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A SCATTERING PROBLEM IN A NONHOMOGENEOUS MEDIUM

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ABSTRACT. In this article, a scattering problem in a nonhomogeneous medium is formulated as an integral equation which contains boundary and volume integrals. The integral equation is solved for suffciently small $\|1-\rho\|, \|k_i^2-k^2\|$ and $\|\nabla\rho\|$ where k, k_i and ρ the wave numbers and the density respectively.

1. Introduction

The reduced wave equation related to the time harmonic acoustic waves has been investigated by many authors (see e.g., [3], [4], [5], [8], [9], [10], [11]). The main results have been given by Werner [10-11] for the reduced wave equation in a non-homogeneous medium. Colton and Wendland [4] have considered the exterior Neumann problem for the reduced wave equation connected with the scattering of acoustic waves in a spherically symmetric medium. They have used the constructive methods to prove the existence of the solution.

The mathematical problem we are about to consider is the scattering of waves in a nonhomogeneous medium. We will assume that, the wave number $k_i(x)$ and the density $\rho_i(x)$ are comlex valued functions in the domain B_i. In $\mathbb{R}^n \backslash \bar{B}_i$, the wave number k and the density ρ_0 will be comlex numbers. If we consider scattering of an acoustic wave, the wave number and the density will be real parameters.

In this paper, we consider a scattering problem in a nonhomogeneous medium. The problem is formulated as an integral equation which contains boundary and volume integrals. Also, the compactness of the

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integral operators is investigated. The integral equation in solved directly as Neumann series and the convergence of the Neumann series is proven for suffciently small $\|1-\rho\|$, $\|k_i^2-k^2\|$ and $\|\nabla\rho\|$.

2. Statement of the problem

Let S be a closed, simply connected, strictly convex Lyapunov surface in $\mathbb{R}^n$ satisfying the two-sided cone condition at each pont. Let B_e and B_i be the exterior and interior domains of S respectively. We also assume that the wave number k and the density ρ_0 are constant in B_e. However, the wave number, k_i and the density ρ_i will be assumed to be functions at position in B_i;

$$k_i = k_i(x), \qquad x \in B_i$$
$$\rho_i = \rho_i(x), \qquad x \in B_i$$

such that $k_i \in C(\bar{B}_i)$ and $\rho_i \in C^1(\bar{B}_i)$.

The problem we consider is that of finding the total field $u(x) = u^i(x) + u^s(x)$ in $\mathbb{R}^n$ when an incident field u^i is given. The solution of the scattering problem is to find a function $u \in C^2(\mathbb{R}^n \backslash S) \cap C^1(S)$ such that

$$u(x) = u^i(x) + u^s(x), \tag{2.1}$$

$$(\nabla^2 + k_i^2)u^s(x) = 0 \quad x \in B_e \tag{2.2}$$

$$[\nabla^2 + k_i^2(x)]u(x) = \frac{1}{\rho_i(x)}\nabla\rho_i(x)\cdot\nabla u(x) \quad x \in B_i \tag{2.3}$$

$$\frac{\partial}{\partial r}u^s(x) - iku^s(x) = 0(r^{-(n-1)/2}) \tag{2.4}$$

uniformly in all directions,

$$x \in S \qquad \begin{cases} u^+(x) = u^-(x) \\ \frac{1}{\rho_0}\frac{\partial u^+(x)}{\partial\nu} = \frac{1}{\rho_i(x)}\frac{\partial u^-(x)}{\partial\nu} \end{cases} \tag{2.5}$$

under the conditions;

$$I_m(k) \geq 0$$
(2.6)
$$I_m(\bar{k}\rho k_i^2) \geq 0$$
$$I_m(k\bar{\rho}) \geq 0$$

where,

(2.7)
$$\rho(x) = \frac{\rho_0}{\rho_i(x)}, \quad \nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

u^i is a given function, the space part of the incident wave, satisfying

(2.8)
$$(\nabla^2 + k^2)u^i(x) = 0, \quad x \in \mathbb{R}^n$$

and u^s is the space part of the scattered wave.

