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A LOCAL CONJUGACY IN LOCALLY FINITE CC-GROUPS

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ABSTRACT. A conjugacy theorem which holds for finite groups is proven to hold for Černikov groups and locally finite CC-groups.

1. Introduction

If G is a locally finite group, by $\pi(G)$ we denote the set of all prime numbers dividing the orders of elements of G. Also we denote the radical part of G by G^0 and the Hirsh-Plotkin radical of G by $\Phi(G)$. If X is locally nilpotent, then for a set σ of primes X has the unique Sylow σ-subgroup, denoted by X_σ. As usual $Linn(G)$ means the set of all locally inner automorphisms of G. If G is a Černikov group, then we define the rank of G, $r(G) = \sum_{p\in\pi(G)} rank(O_p(G^0))$, and we define $i(G) = |G/G^0|$. The pair $(r(G), i(G))$ will be called the size of G, denoted by $|G|$. We can give a well-order on the sizes of Černikov groups lexicographically. We say that two subgroups U and V of a group G are p-conjugate (p-locally conjugate) if a Sylow p-subgroup of U is conjugate (locally conjugate) to a Sylow p-subgroup of V.

In 1979, Losey and Stonehewer proved [4]:

THEOREM 1. *Let G be a finite solvable group. Let U and V be p-conjugate for every prime p. Suppose that U and V have a nilpotent common normal supplement X in G and that one of the following conditions is satisfied:*

1. *X is abelian;*

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2. *G/X is nilpotent;*
3. *the Sylow p-subgroups of G have class at most 2 for every prime p.*

Then U and V are conjugate.

In this theorem, the solvability of G is not necessary [3]. This paper is devoted to obtain the similar result for the locally finite CC-groups. We consider the Černikov groups first and prove:

THEOREM 2. *Let G be a Černikov group. Let U and V be p-conjugate for every prime p. Suppose that U and V have a locally nilpotent common normal supplement X in G and that one of the following conditions is satisfied:*

1. *X is abelian;*
2. *G/X is locally nilpotent;*
3. *the Sylow p-subgroups of G have class at most 2 for every prime p.*

Then U and V are conjugate.

In the next section we will prove Theorem 2. For the proof of this theorem we need the following fact:

LEMMA 1. *Let G be a Černikov group. If $G = XU = XV$, for some normal subgroup X of G and subgroups U and V satisfying $\pi(X) \cap \pi(U) = \pi(X) \cap \pi(V) = \emptyset$, then U and V are conjugate.*

Proof. Clearly $U^0 = V^0$ is normal in G. Consider

$$G/U^0 = (XU^0/U^0)(U/U^0) = (XU^0/U^0)(V/U^0).$$

If we set $H/U^0 = < U/U^0, V/V^0 >$, then by the Dedekind Law $H/U^0 = (XU^0/U^0 \cap H/U^0)(U/U^0) = (XU^0/U^0 \cap H/U^0)(V/U^0)$. However H/U^0 is finite, so by the Schur-Zassenhaus theorem, U and V are conjugate. □

Using Theorem 2 we will prove the following main theorem in section 3.

THEOREM 3. *Let G be a locally finite CC-group. Let U and V be p-locally conjugate for every prime p. Suppose that U and V have a locally nilpotent common normal supplement X in G and that one of the following conditions is satisfied:*

1. *X is abelian;*
2. *G/X is locally nilpotent;*
3. *the Sylow p-subgroups of G have class at most 2 for every prime p.*

Then U and V are locally conjugate.

As an application of Theorem 3 we show:

THEOREM 4. *The Sylow π-subgroups of a locally finite and locally solvable CC-group are locally conjugate, where π is a set of primes.*

2. Proof of Theorem 2

In this section we prove Theorem 2. First, we obtain various reductions that enable us to strengthen the hypotheses.

Throughout this section G is assumed to be a Černikov group. Moreover, $G = XU = XV$ is assumed to be a counter example to Theorem 2.

