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# GAUSS SUMS FOR $\mathbf{G}_2(q)$

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## 1. Introduction

Let $\lambda$ be a nontrivial additive character of the finite field $\mathbb{F}_q$ to $\mathbb{C}^\times$ and let $\chi$ be a multiplicative character of $\mathbb{F}_q^\times$ to $\mathbb{C}^\times$. For a reductive group (or a finite group of Lie type) $G$ defined over $\mathbb{F}_q$ (see [1] and [2]) and its finite dimensional linear representation $\phi$ over $\mathbb{F}_q$, we define the Gauss sum $\mathcal{G}(G,\phi,\lambda,\chi)$ as follows;

$$\mathcal{G}(G,\phi,\lambda,\chi) = \sum_{g\in G} \chi(\det(\phi(g)) \cdot \lambda(\mathrm{tr}(\phi(g)).$$

When $G$ are various finite classical groups and $\phi$ are the natural representations, the explicit expression for the above sum has been obtained ([3]–[7]). In these cases, the above sum turned out to be polynomials in $q$ with coefficients involving certain *classical* exponential sums such as the Gauss sums, the Kloosterman sums and the hyperkloosterman sums (See §.2 for the definitions.)

The main purpose of this paper is to find the explicit expression for the above sum when $G = \mathbf{G}_2(q)$ is a simple group of exceptional type and $\phi$ is its 7-dimensional faithful representation over $\mathbb{F}_q$.

THEOREM A. *If $\phi$ is the 7-dimensional faithful representation of $\mathbf{G}_2(q)$ over $\mathbb{F}_q$, then $\mathcal{G}(\mathbf{G}_2(q),\phi,\lambda,\chi)$ is equal to*

$$\lambda(1)\mathcal{E}(\lambda)q^6 + q^7(q-1)\{\lambda(-1)(q+1)(q^2+2) + \lambda(-2)(q^2+q+1)\},$$

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*where*

$$\mathcal{E}(\lambda) = \sum_{u,v \in \mathbb{F}_q^\times} \lambda(u + v + uv + \frac{1}{u} + \frac{1}{v} + \frac{1}{uv}).$$

Comparing with the results for the finite classical groups, the sum $\mathcal{E}(\lambda)$ appeared in the above theorem might be considered as an *exceptional* exponential sum. We can find an upperbound for $\mathcal{E}(\lambda)$ using the Weil's well-known result on the upperbound of the Kloosterman sum.

PROPOSITION B. *We have*

$$|\mathcal{E}(\lambda)| \leqq 2\sqrt{q}(q-2) + q - 1.$$

In addition, we provide, in §.4, a very simple way to compute $\mathcal{G}(G, \phi, \chi, \lambda)$ for $\mathbf{GL}_n(q)$ and $\mathbf{SL}_n(q)$ with respect to their natural representations. (See [3] for the computation using the maximal parabolic subgroups.)

## 2. Notations and preliminaries

Let $\chi$ and $\lambda$ be as before. The (classical) Gauss sum is denoted by

$$\mathcal{G}(\chi, \lambda) = \sum_{u \in \mathbb{F}_q^\times} \chi(u)\lambda(u),$$

and the hyperkloosterman sum is denoted by

$$\mathcal{K}_r(\lambda; a_1, \dots, a_r; b) = \sum_{u_1, \dots, u_r \in \mathbb{F}_q^\times} \lambda(a_1 u_1 + \dots + a_r u_r + b u_1^{-1} \cdots u_r^{-1}),$$

for $a_1, \dots, a_r, b \in \mathbb{F}_q^\times$.

Let $G$ be a reductive group (or a finite group of Lie type) defined over the finite field $\mathbb{F}_q$ with $q$ elements. We shall use the following notations: $\Phi$ is the root system of $G$ with a fixed positive(negative) system $\Phi^+(\Phi^-)$ (resp.); $H$ is a (split) maximal torus; $B = B(\Phi^+) = HU$ is a Borel subgroup with the unipotent radical $U = \langle x_r(t) \mid r \in \Phi^+,\ t \in \mathbb{F}_q \rangle$; $W = N/H$ is the Weyl group; $U_w = \langle x_r(t) \mid r \in \Phi^+,\ w(r) \in \Phi^-,\ t \in \mathbb{F}_q \rangle$. (See [1] and [2] for the details.)

