K > 0$ such that $\|\Xi_n\|_{\mathcal{L}((E_{p+q'}),(E_p))} \leq K$ for all $n \in \mathbb{N}$. Thus we can see that there exists $q \geq 0$ such that

$$|\widehat{\Xi}_n(\xi,\eta)| \leq K e^{a(|\xi|_{p+q}^2+|\eta|_{-p}^2)}, \quad \xi,\eta \in E_{\mathbb{C}}, n \in \mathbb{N}.$$

So, (U2′) is satisfied. Since (U1′) is clear, we complete the proof of "only if" part.

Conversely, assume that (U1′) and (U2′) are satisfied. Let $a > 0$ with $2ae^2\|A^{-1}\|_{\mathrm{HS}}^2 < 1$. Then in view of Theorem 3.2, we have

$$\|(\Xi_n - \Xi)\phi\|_{p-1} \leq M\|\phi\|_{p+q+1}, \quad \phi \in (E), n \in \mathbb{N},$$

where $M = 2K(1 - 2ae^2\|A^{-1}\|_{\mathrm{H}S}^2)^{-1}$. Let V be the linear span of $\{\varphi_\eta \ ; \ \eta \in E_{\mathbb{C}}\}$. Then from (U1′) we see that $\lim_{n\to\infty}\langle\langle\psi,(\Xi_n-\Xi)\varphi_\xi\rangle\rangle = 0$, for any $\xi \in E_{\mathbb{C}}$ and $\psi \in V$. Now fix $\xi \in E_{\mathbb{C}}$ and $\Phi \in (E)^*$. Then there exists a sequence $\{\psi_k\} \subset V$ and $p \geq 1$ such that $\lim_{k\to\infty}\|\Phi - \psi_k\|_{-(p-1)} = 0$. Then by observing the following inequality:

$$\begin{aligned}|\langle\langle\Phi,(\Xi_n - \Xi)\varphi_\xi\rangle\rangle| &\leq |\langle\langle\Phi - \psi_k,(\Xi_n - \Xi)\varphi_\xi\rangle\rangle| + |\langle\langle\psi_k,(\Xi_n - \Xi)\varphi_\xi\rangle\rangle| \\ &\leq Me^{\frac{1}{2}|\xi|_{p+q+1}^2}\|\Phi - \psi_k\|_{-(p-1)} + |\langle\langle\psi_k,(\Xi_n - \Xi)\varphi_\xi\rangle\rangle|,\end{aligned}$$

we obtain that $\lim_{n\to\infty}\langle\langle\Phi,(\Xi_n - \Xi)\varphi_\xi\rangle\rangle = 0$ for any $\xi \in E_{\mathbb{C}}$ and $\Phi \in (E)^*$, and hence $\Xi_n\varphi_\xi$ converges to $\Xi\varphi_\xi$ in (E) for all $\xi \in E_{\mathbb{C}}$.

Let $\phi \in (E)$ be given. Then we see that for any $\psi \in V$,

$$\begin{aligned}\|(\Xi_n - \Xi)\phi\|_{p-1} &\leq \|(\Xi_n - \Xi)(\phi - \psi)\|_{p-1} + \|(\Xi_n - \Xi)\psi\|_{p-1} \\ &\leq M\|\phi - \psi\|_{p+q+1} + \|(\Xi_n - \Xi)\psi\|_{p-1}.\end{aligned}$$

Since $\inf\{\|\phi-\psi\|_{p+q+1} \ ; \ \psi \in V\} = 0$ and $\lim_{n\to\infty}\|(\Xi_n - \Xi)\psi\|_{p-1} = 0$ for all $\psi \in V$, it follows that $\lim_{n\to\infty}\|(\Xi_n - \Xi)\phi\|_{p-1} = 0$, and hence $\Xi_n\phi$ converges to $\Xi\phi$ in (E). This completes the proof. □

5. Fock expansion theorem

From Theorem 3.1, it is shown that for each $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ there exists a unique operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$ such that

$$\widehat{\Xi}_{l,m}(\kappa)(\xi,\eta) = e^{\langle\xi,\eta\rangle}\langle\kappa, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle, \quad \xi,\eta \in E_{\mathbb{C}}.$$

This operator is called the *integral kernel operator with kernel distribution* κ (see [7]). We note that kernel distribution κ is uniquely determined if it is taken from the subspace

$$(E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)} = \{\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^* \ ; \ s_{l,m}(\kappa) = \kappa\},$$

where $s_{l,m}$ is the symmetrizing operator with respect to the first l and the last m variables separately.

