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MORAVA $K-$ THEORY OF THE DOUBLE LOOP SPACES OF QUATERNIONIC STIEFEL MANIFOLDS

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ABSTRACT. In this paper we get the Morava $K-$ theory of the double loop spaces of quarternionic Stiefel manifolds for an odd prime p by computing the Atiyah – Hirzebruch spectral sequence. We also get the homology with $Z/(p)$ coefficients and analyze p torsion in the homology with Z coefficients.

1. Introduction

Let MU be the Thom spectrum for the unitary group. Quillen constructed a multiplicative idempotent map of ring spectra $\epsilon : MU_{(p)} \to MU_{(p)}$ by localizing the spectrum MU at a prime p [5]. Then for a space X, the image of ϵ_* in $MU_*(X)_p$ becomes a natural direct summand of $MU_*(X)_p$ and satisfies all the axioms for a generalized homology theory. So by the Brown's representability theorem in [2] it has the representing spectrum. This representing spectrum is denoted by BP with $\pi_*(BP) = BP_* = Z_{(p)}[v_1, v_2, \dots]$, deg $v_i = 2(p^i - 1)$. The spectra $k(n)$ can be obtained from the spectrum BP by killing certain bordism classes $(p, v_1, \dots, v_{n-1}, v_{n+1}, \dots)$ in BP_* via Bass-Sullivan construction in [1]. These $k(n)$ are the spectra for the connective Morava K-theories. The spectra $K(n) = \varinjlim_{v_n} \sum^{-2i(p^n-1)} k(n)$ are the representing spectra for Morava K–theories where $\pi_*(K(n)) = Z/(p)[v_n, v_n^{-1}]$.

So there is a sequence of homology theories for each n. Morava $K-$ theories satisfy many nice properties. Since $K(n)_*$ is the graded field in

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the sense that every graded module over $K(n)_*$ is free, $Tor_1^{K(n)_*}(K(n)_*(X), K(n)_*(Y)) = 0$ for all spaces X, Y. Hence from the Künneth formula,

$$K(n)_*(X \times Y) = K(n)_*(X) \otimes K(n)_*(Y).$$

For the case $n = 0$, $K(0)_*(X) = H_*(X; Q)$ and $K(1)_*(X)$ is one of $p-1$ isomorphic summands of mod p complex K–theory for all p.

In this paper we study the Morava K–theory for an odd prime p of the double loop spaces of the quaternionic Stiefel manifolds by computing the Atiyah–Hirzebruch spectral sequence with the structure of the Morava K–theory of the double loop spaces of the spheres in [7].

Besides the rational homology and the mod p complex K-theory, we get the homology with $Z/(p)$ coefficients. Owing to the identification between the Atiyah-Hirzebruch spectral sequence with $E_2 = H_*(X; Z/(p)) \otimes k(m)_*$ and the Bockstein spectral sequence which analyzes the v_m torsion in $k(m)_*(X)$, we analyze the torsion in the connective Morava K–theory and the p torsion in the homology with Z coefficients from the actions of the higher order Milnor operators and the actions of the higher order Bockstein operators on the homology with $Z/(p)$ coefficients. As a special case, the Morava K–theory of the double loop space of the symplectic group can be obtained from above results.

We consider only the odd primary cases so that the spectra $K(n)$ are commutative ring spectra. Hence in this paper p always denotes an odd prime.

2. Main contents

Let $E(x)$ be the exterior algebra on x and $P(x)$ be the polynomial algebra on x and $\Gamma(x)$ be the divided power algebra on x. Let $\Omega^n X$ be the space of all pointed continuous maps from S^n to a space X. Let $V_{n,n-k}$ be the space of all $n-k$ frames in H^n where H is the algebra of quaternionic. Then we call $V_{n,n-k}$ the quaternionic Stiefel manifold which can be identified with $Sp(n)/Sp(k)$. Throughout this paper the subscript of an element always means the degree of an element.