The results in this paper are based on the following well-known fact.

LEMMA 1. (Bruhat decomposition) *Every element $x$ of $G$ is written uniquely in the form $x = hun_wv$, where $h \in H$, $u \in U$, $v \in U_w$ and $n_w \in N$ maps to $w \in W$ under the canonical map $N \to W$.*

The following elementary observations are also useful in the sequel.

LEMMA 2. *We have*

$$\mathcal{G}(G, \phi, \chi, \lambda) = \sum_{w \in W} |U_w| \sum_{b \in B} \chi(\det(\phi(bn_w))) \cdot \lambda(\mathrm{tr}(\phi(bn_w))).$$

*Proof.* This follows from Lemma 1 and the facts that $\det(\phi(bn_wv)) = \det(\phi(vbn_w))$, $\mathrm{tr}(\phi(bn_wv)) = \mathrm{tr}(\phi(vbn_w))$ and $v \in B$, where $b \in B$, $w \in W$, $v \in U_w$. □

LEMMA 3. *For any $a, b \in \mathbb{F}_q$, $b \neq 0$, we have $\sum_{t \in \mathbb{F}_q} \lambda(a + bt) = \sum_{t \in \mathbb{F}_q} \lambda(t) = 0$.*

*Proof.* This is obvious. (Recall that $\lambda$ is nontrivial.) □

## 3. Gauss sums for $\mathbf{G}_2(q)$

Let $\alpha$ and $\beta$ be the simple roots of the root system $\Phi$ of type $\mathbf{G}_2$, where $\beta$ is a long root. Thus $\Phi^+ = \{\alpha,\ \beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ 3\alpha+2\beta\}$.

The faithful 7-dimensional representation of $\mathbf{G}_2(q)$ over $\mathbb{F}_q$ to be used in this paper is slightly modified from that of [8, pp.399–400]. We shall briefly describe our 7-dimensional representation.

We consider the Lie algebra $\mathfrak{g}_2(\mathbb{C})$ as a subalgebra of the orthogonal Lie algebra $\mathfrak{b}_3(\mathbb{C})$ defined as in [1, 11.2.4]. Then the following vectors in $\mathfrak{b}_3(\mathbb{C})$ form a Chevalley basis of $\mathfrak{g}_2(\mathbb{C})$:

$$\begin{aligned}
e_\alpha &= 2e_{61} - e_{13} + e_{27} - e_{45}\\
e_\beta &= -e_{32} + e_{56}\\
e_{\alpha+\beta} &= 2e_{51} - e_{12} - e_{37} + e_{46}\\
e_{2\alpha+\beta} &= 2e_{41} - e_{17} - e_{53} + e_{62}\\
e_{3\alpha+\beta} &= e_{43} - e_{67}\\
e_{3\alpha+2\beta} &= e_{42} - e_{57}\\
e_{-\alpha} &= -2e_{31} + e_{16} - e_{54} + e_{72}\\
e_{-\beta} &= -e_{23} + e_{65}\\
e_{-\alpha-\beta} &= -2e_{21} + e_{15} + e_{64} - e_{73}\\
e_{-2\alpha-\beta} &= -2e_{71} + e_{14} + e_{26} - e_{35}\\
e_{-3\alpha-\beta} &= e_{34} - e_{76}\\
e_{-3\alpha-2\beta} &= e_{24} - e_{75}\\
h_\alpha &= [e_\alpha, e_{-\alpha}]\\
h_\beta &= [e_\beta, e_{-\beta}],
\end{aligned}$$

where $\{e_{ij} \mid 1 \le i,j \le n\}$ denotes the standard basis of $\mathfrak{gl}_n(\mathbb{F}_q)$. For $r \in \Phi$ and $t \in \mathbb{F}_q$, set $x_r(t) = \exp(t \cdot \operatorname{ad} e_r)$. After renumbering the indices (by the permutation $(14)(5263) \in S_7$), we obtain