THEOREM 5.1. *For any* $\Xi \in \mathcal{L}((E),(E)^*)$ *there exists a unique family of kernel distributions* $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ *such that*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{5.1}$$

where the right hand side of (5.1) converges to Ξ *in* $\mathcal{L}((E),(E)^*)$.

Proof. We first put $F(\xi,\eta) = e^{-\langle\xi,\eta\rangle}\langle\langle\Xi\varphi_\xi,\varphi_\eta\rangle\rangle, \xi,\eta \in E_{\mathbb{C}}$. Since $|e^{-\langle\xi,\eta\rangle}| \le e^{\frac{1}{2}\rho^{2p}(|\xi|_p^2+|\eta|_p^2)}$ for all $p \ge 0$, there exist $p \ge 0, a > 0$ and $K > 0$ such that

$$|F(\xi,\eta)| \le Ke^{a(|\xi|_p^2+|\eta|_p^2)}, \quad \xi,\eta \in E_{\mathbb{C}}.$$

Note that F satisfies $(S1)$ in Theorem 3.1. Hence $F(z_1\xi_1 + \cdots + z_m\xi_m, w_1\eta_1 + \cdots + w_l\eta_l)$ is entire on $\mathbb{C}^m \times \mathbb{C}^l$. Fix $l, m \ge 0$, define a symmetric $(l+m)$-linear functional $\alpha_{l,m}$ on $E_{\mathbb{C}}^{l+m}$ by

$$\alpha_{l,m}(\xi_1,\cdots,\xi_m,\eta_1,\cdots,\eta_l) = \frac{1}{l!m!}\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_m}\frac{\partial}{\partial w_1}\cdots\frac{\partial}{\partial w_l}$$
$$F(z_1\xi_1 + \cdots + z_m\xi_m, w_1\eta_1 + \cdots + w_l\eta_l)\Big|_{\substack{z_1=\cdots=z_m=0\\ w_1=\cdots=w_l=0}}$$

with the convention $\alpha_{0,0} = F(0,0)$.

By the Cauchy formula and $(S2)$ in Theorem 3.1 we obtain the following inequality : for positive real numbers $r_1, \cdots, r_m$, $s_1, \cdots, s_l$,

$$\begin{aligned}
&|\alpha_{l,m}(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_l)| \\
&\leq \frac{1}{l!m!}\left(\frac{1}{2\pi}\right)^{l+m} \int_{|z_1|=r_1} \cdots \int_{|z_m|=r_m} \int_{|w_1|=s_1} \cdots \int_{|w_l|=s_l} \\
&\quad \frac{|F(z_1\xi_1 + \cdots + z_m\xi_m, w_1\eta_1 + \cdots + w_l\eta_l)|}{|z_1^2 \cdots z_m^2 w_1^2 \cdots w_l^2|} |dz_1 \cdots dz_m dw_1 \cdots dw_l| \\
&\leq K\frac{1}{l!m!}(r_1 \cdots r_m s_1 \cdots s_l)^{-1} \exp\Big\{a\Big[\Big(\sum_{j=1}^{m} r_j|\xi_j|_p\Big)^2 + \Big(\sum_{k=1}^{l} s_k|\eta_k|_p\Big)^2\Big]\Big\}.
\end{aligned}$$