We have the following well known fact [6], [7]. For an odd prime p,

$$K(m)_*(\Omega^2 S^{4n+3}) = E(x_{(4n+2)p^i-1} : 0 \leq i \leq m) \otimes T_m(y_{(4n+2)p^j-2} : j \geq 1)$$

where $T_m(y)$ denotes the truncated polynomial algebra on y of height p^m, that is, $P(y)/(y^{p^m})$.

First we calculate the homology of the triple loop space of $Sp/Sp(n)$.

THEOREM 2.1.

$$H_*(\Omega^3 Sp/Sp(n+1); Z/(p)) = P(y_{4n+4i+4} : i \geq 0).$$

Proof. Consider the following fibration:

$$Sp(n+1) \longrightarrow Sp \longrightarrow Sp/Sp(n+1)$$

It is well-known that

$$\begin{aligned} &H^*(Sp(n); Z/(p)) = E(a_{4i+3} : 0 \leq i \leq n-1) \\ &\mathcal{P}^j(a_{4i+3}) = (-1)^{j(p-1)/2}\tbinom{2i+1}{j} a_{4i+3+2j(p-1)} \end{aligned}$$

where $\mathcal{P}^j$ is the Steenrod operation. Note that $\mathcal{P}^n x = x^p$ for $x \in H^{2n}$.

So we get

$$\begin{aligned} &H^*(Sp/Sp(n+1); Z/(p)) = E(a_{4n+4i+7} : i \geq 0) \\ &\mathcal{P}^j(a_{4i+3}) = (-1)^{j(p-1)/2}\tbinom{2i+1}{j} a_{4i+3+2j(p-1)}\,. \end{aligned}$$

We have the Eilenberg–Moore spectral sequence of the Steenrod module converging to $H^*(\Omega Sp/Sp(n+1); Z/(p))$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(Sp/Sp(n+1);Z/(p))}(Z/(p), Z/(p)) \\ &= \mathrm{Tor}_{E(a_{4n+4i+7}:i\geq 0)}(Z/(p), Z/(p)) \\ &= \Gamma(b_{4n+4i+6} : i \geq 0)\,. \end{aligned}$$

Since E_2 is even–dimensional, $E_2 = E_\infty$ and

$$\mathcal{P}^j(b_{4i+2}) = (-1)^{j(p-1)/2}\binom{2i+1}{j} b_{4i+2+2j(p-1)}.$$

Hence $b_{4i+2}^p = \mathcal{P}^{2i+1}(b_{4i+2}) = (-1)^{(2i+1)(p-1)/2} b_{(4i+2)p}$. So we have the choices of generators c_i such that

$$H^*(\Omega Sp/Sp(n+1); Z/(p)) = P(c_{4n+4i+6} : i \geq 0).$$

Consider the Eilenberg–Moore spectral sequence again converging to $H^*(\Omega^2 Sp/Sp(n+1); Z/(p))$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(\Omega Sp/Sp(n+1);Z/(p))}(Z/(p), Z/(p)) \\ &= \mathrm{Tor}_{P(c_{4n+4i+6}:i\geq 0)}(Z/(p), Z/(p)) \\ &= E(z_{4n+4i+5} : i \geq 0). \end{aligned}$$

This spectral sequence is the spectral sequence of a Hopf algebra so that the source of the first non trivial differential should be indecomposable and the target should be primitive. Since the target must be even-dimensional and every primitive element of E_2 is odd-dimensional, $E_2 = E_\infty$. Hence we get

$$H^*(\Omega^2 Sp/Sp(n+1); Z/(p)) = E(z_{4n+4i+5} : i \geq 0).$$

Now we apply the Eilenberg–Moore spectral sequence again converging to $H_*(\Omega^3 Sp/Sp(n+1); Z/(p))$ with

$$\begin{aligned} E_2 &= \mathrm{Ext}_{H^*(\Omega^2 Sp/Sp(n+1);Z/(p))}(Z/(p), Z/(p)) \\ &= \mathrm{Ext}_{E(z_{4n+4i+5}:i\geq 0)}(Z/(p), Z/(p)) \\ &= P(y_{4n+4i+4} : i \geq 0). \end{aligned}$$