$$u^+(x) := \lim_{x \in B_e \to x \in S} u(x)$$
$$u^-(x) := \lim_{x \in B_i \to x \in S} u(x).$$

$\frac{\partial u^+}{\partial \nu}$ and $\frac{\partial u^-}{\partial \nu}$ will have similar interpretations for the exterior normal derivatives; $r = |x|$ and $\partial/\partial r$ indicates the derivative in the outward radial direction. Note that B_i may be the union of more than one disjoint domains.

The scattering problem defined by equations (2.1) - (2.5) has a unique solution (see Anar and Celebi [2]) under the conditions (2.6)

3. Integral representation

THEOREM 3.1. *Let*

(3.1)
$$G_n(x, y, k) = -\frac{i}{4}\left(\frac{k}{2\pi|x-y|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(k|x-y|), n > 2$$

be the unmodified free sapce Green's function for $(\nabla^2 + k^2)u = 0$ *in* $\mathbb{R}^n$ *where* $H^{(1)}_{(n-2)/2}$ *is the Hankel function of the first kind of order* $(n-2)/2$.

If u *is the solution of the scattering problem* (2.1) - (2.5) *then* u *has the integral representation for* $x \in \mathbb{R}^n$;

(3.2)

$$\begin{aligned} u(x) =& u^i(x) + \int_S \{[1-\rho(x)]u(x) - [1-\rho(y)]u(y)\} \frac{\partial G_n(x,y,0)}{\partial \nu_y} ds(y) \\ &+ \int_S [1-\rho(y)]u(y) \frac{\partial}{\partial \nu_y}[G_n(x,y,0) - G_n(x,y,k)]ds(y) \\ &- \int_{B_i} \{[k_i^2(y) - k^2]\rho(y) G_n(x,y,k)u(y) \\ &+ \nabla \rho(y) \cdot \nabla_y G_n(x,y,k)u(y)\}dy \end{aligned}$$

where

$$G_n(x,y,0) = \frac{\Gamma(n2)}{(2-n)2\pi^{n/2}} \frac{1}{|x-y|^{n-2}} \tag{3.3}$$

Proof. We know in [2]

$$\begin{aligned} \alpha_n(x) &= -\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x) \cap B_e} \frac{\partial}{\partial \nu_y} G_n(x,y,0) ds(y), \\ &= \begin{cases} -1, & \text{if} \quad x \in B_e \\ -\frac{1}{2}, & \text{if} \quad x \in S \\ 0, & \text{if} \quad x \in B_i \end{cases} \end{aligned} \tag{3.4}$$

where $B_\epsilon(x) = \{y : |x-y| < \epsilon\}$, and $\partial B_\epsilon(x)$ is the boundary of $B_\epsilon(x)$.

In [2] we have the integral relation for the problem (2.1) - (2.5) is that

(3.5)

$$u^i(x) + \alpha_n(x)u(x) = \int_S \left[u^+(y) \frac{\partial G_n(x,y,k)}{\partial \nu_y} - G_n(x,y,k) \frac{\partial u^+(y)}{\partial \nu_y}\right] ds(y).$$

Since on $S u^+ = u^-$ and $\frac{\partial u^+}{\partial \nu} = \rho \frac{\partial u^-}{\partial \nu}$ the equation (3.5) takes the form:

$$(3.6)\qquad \begin{aligned} u^i(x) + \alpha_n(x) u(x) &= \int_S [1-\rho(y)] u^-(y) \frac{\partial G_n(x,y,k)}{\partial \nu_y} ds(y) \\ &+ \int_S \rho(y) \Big[u^-(y) \frac{\partial G_n(x,y,k)}{\partial \nu_y} - G_n(x,y,k) \frac{\partial u^-(y)}{\partial \nu_y} \Big] ds(y). \end{aligned}$$