Suppose that $|\xi_j|_p \neq 0$ for $j = 1, \cdots, m$ and $|\eta_k|_p \neq 0$ for $k = 1, \cdots, l$. For $l \geq 1$, $m \geq 1$, we take $r_j = (\sqrt{2am}|\xi_j|_p)^{-1}$ and $s_k = (\sqrt{2al}|\eta_k|_p)^{-1}$. Then we obtain

(5.2)
$$\begin{aligned}
&|\alpha_{l,m}(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_l)| \\
&\qquad \leq K\frac{1}{l!m!}(2ael)^{l/2}(2aem)^{m/2}|\eta_1|_p \cdots |\eta_l|_p |\xi_1|_p \cdots |\xi_m|_p.
\end{aligned}$$

But we can easily check that (5.2) is valid for any $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_l \in E_{\mathbb{C}}$ and that (5.2) is also valid for $l = 0$ or $m = 0$ with the convention $0^0 = 1$. Therefore $\alpha_{l,m}$ is continuous on $E_{\mathbb{C}}^{l+m}$ and hence by the nuclear theorem there exists $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ such that

$$(5.3)\quad \langle \kappa_{l,m}, \eta_1 \otimes \cdots \otimes \eta_l \otimes \xi_1 \otimes \cdots \otimes \xi_m \rangle = \alpha_{l,m}(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_l),$$

for $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_l \in E_{\mathbb{C}}$. It is clear that $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$. Then by (5.2) and (5.3) we have

$$|\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle| \leq K\frac{1}{l!m!}(2ael)^{l/2}(2aem)^{m/2}|\eta|_p^l |\xi|_p^m, \quad \xi, \eta \in E_{\mathbb{C}}.$$

Now we observe that

$$\sum_{l,m=0}^{\infty} |\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle|$$
$$\leq K \left(\sum_{l=0}^{\infty} \frac{1}{l!}(2ael)^{l/2}|\eta|_p^l \right) \left(\sum_{m=0}^{\infty} \frac{1}{m!}(2aem)^{m/2}|\xi|_p^m \right).$$

But

$$\sum_{l=0}^{\infty} \frac{1}{l!}(2ael)^{l/2}|\eta|_p^l \leq \left(\sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^l \right)^{1/2} \left(\sum_{l=0}^{\infty} \frac{1}{l!} \left(4ae^2|\eta|_p^2\right)^l \right)^{1/2}$$
$$\leq \sqrt{2} e^{2ae^2|\eta|_p^2}$$

Here we have used the inequality $n^n \leq e^n n!, n \geq 0$. In all, we have

$$\sum_{l,m=0}^{\infty} |\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle| \leq 2Ke^{2ae^2(|\xi|_p^2+|\eta|_p^2)}.$$

Hence we see that

$$\sum_{l,m=0}^{\infty} |\widehat{\Xi}_{l,m}(\kappa_{l,m})(\xi,\eta)| \leq 2Ke^{(2ae^2+\frac{1}{2}\rho^{2p})(|\xi|_p^2+|\eta|_p^2)}. \tag{5.4}$$

We here note that

$$\alpha_{l,m}(\xi,\cdots,\xi,\eta,\cdots,\eta) = \frac{1}{l!m!}\frac{\partial^{l+m}}{\partial z^m \partial w^l} F(z\xi, w\eta)\Big|_{z=w=0}.$$

And hence we have $F(z\xi, w\eta) = \sum_{l,m=0}^{\infty} \alpha_{l,m}(\xi,\cdots,\xi,\eta,\cdots,\eta) z^m w^l$. Thus we obtain

$$\begin{aligned}
\widehat{\Xi}(\xi,\eta) &= e^{\langle \xi,\eta\rangle} F(\xi,\eta) \\
&= e^{\langle \xi,\eta\rangle} \sum_{l,m=0}^{\infty} \alpha_{l,m}(\xi,\cdots,\xi,\eta,\cdots,\eta) \\
&= e^{\langle \xi,\eta\rangle} \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle \\
&= \sum_{l,m=0}^{\infty} \widehat{\Xi}_{l,m}(\kappa_{l,m})(\xi,\eta).
\end{aligned}$$