Since E_2 is even-dimensional, $E_2 = E_\infty$ and we get

$$H_*(\Omega^3 Sp/Sp(n+1); Z/(p)) = P(x_{4n+4i+4} : i \geq 0).$$

□

Now consider the inclusion map $\iota : Sp(n)/Sp(k) \to Sp/Sp(k)$. We can convert this map to a homotopy equivalent fiber map by using the fact that $Sp(n)/Sp(k)$ is homotopy equivalent to the space of all paths in $Sp/Sp(k)$ with initial points in $Sp(n)/Sp(k)$ under a homotopy which retracts all paths back to their initial points. Let $S_{n,k}$ be the fiber of this fiber map. So we have the following fibration up to homotopy:

$$S_{n,k} \longrightarrow Sp(n)/Sp(k) \longrightarrow Sp/Sp(k)$$

COROLLARY 2.2.

$$H_*(\Omega^2 S_{n+1,k}; Z/(p)) = P(y_{4n+4i+4} : i \geq 0).$$

Proof. We have the map of the fibrations:

$$\begin{array}{ccccc}
\Omega^2 Sp/Sp(n+1) & \longrightarrow & \Omega Sp(n+1) & \longrightarrow & \Omega Sp \\
\downarrow & & \downarrow & & \downarrow \\
\Omega S_{n+1,k} & \longrightarrow & \Omega Sp(n+1)/Sp(k) & \longrightarrow & \Omega Sp/Sp(k) \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & Sp(k) & \longrightarrow & Sp(k)
\end{array}$$

So we have the fibration:

$$\Omega^2 Sp/Sp(n+1) \longrightarrow \Omega S_{n+1,k} \longrightarrow *$$

Hence $\Omega^2 S_{n+1,k}$ is homotopy equivalent to $\Omega^3 Sp/Sp(n+1)$. □

COROLLARY 2.3.

$$K(m)_*(\Omega^3 Sp/Sp(n+1)) = K(m)_*(\Omega^2 S_{n+1,k}) = P(y_{4n+4i+4} : i \geq 0)$$

where $P(y_{4n+4i+4} : i \geq 0)$ *means* $K(m)_*[y_{4n+4i+4} : i \geq 0]$.

Proof. We consider the Atiyah–Hirzebruch spectral sequence converging to $K(n)_*(\Omega^3 Sp/Sp(n+1))$ with

$$\begin{aligned}
E_2 &= H_*(\Omega^3 Sp/Sp(n+1); K(m)_*) \\
&= H_*(\Omega^3 Sp/Sp(n+1); Z/(p)) \otimes K(m)_*.
\end{aligned}$$

Since $H_*(\Omega^3 Sp/Sp(n+1); Z/(p))$ is even dimensional, the spectral sequence collapses from the E_2–term. □

From now on, the element $y_{4n+4i+4}$ in $H_*(\Omega^2 S_{n+1,k})$ or $K(m)_*(\Omega^2 S_{n+1,k})$ will be denoted by $y_{n+1,4n+4i+4}$.

For each odd number j with $j \neq 0 \bmod p$, let $t(n+1,j)$, $n \geq k$, be the non negative integer satisfying the following condition:

$$2k+1 \leq jp^{t(n+1,j)-1} \leq 2n+1 < jp^{t(n+1,j)}$$

and let $t(k,j)$ be the smallest non negative integer satisfying the following condition: $2jp^{t(k,j)} - 2 > 4k+3$ which implies $jp^{t(k,j)} \geq 2k+1$ and let

$$t(n+1,k,j) = t(n+1,j) - t(k,j).$$

THEOREM 2.4.