Now, apply Green's theorem to B_i and using the relations

$$(3.7)\qquad \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} \rho(y) u(y) \frac{\partial G_n(x,y,k)}{\partial \nu_y} ds(y) = [1+\alpha_n(x)] \rho(x) u(x)$$

and

$$(3.8)\qquad \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} \rho(y) G_n(x,y,k) \frac{\partial u(y)}{\partial \nu_y} ds(y) = 0$$

we obtain
(3.9)

$$\begin{aligned} \int_S \rho(y) \Big[u^-(y) \frac{\partial G_n(x,y,k)}{\partial \nu_y} - G_n(x,y,k) \frac{\partial u^-(y)}{\partial \nu_y} \Big] ds(y) \\ = [1+\alpha_n(x)] \rho(x) u(x) \\ + \int_{B_i} \{ [k_i^2(y) - k^2] \rho(y) G_n(x,y,k) + \nabla \rho(y) \cdot \nabla_y G_n(x,y,k) \} u(y) dy. \end{aligned}$$

Substitute (3.9) in (3.6) we have

$$(3.10)\qquad \begin{aligned} & u^i(x) + \alpha_n(x) u(x) \\ &\quad = \int_S [1 - \rho(y) u^-(y) \frac{\partial G_n(x,y,k)}{\partial \nu_y} ds(y) \\ &\quad + [1+\alpha_n(x)] \rho(x) u(x) + \int_{B_i} \{ [k_i^2(y) - k^2] \rho(y) G_n(x,y,k) \\ &\quad + \nabla \rho(y) \cdot \nabla_y G_n(x,y,k) \} u(y) dy. \end{aligned}$$

Since

$$\int_S \frac{\partial G_n(x,y,0)}{\partial \nu_y} ds(y) = 1 + \alpha_n(x) \tag{3.11}$$

the integral equation (3.10) takes the form:
(3.12)

$$\begin{aligned} u(x) = \ & u^i(x) + \int_S \{[1-\rho(x)]u(x) - [1-\rho(y)]u(y)]\} \frac{\partial G_n(x,y,0)}{\partial \nu_y} ds(y) \\ & + \int_S [1-\rho(y)]u(y) \frac{\partial}{\partial \nu_y} [G_n(x,y,0) - G_n(x,y,k)] ds(y) \\ & - \int_{B_i} \{[k_i^2(y) - k^2]\rho(y) G_n(x,y,k) + \nabla \rho(y) \cdot \nabla_y G_n(x,y,k)\} u(y) dy. \end{aligned}$$

□

4. The integral operators

We introduce the following integral operators;

$$(L_1 u)(x) := \int_S \{[1-\rho(x)]u(x) - [1-\rho(y)]u(y)\} \frac{\partial G_n(x,y,0)}{\partial \nu_y} ds(y) \tag{4.1}$$

$$(L_2 u)(x) := \int_S [1-\rho(y)]u(y) \frac{\partial}{\partial \nu_y} [G_n(x,y,0 - G_n(x,y,k)] ds(y) \tag{4.2}$$

$$(L_3 u)(x) := \int_{B_i} [k^2 - k_i^2(y)]\rho(y) G_n(x,y,k) u(y) dy, \tag{4.3}$$

$$(L_4 u)(x) := -\int_{B_i} \nabla \rho(y) \cdot \nabla_y G_n(x,y,k) u(y) dy. \tag{4.4}$$