By Theorem 4.2 and equation (5.4) it follows that $\sum_{l,m}^{\infty}\Xi_{l,m}(\kappa_{l,m})$ converges to Ξ in $\mathcal{L}((E),(E)^*)$. □

THEOREM 5.2. *Let $\Xi \in \mathcal{L}((E),(E))$ be given by its Fock expansion:*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}). \tag{5.5}$$

Then $\kappa_{l,m} \in E_{\mathbb{C}}^{\otimes l} \otimes (E_{\mathbb{C}}^{\otimes m})^*$ and the right hand side (5.5) converges to Ξ in $\mathcal{L}((E),(E))$.

Proof. We can easily check that the function $F(\xi,\eta) = e^{-\langle \xi,\eta\rangle}\langle\langle \Xi\varphi_\xi, \varphi_\eta\rangle\rangle$ satisfies the condition (S2′) : for any $p \geq 0$ and $a > 0$, there exist $q \geq 0$ and $K > 0$ such that

$$|F(\xi,\eta)| \leq K e^{a(|\xi|_{p+q}^2 + |\eta|_{-p}^2)}, \quad \xi,\eta \in E_{\mathbb{C}}. \tag{5.6}$$

As we have shown in the proof of Theorem 5.1, there exists $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes l+m})^*$ such that

$$\langle \kappa_{l,m}, \eta_1 \otimes \cdots \otimes \eta_l \otimes \xi_1 \otimes \cdots \otimes \xi_m\rangle = \alpha_{l,m}(\xi_1,\cdots,\xi_m,\eta_1,\cdots,\eta_l).$$

By modifying the proof of Theorem 5.1 with (5.6), we obtain

$$\begin{aligned} &|\langle \kappa_{l,m}, \eta_1 \otimes \cdots \otimes \eta_l \otimes \xi_1 \otimes \cdots \otimes \xi_m\rangle| \\ &\qquad \leq K\frac{1}{l!m!}(2ael)^{l/2}(2aem)^{m/2}|\eta_1|_{-p}\cdots|\eta_l|_{-p}|\xi_1|_{p+q}\cdots|\xi_m|_{p+q}, \end{aligned} \tag{5.7}$$

for $\xi_1,\cdots,\xi_m,\eta_1,\cdots,\eta_l \in E_{\mathbb{C}}$. To prove $\kappa_{l,m} \in E_{\mathbb{C}}^{\otimes l} \otimes (E_{\mathbb{C}}^{\otimes m})^*$, fix $l,m \geq 0$ and choose arbitrary $f \in E_{\mathbb{C}}^{\otimes l}$ and $g \in E_{\mathbb{C}}^{\otimes m}$. Then

$$f = \sum_{i_1,\cdots,i_l=0}^{\infty} \langle f, e_{i_1}\otimes\cdots\otimes e_{i_l}\rangle e_{i_1}\otimes\cdots\otimes e_{i_l}$$

and

$$g = \sum_{j_1,\cdots,j_m=0}^{\infty} \langle g, e_{j_1}\otimes\cdots\otimes e_{j_m}\rangle e_{j_1}\otimes\cdots\otimes e_{j_m}.$$