$$\begin{gathered} K(m)_*(\Omega^2 Sp(n+1)/Sp(k)) = \\ E(x_{2jp^{t(k,j)+i}-1} : j : odd,\ p \nmid j,\ 0 \leq i < t(n+1,k,j)(m+1)) \\ \otimes T_{t(n+1,k,j)m}(y_{2jp^{i+t(n+1,j)}-2} : j : odd,\ p \nmid j,\ i \geq 0)\,. \end{gathered}$$

Proof. Consider the following map of the fibrations:

$$\begin{array}{ccccc} \Omega^2 S_{n,k} & \longrightarrow & \Omega^2 Sp(n)/Sp(k) & \longrightarrow & \Omega^2 Sp/Sp(k) \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow & & \| \\ \Omega^2 S_{n+1,k} & \longrightarrow & \Omega^2 Sp(n+1)/Sp(k) & \longrightarrow & \Omega^2 Sp/Sp(k) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^2 S^{4n+3} & = & \Omega^2 S^{4n+3} & \longrightarrow & * \end{array}$$

Now we will illustrate the behavior of the composite of f_n's. Consider the following map of the fibrations:

$$\begin{array}{ccccc} \Omega^2 S_{l,k} & \longrightarrow & * & \longrightarrow & \Omega S_{l,k} \\ {\scriptstyle f_{l+1}}\downarrow & & \downarrow & & \| \\ \Omega^2 S_{l+1,k} & \longrightarrow & \Omega^2 S^{4l+3} & \longrightarrow & \Omega S_{l,k} \end{array}$$

From Morava K-theory $K(m)$ of $\Omega^2 S^{4l+3}$, we get

$$\begin{aligned} K(m)_*(f_{l+1})(y_{l,(4l+2)p^{j+m}-2}) &= v_m (y_{l+1,(4l+2)p^j-2})^{p^m},\ j \geq 1 \\ K(m)_*(f_{l+1})(y_{l,i}) &= y_{l+1,i} \text{ for the other degrees.} \end{aligned}$$

Consider the map of the fibrations:

$$\begin{array}{ccccc}
\Omega^2 S_{lp+\frac{p-1}{2},k} & \longrightarrow & * & \longrightarrow & \Omega S_{lp+\frac{p-1}{2},k} \\
{\scriptstyle f_{lp+\frac{p+1}{2}}}\downarrow & & \downarrow & & \| \\
\Omega^2 S_{lp+\frac{p+1}{2},k} & \longrightarrow & \Omega^2 S^{(4l+2)p+1} & \longrightarrow & \Omega S_{lp+\frac{p-1}{2},k}
\end{array}$$

From Morava K-theory $K(m)$ of $\Omega^2 S^{(4l+2)p+1}$, we get

$$K(m)_*(f_{lp+\frac{p+1}{2}})(y_{lp+\frac{p-1}{2},(4l+2)p^j+m-2}) = v_m(y_{lp+\frac{p+1}{2},(4l+2)p^j-2})^{p^m},\ j\geq 2.$$

Then

$$\begin{aligned}
&K(m)_*(f_{lp+\frac{p+1}{2}}\circ\cdots\circ f_{l+2}\circ f_{l+1})(y_{l,(4l+2)p^j+2m-2}) \\
&= v_m(v_m(y_{lp+\frac{p+1}{2},(4l+2)p^j-2})^{p^m})^{p^m} \\
&= v_m^{p^m+1}(y_{lp+\frac{p+1}{2},(4l+2)p^j-2})^{p^{2m}},\ j\geq 2
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&K(m)_*(f_{lp^2+\frac{p^2+1}{2}}\circ\cdots\circ f_{l+1})(y_{l,(4l+2)p^j+3m-2}) \\
&= v_m^{p^{2m}+p^m+1}(y_{lp^2+\frac{p^2+1}{2},(4l+2)p^j-2})^{p^{3m}},\ j\geq 3.
\end{aligned}$$

Note that in the image of the composite map, the power of v_m depends on the number j of the lowest dimensional element of form $y_{lp+\frac{p+1}{2},(4l+2)p^j-2}$.