Hence, if u is the solution of the problem (2.1) - (2.5), then u has the operator equation representation

$$u = u^i + L_u \tag{4.5}$$

where

$$L_u = (L_1 + L_2 + L_3 + L_4)u. \tag{4.6}$$

We will use the following direct method to solve the equation

$$(I - L)u = u^i.$$

This will lead to the Neumann series

$$\sum_{m=0}^{\infty} L^m u^i.$$

For this it is suffcient to prove that

$$\|L\| := \sup_{C(\bar{B}_i)} \frac{\|L_u\|}{\|u\|} < 1$$

where the norm defined by,

$$\|u\| := \sup_{\bar{B}_i} |u(x)|.$$

We now collect some basic results for the operator L defined by (4.6).
(a) The Operator L_1; We first examine the continuity,
(4.7)

$$\begin{aligned}
&|(L_1 u)(x) - (L_1 u)(x_1)| \\
&= \Big| \int_S \Big[\{[1-\rho(x)]u(x) - [1-\rho(y)]u(y)\} \frac{\partial}{\partial \nu_y} [G_n(x,y,0) - G_n(x_1,y,0)] \\
&+ \{[1-\rho(x)]u(x) - [1-\rho(x_1)]u(x_1)\} \frac{\partial}{\partial \nu_y} G_n(x_1,y,0) \Big] ds(y) \Big| \\
&\leq \int_S |\{[1-\rho(x)]u(x) - [1-\rho(y)]u(y)\}| \Big| \frac{\partial}{\partial \nu_y} [G_n(x,y,0) - G_n(x_1,y,0)] \Big| ds(y) \\
&+ |[1-\rho(x)]u(x) - [1-\rho(x_1)]u(x_1)| [1 + \alpha_n(x_1)]
\end{aligned}$$

We decompose S into a protion lying within a ball of radius δ and center x,

$$\sum_{\delta}(x) = \{y \in S : |x - y| < \delta\}$$

and the remainder $S \setminus \sum_\delta(x)$. So that

$$
\begin{aligned}
(4.8)\qquad & |(L_1u)(x) - (L_1u)(x_1)| \\
& \leq I_1 + I_2 + |[1-\rho(x)]u(x) - [1-\rho(x_1)]u(x_1)|[1+\alpha_n(x_1)],
\end{aligned}
$$

where

$$
\begin{aligned}
(4.9)\qquad I_1 = \int_{\sum_\delta(x)} & |[1-\rho(x)]u(x) - [1-\rho(y)]u(y)| \\
& \times \left| \frac{\partial}{\partial \nu_y} [G_n(x,y,0)] - G_n(x_1,y,0)] \right| ds(y)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.10)\qquad I_2 = \int_{S \setminus \sum_\delta(x)} & |[1-\rho(x)]u(x) - [1-\rho(y)]u(y)| \\
& \times \left| \frac{\partial}{\partial \nu_y} [G_n(x,y,0) - G_n(x_1,y,0)] \right| ds(y).
\end{aligned}
$$

$$
\begin{aligned}
(4.11)\qquad I_1 \leq & \sup_{y \in \sum_\delta(X)} |[1-\rho(x)]u(x) - [1-\rho(y)]u(y)| \\
& \times \int_{\sum_\delta(X)} \left| \frac{\partial}{\partial \nu_y} [G_n(x,y,0) - G_n(x_1,y,0)] \right| ds(y).
\end{aligned}
$$

Since $\sum_\delta(x)$ is a portion of the Lyapunov segment or any finite combination of such segment then (see Mikhlin [7] and, Ahner and Kleinman [1]) there exists a constant C_1 such that,

$$
(4.12)\qquad I_1 \leq C_1 \sup_{y \in \sum_\delta(x)} |[1-\rho(x)]u(x) - [1-\rho(y)]u(y)|.
$$

It is clear that (since u and ρ are continuous on S) there is some constant C_2 such that

$$
(4.13)\qquad I_2 \leq C_2 \int_{S \setminus \sum_{\delta(x)}} \left| \frac{\partial}{\partial \nu_y} [G_n(x,y,0) - G_n(x_1,y,0)] \right| ds(y).
$$

