Bull. Korean Math. Soc. **34** (1997), No. 3, pp. 395–401

EXPONENTIAL RANK OF EXTENSIONS OF C^*-ALGEBRAS

JA A JEONG* AND GIE HYUN PARK

ABSTRACT. We show that if I is an ideal of a C^*-algebra A such that the unitary group of $\tilde{I}$ is connected then $cer(A) \leq cer(I) + cer(A/I)$, where $cer(A)$ denotes the C^*-exponential rank of A.

0. Introduction

Let A be a unital C^*-algebra, $U(A)$ its unitary group, and $U_0(A)$ the connected component of $U(A)$ which contains the unit of A. Then it is well known that $U_0(A)$ is the set of all finite products of exponentials of elements of A, that is, $U_0(A) = \{ \exp(ia_1) \cdots \exp(ia_n) \mid a_j \in A_{sa},\ n = 1, 2, \dots \}$. The unitary group of the C^*-algebra $B(H)$ of all bounded linear operators on a separable infinite dimensional Hilbert space H is connected and each unitary is of the form $\exp(ia)$ for some self adjoint operator a in $B(H)$, and the same assertion is true for all finite dimensional C^*-algebras. But if A is a UHF algebra then there exists a unitary in $U_0(A)$ which can not be expressed as $\exp(ia)$ with $a \in A_{sa}$ although $\{ \exp(ia) \mid a \in A_{sa} \}$ is dense in $U_0(A)$ ([2]).

For a unital C^*-algebra A and a unitary $u \in U_0(A)$, the *C^*-exponential rank* of u, denoted by $cer(u)$, is defined as follows: $cer(u) = n$ if $u \in \exp(iA_{sa})^n$ but $u \notin \overline{\exp(iA_{sa})^{n-1}}$, $cer(u) = n + \varepsilon$ if $u \in \overline{\exp(iA_{sa})^n}$ but $u \notin \exp(iA_{sa})^n$. Note that we can order the possible ranks: $1 < 1 + \varepsilon < 2 < 2 + \varepsilon < \cdots$. Indeed if two unitaries u and v are close so that $\|1 - u^{-1}v\| = \|u - v\| < 2$ then the spectrum $sp(u^{-1}v)$ is a proper subset of the unit circle $\mathbb{T}$. Hence we can find a suitable branch of logarithm function and use continuous functional calculus to express

Received January 22, 1997.
1991 Mathematics Subject Classification: 46L05.
Key words and phrases: exponential rank, real rank.
*Supported by GARC-KOSEF, BSRI-96-1417(1996-1997).

$u^{-1}v = \exp(ia)$, $(a = -i\log(u^{-1}v))$ or $v = u\exp(ia)$. Thus we have $n < n+\varepsilon < n+1$ for each $n = 1, 2, \ldots$. Then the C^*-exponential rank of A, written $cer(A)$, is defined by

$$cer(A) = \sup\{\ cer(u) \mid u \in U_0(A)\ \}.$$

If A is not unital then we define $cer(A) = cer(\tilde{A})$, where $\tilde{A}$ is the unitization of A. See [1] for more on exponential rank.

If $\pi : A \to B$ is a surjective homomorphism then it follows that $cer(B) \le cer(A)$ since $\pi(U_0(A)) = U_0(B)$.

In this short note we get an upper bound of the exponential rank of a C^*-algebra from the ranks of an ideal and the quotient algebra. Before proving the main theorem in section 2, we discuss in section 1 several conditions which have been studied in connection with the notion of C^*-exponential rank.

1. Exponential rank and weak (FU)

Recall that a unital C^*-algebra A has real rank zero, $RR(A) = 0$, if the set of invertible self-adjoint elements is dense in the set A_{sa} of all self-adjoint elements, and a nonunital C^*-algebra A has real rank zero if $RR(\tilde{A}) = 0$ where $\tilde{A}$ denotes the unitization of A. It is shown in [1, 2.6. Theorem] that $RR(A) = 0$ if and only if the set of self-adjoint elements with finite spectra is dense in A_{sa}. In connection with exponential rank, Lin ([4]) proved that if A is a C^*-algebra of real rank zero then $cer(A) \le 1+\varepsilon$ by showing that A has the property weak (FU); every unitary in the identity component $U_0(A)$ (or in $U_0(\tilde{A})$ if A is nonunital) can be approximated by unitaries with finite spectra. Since a C^*-algebra with weak (FU) is always of real rank zero [6, Proposition 1.5] it follows that a C^*-algebra A is of real rank zero if and only if A has weak (FU) [4, Corollary 6].