Let $f=f_{n+1}\circ f_n\cdots\circ f_{k+1}$. We have the map of the fibrations:

$$\begin{array}{ccccc}
\Omega^3 Sp/Sp(k) & \longrightarrow & * & \longrightarrow & \Omega^2 Sp/Sp(k) \\
{\scriptstyle f}\downarrow & & \downarrow & & \| \\
\Omega^2 S_{n+1,k} & \longrightarrow & \Omega^2 Sp(n+1)/Sp(k) & \longrightarrow & \Omega^2 Sp/Sp(k)
\end{array}$$

Consider the Atiyah–Hirzebruch spectral sequence converging to $K(n)_*$ $(\Omega^2 Sp(n+1)/Sp(k))$ with

$$\begin{aligned}
E_2&=H_*(\Omega^2 Sp/Sp(k); K(m)_*(\Omega^2 S_{n+1,k})) \\
&=H_*(\Omega^2 Sp/Sp(k); Z/(p))\otimes K(m)_*(\Omega^2 S_{n+1,k}).
\end{aligned}$$

Note that $\Omega^2 S_{n+1,k}$ is $4n+3$ connected and $H_*(\Omega^2 Sp/Sp(k); Z/(p)) = E(x_{4k+4i+1} : i \geq 0)$. Let j be the odd number and $j \neq 0 \bmod p$ and i be the non-negative integer such that $2jp^i - 1 \geq 4k+1$, that is, $jp^i \geq 2k+1$ so $i \geq t(k,j)$.

Note that $t(n+1,j)$ is defined to satisfy the following condition:

$$4k \leq 2jp^{t(n+1,j)-1} - 2 \leq 4n+3 < 2jp^{t(n+1,j)} - 2,$$

which implies $2k+1 \leq jp^{t(n+1,j)-1} \leq 2n+1 < jp^{t(n+1,j)}$. Since $\Omega^2 S_{n+1,k}$ is $4n+3$ connected, $y_{n+1,2jp^{t(n+1,j)}-2}$ is the lowest degree element of the form $y_{n+1,2jp^t-2}$ in $K(m)_*(\Omega^2 S_{n+1,k})$. Then we have

$$\begin{cases} K(m)_*(f)(y_{k,2jp^{i+t(n+1,j)+t(n+1,k,j)m}}) \\ \quad = v_m^{p^{(t(n+1,k,j)-1)m}+p^{(t(n+1,k,j)-2)m}+\cdots p^{m}+1} \\ \qquad (y_{n+1,2jp^{i+t(n+1,j)}-2})^{p^{t(n+1,k,j)m}}, \ i \geq 0 \\ K(m)_*(f)(y_{k,i}) = y_{n+1,i} \ \text{for the other degrees.} \end{cases}$$

By the naturality, we have the following differentials

$$\begin{cases} d(x_{2jp^{i+t(n+1,j)+t(n+1,k,j)m}-1}) \\ \quad = v_m^{p^{(t(n+1,k,j)-1)m}+p^{(t(n+1,k,j)-2)m}+\cdots p^m+1} \\ \qquad (y_{n+1,2jp^{i+t(n+1,j)}-2})^{p^{t(n+1,k,j)m}}, i \geq 0, \\ d(x_{2jp^i-1}) = 0 \ \text{for} \ 0 \leq i < t(n+1,j) + t(n+1,k,j)m, \ i \geq t(k,j) \\ d(x_i) = y_{n+1,i-1} \ \text{for the other degrees.} \end{cases}$$

Then x_{2jp^i-1} survives permanently for $t(k,j) \leq i < t(n+1,j) + t(n+1,k,j)m$ and $(y_{n+1,2jp^{i+t(n+1,j)}-2})^\ell$ also survives permanently for $i \geq 0$ and $0 \leq \ell < p^{t(n+1,k,j)m}$. Since $t(n+1,k,j) = t(n+1,j) - t(k,j)$, $x_{2jp^{t(k,j)+i}-1}$ survive for $0 \leq i < t(n+1,k,j)(m+1)$. □