LEMMA 2. *There are counter examples in which X is a p-group for some prime p.*

Proof. Suppose no counter example exists with $\pi(X) = \{p\}$. We shall prove by induction on $|\pi(X)|$ that G does not exist. By assumption, the induction starts with $|\pi(X)| = 1$. Suppose that Theorem 2 holds for $K = YU = YV$, with $|\pi(Y)| < |\pi(X)|$. Now $X = X_pX_{p'}$, so $G/X_p = (X/X_p)(UX_p/X_p) = (X/X_p)(VX_p)/X_p)$. However $|\pi(X/X_p)| = |\pi(X)|-1$, and UX_p/X_p and VX_p/X_p are q-conjugate for every prime q. So by the induction hypothesis, there exists $g \in G$ such that $UX_p = (VX_p)^g = V^g = X_p$. Replacing V^g by V, we may assume that $UX_p = VX_p \equiv H$. Then $G = HX_{p'}$. Consider the natural isomorphism

$$G/X_{p'} = HX_{p'}/X_{p'} \longrightarrow H/(H \cap X_{p'}).$$

In $G/X_{p'}$, $UX_{p'}/X_{p'}$, and $VX_{p'}/X_{p'}$ are q-conjugate for every prime q and $\pi(X/X_{p'}) = \{p\}$. By assumption, $UX_{p'}/X_{p'}$ and $VX_{p'}/X_{p'}$ are conjugate and hence their images $(H \cap UX_{p'})/(H \cap X_{p'})$ and $(H \cap VX_{p'})/(H \cap X_{p'})$ are conjugate in $H/(H \cap X_{p'})$. However, if $h \in H \cap UX_{p'} = UX_p \cap UX_{p'}$, then $h = u_1a = u_2b$, for $u_1,u_2 \in U$, $a \in X_p$, and $b \in X_{p'}$. Hence $ab^{-1} = u_1^{-1}u_2 \in U$. Using the fact that a and b are commuting elements of coprime order, it follows that a, $b \in U$. Hence $h = u_1a \in U$ and so $U \leq UX_p \cap UX_{p'} = H \cap UX_{p'} \leq U$. Hence $U = H \cap UX_{p'}$, and similarly $V = H \cap VX_{p'}$. So U and V are conjugate. Hence if there are counter examples to the theorem, there are counter examples in which X is a p-group. □

The last part of the argument in the proof of Lemma 2 is essentially that occurring in [4].

By Lemma 2 we may suppose G is a counter example with X a p-group for some prime p. From now on, we assume that U and V have a common Sylow p-subgroup U_p. This we may do because of the hypothesis of p-conjugacy, so $U_p V_p^g$ for some g and we can replace V by V^g.

LEMMA 3. *We may assume that* $G = < U, V >$.

Proof. Notice that if $H = < U, V >$ then by the Dedekind Law $H = (X \cap H)U = (X \cap H)V$. By the assumption on G, $U_p = V_p$. On the other hand, for $q \neq p$, U_q, $V_q \in Syl_q(H)$, where U_q and V_q are Sylow q-subgroups of U and V. Now

1. if X is abelian, then $X \cap H$ is abelian;
2. G/X is locally nilpotent, then $H/(X \cap H)$ is locally nilpotent;
3. if the Sylow q-subgroups of G have class at most 2, then the Sylow q-subgroups of H have class at most 2.

Since G is a counter example, so is H. □

We now suppose $G = < U, V >$ is a counter example with X a p-group.

LEMMA 4. *We may assume that* $O_p(U) = O_p(V) = 1$ *and hence* $X \cap U = X \cap V = 1$.

Proof. Clearly $\Phi(G) \cap U_p = (\Phi(G) \cap U)_p$ char $\Phi(G) \cap U \leq U$. Hence $\Phi(G) \cap U_p \leq O_p(U)$. On the other hand, $XO_p(U)$ is a normal p-subgroup of G, so $O_p(U) \leq \Phi(G) \cap U_p$. Hence $\Phi(G) \cap U_p = O_p(U)$. Similarly, $\Phi(G) \cap U_p = O_p(V)$. So, $O_p(U) = O_p(V)$, and hence, $O_p(U) \trianglelefteq G = < U, V >$. However if G is a counter example to the theorem, so is $G/O_p(U)$ and $O_p(U/O_p(U)) = 1$. Hence we may suppose $O_p(U) = O_p(V) = 1$. Since $X \cap U \leq O_p(U), X \cap U = 1 = X \cap V$. □

LEMMA 5. $[X, U] = X$.

Proof. For each prime q, pick a Sylow q-subgroup V_q of V. Then $U_q^{ux} = V_q$ for some $U_q \in Syl_q(U)$, $u \in U$ and $x \in X$. Then $V_q \leq [X, U]U$, and hence $V \leq [X, U]U$ since a Černikov group is always generated by a complete set of Sylow q-subgroups. But this means $G = [X, U]U$ since $G = < U, V >$. Hence $X = X \cap [X, U]U = [X, U](X \cap U)$ by the Dedekind Law. However G is a counter example with $X \cap U = 1$. So it follows that $X = [X, U]$. □

Now we prove Theorem 2 through the claims.