PROPOSITION 1.1. *For a C^*-algebra A the following are equivalent*

i) $RR(A) = 0$.

ii) *A has weak (FU); every unitary in $U_0(A)$ ($U_0(\tilde{A})$ if A is nonunital) can be approximated by unitaries with finite spectrum.*

iii) *every unitary in* $U_0(A)$ ($U_0(\tilde{A})$ *if* A *is nonunital) can be approximated by unitaries* v *such that 1 and -1 are isolated points in the spectrum* $sp(v)$.

Proof. We have only to show that iii) implies i), and this can be done by modifying the proof of [6, Proposition 1.5].

Let $a \in A_{sa}$. We may assume that A is unital and $\|a\| < \pi$. Set $u = \exp(ia)$, then $u \in U_0(A)$ and hence u is the limit of unitaries $v_n \in U_0(A)$ whose spectra contain 1 and -1 as isolated points. Let log be the branch of the logarithm function with range $i(-\pi, \pi]$. Then for each v_n, $-i\log(v_n) \in A_{sa}$ and $\mathrm{sp}(-i\log(v_n))$ has 0 as an isolated point. Since $a = \lim_n(-i\log(v_n))$ and each self-adjoint element $-i\log(v_n)$ can be approximated by invertible self-adjoint elements we complete the proof. □

One may consider a property like that the set of all unitaries with finite spectra is dense in the set of all unitaries in a unital C^*-algebra, which has been introduced and studied in [6] by the name of (FU).

Lin introduced the condition (UFS) in [3] and showed that if A is a σ-unital C^*-algebra with stable (UFS) and cacellation of projections and B is a hereditary C^*-subalgebra of the multiplier algebra $M(A)$ of A then $U(\tilde{B})$ is connected and $\mathrm{cer}(\tilde{B}) \leq 2 + \varepsilon$. A unital C^*-algebra A is said to have (UFS) if the set of normal partial isometries with finite spectrum is dense in the set of all normal partial isometries, and stable (UFS) if $M_n(A)$ has (UFS) for each $n = 1, 2, 3, \ldots$. A nonunital C^*-algebra A is said to have (UFS) if $\tilde{A}$ has (UFS).

The following shows the relation between the properties (UFS) and (FU):

PROPOSITION 1.2. *A* C^**-algebra* A *has (UFS) if and only if for each nonzero projection* p *in* A *the hereditary* C^**-subalgebra* pAp *has (FU).*

Proof. ($\Rightarrow$) Let $u \in U(pAp)$ and $\varepsilon > 0$. Since pAp has (UFS) [3, 1.2. Lemma], there exists a normal partial isometry $v \in pAp$ with finite spectrum such that $\|u - v\| < \varepsilon(< 1)$. Then $\|u^*u - v^*v\| = \|p - q\| < \varepsilon$, where $q = v^*v$. Since q is a projection in pAp it follows that $p = q$ and v must be a unitary because it is normal.

($\Leftarrow$) Let v be a normal partial isometry with $v^*v = vv^* = p$. Since pAp has (FU) we can find a unitary $u \in U(pAp)$ with finite spectrum such that $\|u - v\| < \varepsilon$. Thus v can be arbitrarily approximated by normal partial isometries of finite spectrum. □

Recall that a unital C^*-algebra A has topological stable rank 1, $tsr(A) = 1$, if the set of invertible elements is dense in A. For a nonunital C^*-algebra A, we define $tsr(A) = tsr(\tilde{A})$. For precise definition and properties refer [9]. Examples are all AF-algebras, the commutative algebra $C(\mathbb{T})$, Bunce-Deddens algebras and irrational rotaion algebras among others.