COROLLARY 2.5.

$$H_*((\Omega^2 Sp(n+1)/Sp(k); Q) = K(0)_*(\Omega^2 Sp(n+1)/Sp(k)) = \\ E(x_{2j-1} : 2k+1 \leq j \leq 2n+1, \ j \ odd, \ j \neq 0 \bmod p).$$

Proof. Since $m = 0$, the truncated polynomial algebra T_0 disappear. Let n be fixed. Note that Theorem 2.4 holds for any odd prime number p. Since $t(n+1,j)$ is the integer satisfying the condition:

$$2k+1 \leq jp^{t(n+1,j)-1} \leq 2n+1 < jp^{t(n+1,j)},$$

$t(n+1,j)$ is 1 for a sufficiently large odd prime number p. Hence $2k+1 \leq j \leq 2n+1$. □

Now we turn to the mod p homology.

THEOREM 2.6.

$$\begin{aligned} H_*(\Omega^2 Sp(n+1)/Sp(k);Z/(p)) = & E(x_{2jp^{t(k,j)+i}-1} : j : odd,\ p \nmid j,\ i \geq 0) \\ & \otimes P(y_{2jp^{i+t(n+1,j)}-2} :\ j : odd,\ p \nmid j,\ i \geq 0)\,. \end{aligned}$$

Proof. In the proof of Theorem 2.4, we computed the Atiyah–Hirzebruch spectral sequence converging to $K(m)_*(\Omega^2 Sp(n+1))/Sp(k)$ with

$$\begin{aligned} E_2 = & H_*(\Omega^2 Sp/Sp(k)) \otimes K(m)_*(\Omega^2 S_{n+1,k}) \\ = & H_*(\Omega^2 Sp/Sp(k)) \otimes H_*(\Omega^2 S_{n+1,k}) \otimes K(m)_* \end{aligned}$$

and we got the following differentials

$$\begin{cases} d(x_{2jp^{i+t(n+1,j)+t(n+1,k,j)m}-1}) \\ \quad = v_m^{p^{(t(n+1,k,j)-1)m}+p^{(t(n+1,k,j)-2)m}+\cdots p^m+1} \\ \qquad (y_{n+1,2jp^{i+t(n+1,j)}-2})^{p^{t(n+1,k,j)m}}, i \geq 0, \\ d(x_{2jp^i-1}) = 0,\ t(k,j) \leq i < t(n+1,j) + t(n+1,k,j)m \\ d(x_i) = y_{n+1,i-1} \text{ for the other degrees.} \end{cases}$$

For each element x_{2jp^t-1} of any fixed j and t, if we choose m to be large enough that $t(n+1,j) + t(n+1,k,j)m$ is larger than t, then we have $d(x_{2jp^t-1}) = 0$. Now we consider the Serre spectral sequence converging to $H_*(\Omega^2 Sp(n+1))/Sp(k)$ with

$$E_2 = H_*(\Omega^2 Sp/Sp(k)) \otimes H_*(\Omega^2 S_{n+1,k}),$$

then we have

$$\begin{cases} d(x_{2jp^i-1}) = 0, \ i \geq t(k,j) \\ d(x_i) = y_{n+1,i-1} \text{ for the other degrees.} \end{cases}$$

So $x_{2jp^{t(k,j)+i}-1}$ survives permanently for all $i \geq 0$ and $(y_{n+1,2jp^i-2})$ survives permanently for $i \geq t(n+1,j)$. Therefore we get the conclusion. □

Now we consider the connective Morava K theory.