$$\begin{aligned}
x_\alpha(t) &= I + t(2e_{34} - e_{45} + e_{67} - e_{12}) - t^2 e_{35}\\
x_\beta(t) &= I + t(-e_{56} + e_{23})\\
x_{\alpha+\beta}(t) &= I + t(2e_{24} + e_{13} - e_{46} - e_{57}) - t^2 e_{26}\\
x_{2\alpha+\beta}(t) &= I + t(2e_{14} - e_{25} + e_{36} - e_{47}) - t^2 e_{17}\\
x_{3\alpha+\beta}(t) &= I + t(e_{15} - e_{37})\\
x_{3\alpha+2\beta}(t) &= I + t(e_{16} - e_{27})\\
x_{-\alpha}(t) &= I + t(-2e_{54} + e_{43} + e_{76} - e_{21}) - t^2 e_{53}\\
x_{-\beta}(t) &= I + t(-e_{65} + e_{32})\\
x_{-\alpha-\beta}(t) &= I + t(-2e_{64} + e_{42} + e_{31} - e_{75}) - t^2 e_{62}
\end{aligned}$$

$$x_{-2\alpha-\beta}(t) = I + t(-2e_{74} + e_{41} - e_{52} + e_{63}) - t^2 e_{71}$$
$$x_{-3\alpha-\beta}(t) = I + t(e_{51} - e_{73})$$
$$x_{-3\alpha-2\beta}(t) = I + t(e_{61} - e_{72})$$

Since the root system of type $\mathbf{G}_2$ has trivial fundamental group, $\overline{G} = \langle x_r(t) \mid r \in \Phi, t \in \mathbb{F}_q \rangle$ is isomorphic to $\mathbf{G}_2(q)$ (see [2, p.26]). Hence we obtained a faithful 7-dimensional representation $\phi : \mathbf{G}_2(q) \to \overline{G}$. Since we can identify $\mathbf{G}_2(q)$ with $\overline{G}$, we write $\phi(g) = g$ if $g \in \mathbf{G}_2(q)$, for the simplicity.

As usual, we put

$$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t),$$

$$n_r = n_r(1)$$

and

$$h_r(t) = n_r(t)n_r(-1),$$

where $r \in \Phi$ and $t \in \mathbb{F}_q^\times$.

Thus, we have

$$h_\alpha(t) = \text{diag}(t, \frac{1}{t}, t^2, 1, \frac{1}{t^2}, t, \frac{1}{t}),$$
$$h_\beta(t) = \text{diag}(1, t, \frac{1}{t}, 1, t, \frac{1}{t}, 1),$$

for $t \in \mathbb{F}_q^\times$.

The Weyl group $W = N/H$ of $\mathbf{G}_2(q)$ is the dihedral group $D_6$ and thus

$$W = \{1,\ w_\gamma,\ w_\gamma^2,\ w_\gamma^3,\ w_\gamma^4, w_\gamma^5,\ w_\alpha,\ w_\alpha w_\gamma,\ w_\alpha w_\gamma^2,\ w_\alpha w_\gamma^3,\ w_\alpha w_\gamma^4,\ w_\alpha w_\gamma^5\},$$

where $n_\gamma = n_\alpha n_\beta$ and $w_\alpha,\ w_\beta,\ w_\gamma$ is the image of $n_\alpha,\ n_\beta,\ n_\gamma$ under the canonical projection $N \to W$, respectively.

We note that every element of $U$ can be written uniquely as

$$x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3)x_{2\alpha+\beta}(t_4)x_{3\alpha+\beta}(t_5)x_{3\alpha+2\beta}(t_6),$$

for some $t_1, \ldots, t_6 \in \mathbb{F}_q$ (see [1, Theorem 5.3.3]). Since $\mathbf{G}_2(q)$ is a simply connected group, we also note that every element of $H$ is uniquely expressed as

$$h_\alpha(u)h_\beta(v) = \mathrm{diag}(u, \frac{v}{u}, \frac{u^2}{v}, 1, \frac{v}{u^2}, \frac{u}{v}, \frac{1}{u}),$$

for some $u, v \in \mathbb{F}_q^\times$. Thus every element of $B = HU$ can be written uniquely as

$$(*) \quad h_\alpha(u)h_\beta(v)x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3)x_{2\alpha+\beta}(t_4)x_{3\alpha+\beta}(t_5)x_{3\alpha+2\beta}(t_6),$$

for some $t_1, \ldots, t_6 \in \mathbb{F}_q$ and $u, v \in \mathbb{F}_q^\times$.