If a C^*-algebra A contains two orthogonal isometries then $tsr(A) \neq 1$ ($tsr(A) = \infty$ precisely). Cuntz algebras $\mathcal{O}_n$, $B(H)$, and Toeplitz algebra $C^*(S)$ [9, Example 4.13] are examples which have topological stable rank more than one.

PROPOSITION 1.3. *Let A be a unital C^*-algebra with $tsr(A) = 1$. Then A has stable (UFS) if and only if $RR(A) = 0$ and $K_1(A) = 0$.*

Proof. ($\Rightarrow$) Obviously (UFS) implies weak (FU), that is, $RR(A) = 0$. For a C^*-algebra of topological stable rank 1, it is known that $K_1 = U(A)/U_0(A)$. But (UFS) implies that $U(A)$ is connected, so that $K_1(A) = 0$.

($\Leftarrow$) Since the properties, $RR(A) = 0$ and $tsr(A) = 1$, are preserved through the matrix algebras $M_n(A)$ over A it suffices to show that A has (UFS).

We show that for any nonzero projection $p \in A$ the hereditary subalgebra pAp has (FU). Then A has (UFS) by Proposition 1.2. First note that $RR(pAp) = 0$, that is, pAp has weak (FU). By [5, Lemma 2.4] we have $K_1(pAp) = 0$ and hence $K_1(pAp) = U(pAp)/U_0(pAp) = 0$ since $tsr(pAp) = 1$ [2, Corollary 3.6]. Therefore pAp has weak (FU) and has connected unitary group so that it has (FU). □

2. Exponential rank of extensions

As mentioned in section 1, Lin showed that if A is a σ-unital C^*-algebra with stable (UFS) and cancellation of projections then $cer(\tilde{B}) \leq$

$2+\varepsilon$ for any hereditary C^*-subalgebra B in $M(A)$. Especially we have $cer(M(A)) \leq 2+\varepsilon$.

The following theorem gives an upper bound for exponential rank of C^*-extensions. As extremely opposite cases to Lin's, if A is a purely infinite σ-unital (not unital) simple C^*-algebra then it has real rank zero and its corona algebra $(M(A)/A)$ is also purely infinite simple with real rank zero so that both of them have exponential rank $\leq 1+\varepsilon$. Therefore our theorem can be applied to obtain the same bound as Lin's

$$cer(M(A)) \leq 2+\varepsilon$$

if $K_1(A)=0$, that is, $U(\tilde{A})$ is connected.

THEOREM 2.1. *Let I be an ideal of a C^*-algebra A such that the unitary group of $\tilde{I}$ is connected. Then $cer(A) \leq cer(I)+cer(A/I)$ with convention $\varepsilon+\varepsilon=\varepsilon$.*

Proof. Case 1: $cer(I)=m$, $cer(A/I)=n$ for $m,n \in \mathbb{N}$. Let $\pi : A \to A/I$ be the canonical quotient homomorphism. If u is in $U_0(A)$ then $\pi(u) \in U_0(A/I)$. Hence we can write

$$\begin{aligned}\pi(u) &= \exp(i\pi(a_1))\cdots\exp(i\pi(a_n))\\ &= \pi(\exp(ia_1)\cdots\exp(ia_n))\end{aligned}$$

for some self adjoint elements $a_i \in A$. It follows that $u-\exp(ia_1)\cdots\exp(ia_n) \in I$ and we have $u\exp(-ia_n)\cdots\exp(-ia_1)=1+x$ for some $x \in I$. The unitary $1+x$ is the product of m exponentials, therefore we have

$$u=\exp(ix_1)\cdots\exp(ix_m)\exp(ia_1)\cdots\exp(ia_n)$$

for some self adjoint elements $x_i \in I$.

Case 2: $cer(I)=m+\varepsilon$, $cer(A/I)=n$. The proof for this case is similar to that of case 1.