Then it can be shown by using (5.7) that

$$
\begin{aligned}
&|\langle \kappa_{l,m}, f\otimes g\rangle| \\
&\quad \le \sum_{\substack{i_1,\cdots,i_l=0\\ j_1,\cdots,j_m=0}}^{\infty} |\langle f, e_{i_1}\otimes\cdots\otimes e_{i_l}\rangle||\langle g, e_{j_1}\otimes\cdots\otimes e_{j_m}\rangle| \\
&\qquad\qquad \times |\langle \kappa_{l,m}, e_{i_1}\otimes\cdots\otimes e_{i_l}\otimes e_{j_1}\otimes\cdots\otimes e_{j_m}\rangle| \\
&\quad \le C\Big(\sum_{i_1,\cdots,i_l=0}^{\infty} |\langle f, e_{i_1}\otimes\cdots\otimes e_{i_l}\rangle||e_{i_1}|_{-p}\cdots|e_{i_l}|_{-p}\Big) \\
&\quad \times\Big(\sum_{j_1,\cdots,j_m=0}^{\infty} |\langle g, e_{j_1}\otimes\cdots\otimes e_{j_m}\rangle||e_{j_1}|_{p+q}\cdots|e_{j_m}|_{p+q}\Big) \\
&\quad \le C\Big(\sum_{i_1,\cdots,i_l=0}^{\infty} |\langle f, e_{i_1}\otimes\cdots\otimes e_{i_l}\rangle|^2|\lambda_{i_1}|^{-2(p-1)}\cdots|\lambda_{i_l}|^{-2(p-1)}\Big)^{1/2} \\
&\quad \times\Big(\sum_{j_1,\cdots,j_m=0}^{\infty} |\langle g, e_{j_1}\otimes\cdots\otimes e_{j_m}\rangle|^2|\lambda_{j_1}|^{2(p+q+1)}\cdots|\lambda_{j_m}|^{2(p+q+1)}\Big)^{1/2} \\
&\quad \times\Big(\sum_{j=0}^{\infty}\lambda_j^{-2}\Big)^{(l+m)/2} \\
&\quad \le C\|A^{-1}\|_{\mathrm{HS}}^{l+m}|f|_{-(p-1)}|g|_{p+q+1},
\end{aligned}
$$

where $C = K\frac{1}{l!m!}(2ael)^{l/2}(2aem)^{m/2}$. Hence $\kappa_{l,m}\in E_{\mathbb{C}}^{\otimes l}\otimes(E_{\mathbb{C}}^{\otimes m})^*$.

To prove the convergence of (5.5), we first note from (5.7) that

$$
|\langle \kappa_{l,m}, \eta^{\otimes l}\otimes\xi^{\otimes m}\rangle| \le K\frac{1}{l!m!}(2ael)^{l/2}(2aem)^{m/2}|\eta|_{-p}^{l}|\xi|_{p+q}^{m}
$$

for $\xi,\eta\in E_{\mathbb{C}}$. So we obtain that

$$
\sum_{l,m=0}^{\infty}|\langle \kappa_{l,m}, \eta^{\otimes l}\otimes\xi^{\otimes m}\rangle| \le 2Ke^{2ae^2(|\xi|_{p+q}^2+|\eta|_{-p}^2)}.
$$

Hence by Theorem 4.3, it follows that the series $\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})$ converges to Ξ in $\mathcal{L}((E),(E))$. □

References

[1] T. Hida, *Analysis of Brownian functionals*, Carleton Math. Lect. Notes, vol 13, Carleton University, Ottawa, 1975.

[2] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, *White Noise: An Infinite Dimensional Calculus*, Kluwer Academic Publishers, 1993.

[3] I. Kubo and H.-H. Kuo, *A simple proof of Hida distribution characterization theorem*, in A. V. Skorokhod and Yu. V. Borovskikh (eds.): Exploring Stochastic Laws, Festschrift in Honor of the 70th Birthday of V. S. Korolyuk, International Science Publishers, 1995, pp. 243–250.

[4] H.-H. Kuo, *White noise distribution theory*, CRC Press, 1996.

[5] H.-H. Kuo, J. Potthoff and L. Streit, *A characterization of white noise test functionals*, Nagoya Math. J. **121** (1991), 185–194.

[6] N. Obata, *An analytic characterization of symbols of operators on white noise functionals*, J. Math. Soc. Japan **45** (1993), 422–445.

[7] ———, *White Noise Calculus and Fock Space*, Lect. Notes in Math. Vol. 1577, Springer-Verlag, 1994.

[8] J. Potthoff and L. Streit, *A characterization of Hida distributions*, J. Funct. Anal. **101** (1991), 212–229.

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