We now prove Theorem A. We first observe that the generators $x_r(t)$ are unipotent matrices and hence have determinant 1. By Lemma 2, it is enough to compute

$$\mathcal{G}(w) = \sum_{b \in B} \lambda(\mathrm{tr}(bn_w)),$$

for $w \in W$, to prove Theorem A. We note that $|U_w| = q^{\ell(w)}$, where $\ell(w)$ is the length of $w \in W$.

Since $U$ consists of upper-triangular unipotent matrices, $\mathrm{tr}(hu) = \mathrm{tr}(h)$ for all $h \in H$ and $u \in U$. Therefore, we have

$$\begin{aligned}
\mathcal{G}(1) &= |U_1| \sum_{h \in H, u \in U} \lambda(\mathrm{tr}(hu)) \\
&= |U| \sum_{h \in H} \lambda(\mathrm{tr}(h)) \\
&= q^6 \sum_{u,v \in \mathbb{F}_q^\times} \lambda(1 + u + \frac{1}{u} + \frac{u}{v} + \frac{v}{u} + \frac{u^2}{v} + \frac{v}{u^2}) \\
&= q^6 \lambda(1)\mathcal{E}(\lambda).
\end{aligned}$$

We next compute $\mathcal{G}(w_\alpha)$. Note that $\ell(w_\alpha) = 1$. If we compute $\mathrm{tr}(bn_\alpha)$ using $(*)$, then

$$tr(bn_\alpha) = -1 - \frac{t_1}{u} - \frac{t_1 v}{u} + \frac{t_1^2 v}{u}.$$

Therefore, by Lemma 3, we have

$$
\begin{aligned}
\mathcal{G}(w_\alpha) &= q\sum_{b\in B}\lambda(\mathrm{tr}(bw_a)) \\
=& q\cdot q^5 \sum_{\substack{t_1\in\mathbb{F}_q \\ u,v\in\mathbb{F}_q^\times}} \lambda(-1-\frac{t_1}{u}-\frac{t_1 v}{u}+\frac{t_1^2 v}{u^2}) \\
=& q^6(q-1)^2\lambda(-1)+q^6\sum_{t,u,v\in\mathbb{F}_q^\times}\lambda(-1+\frac{t}{u}+(\frac{t}{u}+\frac{t^2}{u^2})v) \\
=& q^6(q-1)^2\lambda(-1) \\
&+q^6\{\sum_{\substack{t,u\in\mathbb{F}_q^\times \\ v\in\mathbb{F}_q}}\lambda(-1+\frac{t}{u}+(\frac{t}{u}+\frac{t^2}{u^2})v)-\sum_{t,u\in\mathbb{F}_q^\times}\lambda(-1+\frac{t}{u})\} \\
=& q^6(q-1)^2\lambda(-1) \\
&+q^6\{\sum_{\substack{t,s\in\mathbb{F}_q^\times \\ v\in\mathbb{F}_q}}\lambda(-1+s+(s+s^2)v)-\sum_{t,s\in\mathbb{F}_q^\times}\lambda(-1+s)\} \\
=& q^6(q-1)^2\lambda(-1) \\
&+q^6(q-1)\{\sum_{\substack{s\in\mathbb{F}_q^\times \\ v\in\mathbb{F}_q}}\lambda(-1+s+(1+s)sv)-\sum_{s\in\mathbb{F}_q^\times}\lambda(-1+s)\} \\
=& q^6(q-1)^2\lambda(-1)+q^7(q-1)\lambda(-2)+q^6(q-1)\lambda(-1).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathcal{G}(w_\alpha w_\gamma) &= q\cdot q^5\sum_{\substack{t_2\in\mathbb{F}_q \\ u,v\in\mathbb{F}_q^\times}}\lambda(1-\frac{1}{u}-u+\frac{t_2 u}{v}-\frac{t_2 u^2}{v}) \\
&= q^7(q-1)\lambda(-1),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{G}(w_\alpha w_\gamma^2) &= q^5 \cdot q^3 \sum_{\substack{t_6 \in \mathbb{F}_q \\ u,v \in \mathbb{F}_q^\times}} \lambda(1 - (\frac{1}{u} + \frac{u}{v})t_6 + \frac{u^2}{v} + \frac{v}{u^2}) \\
&= q^9(q-1)\lambda(-1), \\
\mathcal{G}(w_\alpha w_\gamma^3) &= q^3 \cdot q^4 \sum_{\substack{t_3 \in \mathbb{F}_q \\ u,v \in \mathbb{F}_q^\times}} \lambda(-1 - \frac{t_3}{u} + \frac{t_3^2 u}{v} - \frac{t_3 u^2}{v}) \\
&= q^8(q-1)\lambda(-2) + (q^7(q-1)^2 + q^7(q-1))\lambda(-1), \\
\mathcal{G}(w_\alpha w_\gamma^4) &= q^5 \cdot q^2 \sum_{\substack{t_4 \in \mathbb{F}_q \\ u,v \in \mathbb{F}_q^\times}} \lambda(-1 - \frac{t_4^2}{u} + \frac{t_4 u}{v} - \frac{t_4 v}{u^2}) \\
&= q^9(q-1)\lambda(-2) + (q^8(q-1)^2 + q^8(q-1))\lambda(-1), \\
\mathcal{G}(w_\alpha w_\gamma^5) &= q^3 \cdot q^4 \sum_{\substack{t_5 \in \mathbb{F}_q \\ u,v \in \mathbb{F}_q^\times}} \lambda(1 + (\frac{1}{u} + \frac{v}{u^2})t^5 + \frac{u}{v} + \frac{v}{u}) \\
&= q^8(q-1)\lambda(-1)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(w_\gamma^3) &= q^6 \cdot q^2 \sum_{\substack{t_2, t_4 \in \mathbb{F}_q \\ u,v \in \mathbb{F}_q^\times}} \lambda(-1 + \frac{t_4^2}{u} - \frac{t_2 t_4 u}{v}) \\
&= q^9(q-1)^2\lambda(-1).
\end{aligned}$$