Since $y \in S\backslash \sum_\delta$ then $|x-y| \geq \delta$. With the additional restriction that for any $\delta_1 < \delta$ such that $|x-x_1| < \delta_1 < \delta$ the following expansion (Lebedev [6]) is valid for all $x_1 \in \sum_\delta(x)$ and $y \in S\backslash \sum_\delta(x)$

$$(4.14) \qquad \frac{1}{|x_1-y|} = \sum_{j=0}^{\infty} \frac{|x-x_1|^j}{|x-y|^{j+1}} P_j\Big(\widehat{\frac{x_1-x}{|x_1-x|} \cdot \frac{y-x}{|y-x|}}\Big),$$

where P_j are the Legendre polynomials. The series (4.14) is uniformly and absolutely convergent. It follows form (4.14) that

$$(4.15) \qquad \frac{1}{|x_1-y|} - \frac{1}{|x-y|} = \sum_{j=1}^{\infty} \frac{|x-x_1|^j}{|x-y|^{j+1}} P_j\Big(\widehat{\frac{x_1-x}{|x_1-x|} \cdot \frac{y-x}{|y-x|}}\Big).$$

Now consider the integrant in the inequality (4.13),

$$\Big|\frac{\partial}{\partial \nu_y}[G_n(x,y,0) - G_n(x_1,y,0)]\Big|$$
$$= \Big|\frac{\Gamma(\frac{n}{2})}{(2-n)2\pi^{n/2}} \frac{\partial}{\partial \nu_y}\Big(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x_1-y|^{n-2}}\Big)\Big|$$

We can write

$$(4.16) \qquad \frac{1}{|x-y|^{n-2}} - \frac{1}{|x_1-y|^{n-2}} = \Big(\frac{1}{|x-y|} - \frac{1}{x_1-y|}\Big) F_n\Big(\frac{1}{|x-y|}, \frac{1}{|x_1-y|}\Big)$$

where

$$(4.17) \qquad F_n(|x-y|^{-1}, |x_1-y|^{-1}) = \sum_{i=0}^{n-3} |x-y|^{3-n+i}|x_1-y|^{-i}.$$

Hence

$$\frac{\partial}{\partial \nu_y}\Big(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x_1-y|^{n-2}}\Big)$$
$$(4.18) \qquad = F_n(|x-y|^{-1}, |x_1-y|^{-1}) \sum_{m=1}^{\infty} \frac{|x-x_1|^m}{|x-y|^{m+2}} f_m(x,x_1,y)$$
$$+ \Big[\sum_{m=1}^{\infty} \frac{|x-x_1|^m}{|x-y|^{m+1}} P_m(\mu)\Big] \frac{\partial}{\partial \nu_y} F_n(|x-y|^{-1}, |x_1-y|^{-1}]$$

where (see Ahner and Kleinman [1])
(4.19)

$$f_m(x,x_1,y) = -(m+1)\hat{\nu}_y(\widehat{x-y})P_m(\mu) + [\hat{\nu}_y\cdot(\widehat{x-x_1}) - \hat{\nu}_y(\widehat{x-y})(\widehat{x-x_1})(\widehat{x-y})]P_m(\mu)$$

and

$$\mu = \frac{\widehat{x_1-x}}{|x_1-x|}\cdot\frac{y-x}{|y-x|}. \tag{4.20}$$

So

$$\begin{aligned}&\Big|\frac{\partial}{\partial\nu_y}\Big(\frac{1}{|x_1-y|^{n-2}} - \frac{1}{|x_1-y|^{n-2}}\Big)\Big|\\ &\le |F_n(|x-y|^{-1},|x_1-y|^{-1})\sum_{m=1}^{\infty}\frac{|x-x_1|^m}{\delta^{m+2}}|f_m|\\ &+\Big|\sum_{m=1}^{\infty}\frac{|x-x_1|^m}{\delta^{m+1}}P_m(\mu)\Big|\Big|\frac{\partial}{\partial\nu_y}F_n(|x-y|^{-1},|x_1-y|^{-1})\Big|.\end{aligned} \tag{4.21}$$