CLAIM 1. *If X is finite abelian, Theorem 2 holds.*

Proof. It is clear that $G^0 = U^0 = V^0$. So,

$$G/U^0 = (U/U^0)(XU^0/U^0) = (V/V^0)(XV^0/V^0).$$

Now by the theorem of [3], U and V are conjugate. □

CLAIM 2. *If X is infinite abelian, Theorem 2 holds.*

Proof. Note that X is a direct sum of finitely many quasicyclic groups and cyclic groups of prime power orders. So if F is a finite subset of X, then we can find a finite characteristic subgroup W of X that contains F. Let $\pi(G) = \{p_1, \dots p_k\}, U_{p_i} \in Syl_{p_i}(U), V_{p_i} \in Syl_{p_i}(V)$, for $i = 1, \dots, k$. There exist $x_i, y_i \in X, u_i \in U$, and $v_i \in V$ such that $U_{p_i} = V_{p_i}^{v_i x_i}$, and $V_{p_i} = U_{p_i}^{u_i y_i}$, for $i = 1, \dots, k$. By the above remarks there is a finite characteristic subgroup W of X such that $\{x_1, \dots, x_k, y_1, \dots, y_k\} \subseteq W$. Then $W \trianglelefteq G$. Consider $G^* = UW$. It is clear that $UW = VW$. Also W is a finite abelian normal subgroup of G^*, and U and V are p_i-conjugate in G^* for $i = 1, \dots, k$. By Claim 1, U and V are conjugate in G^*, and hence in= G. □

CLAIM 3. *If G/X is locally nilpotent, Theorem 2 holds.*

Proof. If this claim is false, then by Section 2 there would exist a counter example satisfying $O_p(U) = 1$ and $X \cap U = 1$. Since $U \simeq G/X$ is locally nilpotent, U is a p'-group. Therefore U and V are Sylow p'-subgroups of G. So, by Lemma 1, U and V are conjugate, a contradiction. □

CLAIM 4. *If the Sylow p-subgroups of G have class at most 2 for every prime p, then Theorem 2 holds.*

Proof. If this claim is false, then there would exist a counter example G such that the size of U is minimal. Note that we may assume the lemmas in Section 2 for this group G. If $U = U_p^U$, where U_p^U is the normal closure of U_p in U, then by Lemma 5, $[X, U_p^U] = [X, U] = X$. But $[X, U_p, X] = 1$ since the Sylow p-subgroups have class at most 2, and hence $[X, U_p] \leq Z(X)$. This implies that $X = [X, U_p^U] \leq Z(X)$. Hence X is abelian and by Claim 1 and Claim 2, U and V are conjugate. Therefore we may assume that $U_p^U < U$. As in the proof of Lemma 5, it is clear that $XU_p^U = XU_p^V \equiv N$. It is also clear that U_p^U and U_p^V are q-conjugate in N for every prime q. Now $|U_p^U| < |U|$. By the minimality of U, U_p^U and U_p^V are conjugate in N. Therefore, there exists $n \in N$ such that $U_p^U = (U_p^V)^n$. Consider $M =< U, V^n >$. U_p^U is normalized by U and V^n. So $J \equiv U_p^U \trianglelefteq M$. Consider $M/J = ((X \cap M)J/J)(U/J) = ((X \cap M)J/J)(V^n/J)$. If $J = 1$, then $U_p = 1$ so by Lemma 1, U and V are conjugate. If $J \neq 1$, then $1 \neq U_p \not\leq U^0$ because $O_p(U) = 1$. Therefore $|U/J| < |U|$. Now by the minimality of $U, U/J$ and V^n/J are conjugate, a contradiction. □

This completes the proof when G is a Černikov group.

3. Proof of Theorem 3

Let $\Sigma = \{F_i : i \in I\}$ be the local system of G consisting of all F^G for any finite subset F of G. Note that each F_i is Černikov. Also note that the Sylow p-subgroups of a CC-group are locally conjugate [1]. If $U_p \in Syl_p(U)$, then $U_p \cap F_i = U_p \cap F_i \cap U \in Syl_p(F_i \cap U)$. Now let $\pi(F_i \cap U) = \{p_{i_1}, \dots, p_{i_{k_i}}\}$. Then there exist $\delta_{ij} \in Linn(G)$ such that $\delta_{ij}(U_{p_j}) = V_{p_j}, j = i_1, \dots, i_{k_i}$. Also $\delta_{ij}(U_{p_j} \cap F_i) = V_{p_j} \cap F_i$. Since $U_{p_j} \cap F_i$ is contained in a normal Černikov subgroup of G, there exists $u_{ij}x_{ij}$ such that $(U_{p_j} \cap F_i)^{u_{ij}x_{ij}} = V_{p_j} \cap F_i$ for $j = i_1, \dots, i_{k_i}$. Also there exist $v_{ij} \in V$ and $y_{ij} \in X$ so that $(u_{ij}x_{ij})^{-1} = v_{ij}y_{ij}$. Let