COROLLARY 2.7.

$$k(m)_*(\Omega^2 Sp(n+1)/Sp(k))/(v_m^\infty) =$$
$$E(x_{2jp^{t(k,j)+i}-1} : j : odd, \ p \nmid j, \ 0 \leq i < t(n+1,k,j)(m+1))$$
$$\otimes T_{t(n+1,k,j)m}(y_{2jp^{i+t(n+1,j)}-2} : j : odd, \ p \nmid j, \ i \geq 0).$$

Here we denote $(v_m^\infty) = \bigcup_{i\geq 1}(v_m^i)$ *and* $(v_m^i) = \{x \in k(m)_*(\Omega^2 Sp(n+1)/Sp(k)) | v_m^i x = 0\}$.

In the Atiyah–Hirzebruch spectral sequence which converges to $k(m)_*(X)$ with

$$E_2 = H_*(X; k(m)_*),$$

as the classical K–theory, the first non-trivial differential in $k(m)$ theory is determined by the Milnor operation $\mathcal{Q}_m$ in [4], where $\mathcal{Q}_m$ is defined inductively as the commutator for

$$\mathcal{Q}_0 = \beta,$$
$$\mathcal{Q}_{k+1} = [\mathcal{Q}_k, \mathcal{P}_*^{p^k}]$$

where β is the mod p homology Bockstein operation. Let $\mathcal{Q}_m^{(r)}$ be the r-th order Milnor operation defined by the relations $\mathcal{Q}_m \mathcal{Q}_m^{(r-1)} = 0$ where $deg\, \mathcal{Q}_m^{(r)} = -2r(p^m-1)-1$.

In particular, the differentials in the Atiyah–Hirzebruch spectral sequence for $k(m)_*(X)$ are given by the k-invariants of $k(m)$ [3], so all the higher order non-trivial differentials are determined by the higher order Milnor operations given by $d_{2r(p^m-1)+1}(x \otimes v_m^i) = c\mathcal{Q}_m^{(r)} x \otimes v_m^{i+r}$, $c \neq 0$ mod p. That means that there is the identification between the Atiyah-Hirzebruch spectral sequence with $E_2 = H_*(X; Z/(p)) \otimes k(m)_*$ and the Bockstein spectral sequence which analyzes the v_m torsion in $k(m)_*(X)$.

COROLLARY 2.8. *Let* $s = p^{(t(n+1,k,1)-1)m} + p^{(t(n+1,k,1)-2)m} + \cdots p^m + 1$. *Then* v_m^s *annihilates all the* v_m *torsions in* $k(m)_*(\Omega^2 Sp(n+1)/Sp(k))$.

Proof. As we mentioned, there is the identification between the Atiyah-Hirzebruch spectral sequence with $E_2 = H_*(X; Z/(p)) \otimes k(m)_*$ and the Bockstein spectral sequence. We can interpret the Atiyah-Hirzebruch spectral sequence for Theorem 2.4 as the Atiyah-Hirzebruch spectral sequence with $E_2 = H_*(\Omega^2 Sp(n+1))/Sp(k); Z/(p)) \otimes k(m)_*$. From the proof of Theorem 2.4, the sets $(v_m^{p^{(t(n+1,k,j)-1)m} + p^{(t(n+1,k,j)-2)m} + \cdots p^m + 1})$ are non–empty. For example, there are elements of degree $(2jp^t - 2)p^{t(n+1,k,j)m}$, $t \geq t(n+1,j)$ of v_m torsion of order $p^{(t(n+1,k,j)-1)m} + p^{(t(n+1,k,j)-2)m} + \cdots p^m + 1$. Since $max\{t(n+1,k,j) : j\} = t(n+1,k,1)$, $v_m^{p^{(t(n+1,k,1)-1)m} + p^{(t(n+1,k,1)-2)m} + \cdots p^m + 1}$ annihilates all the v_m torsions. □

COROLLARY 2.9. *There exist nontrivial actions of the higher order Milnor operators* $Q_m^{(r)}$ *on* $H_*(\Omega^2 Sp(n+1)/Sp(k); Z/(p))$ *such that*

$$Q_m^{(p^{(t(n+1,k,j)-1)m} + p^{(t(n+1,k,j)-2)m} + \cdots p^m + 1)}((x_{2jp^{i+t(n+1,j)+t(n+1,k,j)m}-1}) = (y_{2jp^{i+t(n+1,j)}-2})^{p^{t(n+1,k,j)m}}, i \geq 0.$$

From above information we analyze the p torsion in the homology with Z coefficients.