Case 3: $cer(I)=m$, $cer(A/I)=n+\varepsilon$. For the sake of simplicity we may assume that $cer(I^+)$=1 and $cer(A/I) \leq 1+\varepsilon$. For a unitary $u \in U_0(A)$ and $\varepsilon>0$ ($\varepsilon<1$) we can find a self adjoint element $a \in A_{sa}$

such that $\|\pi(u) - \exp(i\pi(a))\| < \varepsilon$. Choose $x' \in I$ such that $\|(u - \exp(ia)) - x'\| < \varepsilon$, so

$$(*) \qquad \|u \exp(-ia) - (1 + x)\| < \varepsilon$$

where $x = x' \exp(ia)$. Then $1 + x$ is invertible and its polar decomposition is of the form $1 + x = \exp(ib)|1 + x|$ for some $b \in I_{\text{sa}}$ because $cer(I)$=1. It follows that $\|u - \exp(ib)|1 + \exp x| \exp(ia)\| < \varepsilon$. Then we have

$$\begin{aligned} \|u - \exp\exp(ib)\exp(ia)\| &\leq \|u - \exp(ib)|1 + x| \exp(ia)\| \\ (**) \qquad &+ \|\exp(ib)|1 + x| \exp(ia) - \exp(ib)\exp(ia)\| \\ &< \varepsilon + \|1 - |1 + x|\|. \end{aligned}$$

From $(*)$ we see that $\text{sp}(1 + x) \subset \{\lambda \in \mathbb{C} |\ 1 - \varepsilon \leq |\lambda| \leq 1 + \varepsilon\}$, hence $\|1 + x\| \leq 1 + \varepsilon$.

Put $v = u \exp(-ia)$ then

$$\begin{aligned} \|1 - (1 + x)^*(1 + x)\| &\leq \|v^*v - (1 + x)^*(1 + x)\| \\ &\leq \|v^*v - (1 + x)^*v\| \\ &\qquad + \|(1 + x)^*v - (1 + x)^*(1 + x)\| \\ &\leq \varepsilon + \|(1 + x)^*\|\|v - (1 + x)\| \\ &\leq \varepsilon + (1 + \varepsilon)\varepsilon \\ &< 3\varepsilon. \end{aligned}$$

Therefore $\text{sp}(|1+x|)=\text{sp}(((1+x)^*(1+x))^{1/2}) \subset (1-3\varepsilon, 1+3\varepsilon)$, that is, $\|1-|1+x|\| < 3\varepsilon$ and it follows from $(**)$ that $\|u-\exp(ib)\exp(ia)\| < 4\epsilon$.

Case 4: $cer(I) = m + \varepsilon$, $cer(A/I) = n + \varepsilon$. It is obvious from the above cases. □

Let $C^*(S)$ be the C^*-algebra generated by the unilateral shift S on an infinite dimensional separable Hilbert space. Then $C^*(S)$ contains the C^*-algebra $\mathcal{K}$ of compact operators as an ideal and the quotient algebra $C^*(S)/\mathcal{K}$ is isomorphic to the commutative C^*-algebra $C(\mathbb{T})$ where $\mathbb{T}$ is the unit circle. Since the unitary group of $\tilde{K}$ is connected with $cer(\mathcal{K})$=1 and C^*-exponential of every commutative C^*-algebra is obviously 1, we obtain the following.

COROLLARY 2.2. $cer(C^*(S)) \leq 2$.

References

[1] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.

[2] ———, *On the geometry of the unit ball of a C^*-algebra*, J. reine angew. **469** (1995), 113–147.

[3] H. Lin, *per Generalized Weyl-von Neumann theorems*, Inter. J. of Math. **2** (1991), 725–739.

[4] ———, *Exponential rank of C^*-algebras with real rank zero and Brown-Pedersen conjectures*, J. Funct. Anal. **114** (1993), 1-11.

[5] ———, *Generalized Weyl-von Neumann theorems* (II), Math. Scand. **77** (1995), 129–147.

[6] N. C. Phillips, *Simple C^*-algebras with the property weak* (FU), Math. Scand. **69** (1991), 127–151.

[7] ———, *C^*-algebras: 1943-1993 A fifty year celebration, C ontemporary Math.*, vol. **167**, Amer. Math. Soc., 1994, pp. 353-399.

[8] ———, *UHF algebras have neither property* (FS') *nor exponential rank 1*, preprint.

[9] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), 301–333.

Ja A Jeong
Department of Mathematics, Kyung Hee University, Seoul 130-701, Korea

Gie Hyun Park
Department of Mathematics, Hanshin University, Osan 447-791, Korea