We have that estimates,

$$|F_n(|x-y|^{-1},|x_1-y|^{-1})| \le \frac{n-2}{(\delta-\delta_1)^{n-3}} \tag{4.22}$$

$$\Big|\frac{\partial}{\partial\nu_y}F_n(|x-y|^{-1},|x_1-y|^{-1})\Big| \le \frac{(n-2)(n-3)}{(\delta-\delta_1)^{n-2}}. \tag{4.23}$$

Hence we have,

$$\begin{aligned}&\Big|\frac{\partial}{\partial\nu_y}\Big(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x_1-y|^{n-2}}\Big)\Big|\\ &\le \frac{n-2}{(\delta-\delta_1)^{n-3}}\sum_{m=1}^{\infty}\frac{|x-x_1|^m}{\delta^{m+2}}|f_m|\\ &+\frac{(n-2)(n-3)}{(\delta-\delta_1)^{n-2}}\sum_{m=1}^{\infty}\frac{|x-x_1|^m}{\delta^{m+1}}|P_m(\mu)|.\end{aligned} \tag{4.24}$$

Assuming $|x-x_1|<1$. Then for $0<\alpha<1$ and $x\neq x_1$ choosing

$$\delta=|x-x_1|^{(1-\alpha)/3}>|x-x_1|$$

then

$$\frac{|x-x_1|^m}{\delta^{m+2}}=|x-x_1|^{\alpha}|x-x_1|^{(m-1)(\alpha+2)/3} \tag{4.25}$$

and

$$\frac{|x-x_1|^m}{\delta^{m+1}}=|x-x_1|^{\alpha/3}|x-x_1|^{\frac{2m-1+\alpha m}{3}}. \tag{4.26}$$

Hence the inequality (2.24) takes the form;

$$\begin{aligned} &\left|\frac{\partial}{\partial\nu_y}\Big(\frac{1}{|x-y|^{n-2}}-\frac{1}{|x_1-y|^{n-2}}\Big)\right| \\ &\leq \frac{n-2}{(\delta-\delta_1)^{(n-3)}}\sum_{m=1}^{\infty}|x-x_1|^{\alpha}|x-x_1|^{(m-1)(\alpha+2)/3}|f_m| \\ &+\frac{(n-2)(n-3)}{(\delta-\delta_1)^{n-2}}\sum_{m=1}^{\infty}|x-x_1|^{\frac{\alpha}{3}}|x-x_1|^{\frac{(2m-1+\alpha m)}{3}}|P_m(\mu)| \end{aligned} \tag{4.27}$$

Finally we have inequality,

(4.28)

$$\begin{aligned} I_2\leq C_3\Bigg\{&(n-2)|x-x_1|^{\alpha}\int_{S\setminus\sum_\delta(x)}\sum_{m=1}^{\infty}|f_m|\frac{|x-x_1|^{(m-1)(\alpha+2)/3}}{(\delta-\delta_1)^{n-3}}ds \\ &+(n-2)(n-3)|x-x_1|^{\alpha/3}\int_{S\setminus\sum_\delta(x)}\sum_{m=1}^{\infty}|P_m(\mu)|\frac{|x-x_1|^{(2m-1+\alpha m)/3}}{(\delta-\delta_1)^{n-2}}ds\Bigg\} \end{aligned}$$

Since $|x-x_1|<1$, the integrals of the power series in (4.28) are bounded and hence there are some constants C_4 and C_5 such that

$$I_2\leq C_4|x-x_1|^{\alpha}+C_5|x-x_1|^{\alpha/3}. \tag{4.29}$$

Utilizing the results (4.12) and (4.29) in (4.8) it follows that

$$
\begin{aligned}
|(L_1u)(x) - (L_1u)(x_1)| \le C_1 \sup_{y\in\sum_\delta(x)} &|[1-\rho(x)]u(x) - [1-\rho(y)]u(y)| \\
&+ C_4|x-x_1|^\alpha + C_5|x-x_1|^{\alpha/3} \\
&+ |[1-\rho(x)]u(x) - [1-\rho(x_1)]u(x_1)|[1+\alpha_n(x_1)].
\end{aligned} \tag{4.30}
$$

The right-hand side may be made arbitrarily small by making $|x-x_1|$, small enough, provided u and ρ are continuous at x.