COROLLARY 2.10. $p^{t(n+1,k,1)}$ *annihilates all the* p *torsion in* $H_*(\Omega^2 Sp(n+1)/Sp(k); Z)$. *That is, if we let* $p^{r-1} \leq 2n+1 < p^r$ *and* s *be the smallest integer satisfying the condition:* $p^{s-1} < 2k+1 \leq p^s$, *then* p^{r-s} *annihilates all the* p *torsions in* $H_*(\Omega^2 Sp(n+1)/Sp(k); Z)$.

Proof. $k(0)_*(\Omega^2 Sp(n+1)/Sp(k))$ consists of the torsion free part and the torsion part. The torsion free part is that $k(0)_*(\Omega^2 Sp(n+1)/Sp(k))/(v_0^\infty) = E(x_{2j-1} : 2k+1 \leq j \leq 2n+1,\ j\, odd,\ j \neq 0 \, mod\, p)$. We have already computed in Theorem 2.4 :

$$\begin{cases} K(m)_*(f)(y_{k,2jp^{i+t(n+1,j)+t(n+1,k,j)m}}) \\ \quad = v_m^{p^{(t(n+1,k,j)-1)m} + p^{(t(n+1,k,j)-2)m} + \cdots p^m + 1} \\ \quad\quad (y_{n+1,2jp^{i+t(n+1,j)}-2})^{p^{t(n+1,k,j)m}}, \; i \geq 0 \\ K(m)_*(f)(y_{k,i}) = y_{n+1,i} \text{ for the other degrees.} \end{cases}$$

We know that $k(0)_*(X) = H_*(X;Z)_{(p)}$ with $v_0 = p$. From above we get that

$$\begin{cases} k(0)_*(f)(y_{k,2jp^{i+t(n+1,j)}}) = v_0^{t(n+1,k,j)}(y_{n+1,2jp^{i+t(n+1,j)}-2}),\ i \geq 0 \\ = p^{(t(n+1,k,j)}(y_{n+1,2jp^{i+t(n+1,j)}-2}),\ i \geq 0 \\ k(0)_*(f)(y_{k,i}) = y_{n+1,i} \text{ for the other degrees.} \end{cases}$$

Owing to the identification between the Atiyah-Hirzebruch spectral sequence and the Bockstein spectral sequence, we have the nontrivial higher order differentials $\beta^{t(n,j)}$ in $H_*(\Omega^2 Sp(n+1)/Sp(k); Z/(p))$ such that

$$\beta^{t(n+1,k,j)}(x_{2jp^i-1}) = y_{2jp^i-2} \text{ for } i \geq t(n+1,j).$$

Hence in $H_*(\Omega^2 Sp(n+1)/Sp(k); Z)$ we have the elements of degree $(2jp^i - 2)$ for $i \geq t(n+1,j)$ of the p torsion of order $t(n+1,k,j)$. Since $max\{t(n+1,k,j) : j\} = t(n+1,k,1)$, $t(n+1,k,1)$ annihilates all the p torsions in $H_*(\Omega^2 Sp(n+1)/Sp(k); Z)$. □

COROLLARY 2.11. *There exist nontrivial higher order Bockstein actions $\beta^{t(n,j)}$ on $H_*(\Omega^2 Sp(n+1)/Sp(k); Z/(p))$ such that $\beta^{t(n+1,k,j)}$ $(x_{2jp^i-1}) = y_{2jp^i-2}$ for $i \geq t(n+1,j)$.*

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