$$X_i^* = x_{i_1}^G \dots x_{i_{k_i}}^G y_{i_1}^G \dots y_{i_{k_i}}^G (X \cap F_i \cap U)^G (X \cap F_i \cap V)^G \leq X.$$

Then X_i^* is a Černikov, normal, and locally nilpotent subgroup of G. Consider $G_i^* = (F_i \cap U)X_i^* = (F_i \cap V)X_i^*$. It is clear that $F_i \cap U$ and $F_i \cap V$ are p-conjugate, for all primes $p \in \pi(F_i \cap U)$, in G_i^*. Now

1. if X is abelian, then X_i^* is abelian,
2. if G/X is locally nilpotent, then G_i^*/X_i^* is locally nilpotent,
3. if the Sylow p-subgroups of G have class at most 2, then the Sylow p-subgroups of G_i^* have class at most 2.

But G_i^* is Černikov. So, by Theorem 2, $(F_i \cap U)^{x_i} = (F_i \cap V)$ for some $x_i \in G$. Now let Γ_i be the set of automorphisms of F_i induced by the inner automorphisms of G such that $\alpha(F_i \cap U) = (F_i \cap V)$. Then Γ_i is non-empty. Suppose $F_i \leq F_j$. Define a map $\theta_{ji} : \Gamma_j \longrightarrow \Gamma_i$ by

$\theta_{ji}(\alpha) = \alpha \mid_{F_i}$ for $\alpha \in \Gamma_j$. If we give an ordering on I by $i \leq j$ if $F_i \leq F_j$, then $\{\Gamma_i, \theta_{ji} : i, j \in I, i \leq j\}$ is an inverse system of sets and mappings. We endow Γ_i with a suitable topology, to facilitate the use of a theorem from general topology. Suppose $\alpha, \beta \in \Gamma_i$. Then there exist $x, y \in G$ so that $\alpha = \lambda_x \mid_{F_i}, \beta = \lambda_y \mid_{F_i}$, where λ_x and λ_y are inner automorphisms of G induced by x and y, respectively. Thus $\beta^{-1}\alpha = \lambda_{xy^{-1}} \mid_{F_i}$. Moreover, $\beta^{-1}\alpha(F_i \cap U) = F_i \cap U$. So, $xy^{-1} \in N_G(F_i \cap U)$. Let $K_i = N_G(F_i \cap U)/C_G(F_i)$, a Černikov group. It is easy to show that there is a 1-1 correspondence between Γ_i and K_i. Now using the same argument as in the proof of theorem 3.9 in [2], $\{\Gamma_i, \theta_{ji} : i, j \in I, i \leq j\}$ is an inverse system of non-empty compact topological T_1-spaces and closed, continuous maps. So,

$$\Gamma = \varprojlim \Gamma_i \neq \emptyset.$$

If $(\alpha_i) \in \Gamma$, define $\alpha : G \longrightarrow G$ by $\alpha(x) = \alpha_i(x)$ for $x \in F_i$. Then α is a well-defined, locally inner automorphism of G such that $\alpha(U) = V$. So U and V are locally conjugate.

4. An application

In CC-groups, a locally inner automorphism can be extended from a subgroup H of G to G or lifted from a factor group G/N to G [6]. Using this fact we prove Theorem 4 as an application of Theorem 3.

Proof of Theorem 4. Let S and T be Sylow π-subgroups of G. Note that G/G^0 is an FC-group, where G^0 is the radicable part of G [5]. Since SG^0/G^0 and TG^0/G^0 are locally conjugate by Theorem 5.2 in [7], we can find $\theta \in Linn(G)$ such that $G^* \equiv SG^0 = \theta(T)G^0$. Clearly S and $\theta(T)$ are p-locally conjugate for every prime p. Since G^0 is abelian normal in G^*, by Theorem 3, S and $\theta(T)$ are locally conjugate in G^*. It is obvious that S and T are also locally conjugate in G.

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