The above analysis yields the following theroem.

THEOREM 4.1. *If S is a piecewise Lyapunov surface then*

$$L_1 : C(S) \to C(S) \tag{4.31}$$

where $C(S)$ is the space of continuous functions on S.
Moreover, if $u, \rho, \in C(S)$ then

$$|[1-\rho(x)]u(x) - [1-\rho(y)]u(y)| \tag{4.32}$$

is continuous for $y \in S$. $\forall x, y \in S$ we have

$$|[1-\rho(x)]u(x) - [1-\rho(y)]u(y)| \le 2\|1-\rho\|\|u\|. \tag{4.33}$$

Hence

$$|(L_1u)(x) \le 2\|1-\rho\|\|u\| \int_S \left|\frac{\partial G_n(x,y,0)}{\partial \nu_y}\right| ds(y). \tag{4.34}$$

Also,

$$
\begin{aligned}
\int_S \left|\frac{\partial G_n(x,y,0)}{\partial \nu_y}\right| ds(y) &= \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \int_S \frac{1}{|x-y|^{n-1}} ds(y) \\
&\le \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \int_{\sum_a(x)} \frac{1}{|x-y|^{n-1}} ds(y) \\
&\le 1
\end{aligned} \tag{4.35}
$$

where $\sum_a(x)$ is the sphere radius a and center x which contains S. So that

$$|(L_1u)(x) \le 2\|1-\rho\|\|u\| \tag{4.36}$$

and we have the lemma :

LEMMA 4.1. *Let S be a closed, simply connected, strictly convex Lyapunov surface, then*

$$\|L_1 u\| \leq 2\|1-\rho\|\|u\|. \tag{4.37}$$

(b) The Operator L_2 :

Now examine the operator L_2 in defined by

$$\begin{aligned}(L_2)(x) := &\int_S [1-\rho(y)]u(y)\frac{\partial}{\partial \nu_y}G_n(x,y,0)ds(y)\\ &-\int_S [1-\rho(y)]u(y)\frac{\partial}{\partial \nu_y}G(x,y,k)ds(y).\end{aligned} \tag{4.38}$$

So,

$$\begin{aligned}|(L_2u)(x) \leq &\|1-\rho\|\|u\| \int_S \left|\frac{\partial}{\partial \nu_y}G_n(x,y,0)\right|ds(y)\\ &+ \|1-\rho\|\|u\| \int_s \left|\frac{\partial}{\partial \nu_y}G_n(x,y,k)\right|ds(y)\end{aligned} \tag{4.39}$$

Also we have in [2] this estimate

$$\left|\frac{\partial}{\partial \nu_y}G_n(x,y,k)\right| \leq \frac{C}{|x-y|^{n-1}}, x \neq y. \tag{4.40}$$

Hence we obtain

(4.41)

$$\begin{aligned}|(L_2u)(x)| \leq &\|1-\rho\|\|u\| \int_S \left| -\frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}\hat{\nu}_y \widehat{\frac{x-y}{|x-y|}} \cdot \frac{1}{|x-y|^{n-1}}\right|ds(y)\\ &+ \|1-\rho\|\|u\| \int_S \frac{C}{|x-y|^{n-1}}ds(y)\end{aligned}$$

(4.42)

$$\begin{aligned}&\leq \|1-\rho\|\|u\|\left\{\frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}\int_{\sum_{a(x)}} \frac{1}{|x-y|^{n-1}}ds(y) + C\int_{\sum_{a(x)}} \frac{1}{|x-y|^{n-1}}ds(y)\right\}\\ &\leq \|1-\rho\|\|u\|(1+Cw_n),\end{aligned}$$

where

$$w_n = \frac{\Gamma(\frac{n}{2})}{w\pi^{n/2}}.$$

Hence we have the lemma.

LEMMA 4.2.

$$\|L_2 u\| \leq \|1-\rho\|\|u\|(1+Cw_n). \tag{4.43}$$

It is obvious that L_2 is weakly singular then L_2 is a compact operator on $C(S)$ (see Colton and Kress [3]).
(c) The Operators L_3 and L_4 :

Anar and Celebi [2] have proved L_3 and L_4 are compact operators on $C(\bar{B}_i)$ that is

$$L_3, L_4 : C(\bar{B}_i) \to C(\bar{B}_i).$$

It is easy to see that estimates ;

$$\begin{aligned} |(L_3 u)(x) &= \Big| \int_{B_i} [k^2 - k_i^2(y)]\rho(y) G_n(x,y,k) u(y) dy \Big| \\ &\leq \|k_i^2 - k^2\|\|\rho\|\|u\| \int_{B_i} \frac{M}{|x-y|^{n-2}} dy \\ &\leq \|k_i^2 - k^2\|\|\rho\|\|u\| \int_0^a \int_{\sum_a(x)} \frac{M\,ds(y)}{|x-y|^{n-2}} d|x-y| \\ &\leq \|k_i^2 - k^2\|\|\rho\|\|u\| \frac{M}{2} a^2 w_n \end{aligned} \tag{4.44}$$

where M is some constant. Also

$$\begin{aligned} |(L_4 u)(x)| &= \Big| - \int_{B_i} \nabla\rho(y) \cdot \nabla_y G_n(x,y,k) u(y) dy \Big| \\ &\leq \|\nabla\rho\|\|u\| \int_{B_i} |\nabla_y G_n(x,y,k)| dy. \end{aligned}$$

Since [2]

$$|\nabla_y G_n(x,y,k)| \le \frac{C_2}{|x-y|^{n-1}} \tag{4.45}$$

then we have

$$|(L_4u)(x)| \le \|\nabla\rho\|\|u\|aw_nC_2. \tag{4.46}$$

Hence we have the lemma :

LEMMA 4.3.
$$\|L_3u\| \le C_1\|k_i^2-k^2\|\|\rho\|\|u\|$$
and
$$\|L_4u\| \le C_3\|\triangle\rho\|\|u\|$$

Hence we have the estimate,

$$\|Lu\| \le [C_1\|k_i^2-k^2\|\|\rho\| + C_2\|1-\rho\| + C_3\|\nabla\rho\|]\|u\|. \tag{4.47}$$

Hence, for any $\delta_0 > 0$ *it is always possible to choose* $\|k_i^2-k^2\|$, $\|1-\rho\|$ *and* $\|\nabla\rho\|$ *small enough*

$$(e.g., \|k_i^2-k^2\| < \frac{\delta_0}{3C_1\|\rho\|}, \|1-\rho\| < \frac{\delta_0}{3C_2} \text{ and } \|\nabla\rho\| < \frac{\delta_0}{3C_3})$$

so that

$$\|Lu\| \le \delta_0\|u\|. \tag{4.48}$$

Choosing $\delta_0 < 1$, *we have*

$$\|L\| = \sup_{C(\bar{B}_i)} \frac{\|Lu\|}{\|u\|} \le \delta_0 < 1. \tag{4.49}$$

This conclusion permit us to establish the main result.

THEOREM 4.2. *If* S *is Lyapunov (not piecewise Lyapunov) the* $\|L\|$ < 1 *for sufficiently small* $\|1-\rho\|$, $\|k_i^2-k^2\|$ *and* $\|\nabla\rho\|$.

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