We also have

$$\begin{aligned}
\mathcal{G}(w_\gamma) &= q^2 \sum_{\substack{t_1, t_3, t_4 \in \mathbb{F}_q \\ u,v \in \mathbb{F}_q^\times}} \lambda(-1 + \frac{t_3}{u} - \frac{t_1^2 v}{u^2} - \frac{t_1 v}{u}) \\
&= 0
\end{aligned}$$

and similarly we have

$$\mathcal{G}(w_\gamma^2) = \mathcal{G}(w_\gamma^4) = \mathcal{G}(w_\gamma^5) = 0.$$

Adding up the above 12 terms, we complete the proof of Theorem A.

Next, we prove Proposition B. First, we note that

$$\begin{aligned}
\mathcal{E}(\lambda) &= \sum_{u,v\in\mathbb{F}_q^\times} \lambda(u+v+uv+\frac{1}{u}+\frac{1}{v}+\frac{1}{uv}) \\
&= \sum_{u,k\in\mathbb{F}_q^\times} \lambda(u+\frac{k}{u}+k+\frac{1}{u}+\frac{u}{k}+\frac{1}{k}) \\
&= \sum_{u,k\in\mathbb{F}_q^\times} \lambda(k+\frac{1}{k})\cdot\lambda((1+\frac{1}{k})u+(1+k)\frac{1}{u}) \\
&= (q-1)\lambda(-2) + \sum_{k\in\mathbb{F}_q^\times-\{-1\}} \lambda(k+\frac{1}{k})\cdot\mathcal{K}_1(\lambda;1+\frac{1}{k};1+k).
\end{aligned}$$

For $a,b\in\mathbb{F}_q^\times$, it is well-known (the Weil's theorem) that

$$|\mathcal{K}_1(\lambda;a;b)| \leqq 2\sqrt{q}.$$

Thus,

$$|\mathcal{E}(\lambda) \leqq 2\sqrt{q}(q-2)+q-1$$

This proves Proposition B.

## 4. Gauss sums for $\mathbf{GL}_n(q)$ and $\mathbf{SL}_n(q)$

In this section, let $G = \mathbf{GL}_n(q)$ or $\mathbf{SL}_n(q)$ and let $\phi$ be the natural $n$-dimensional representation over $\mathbb{F}_q$.

In this case, we may assume the followings: $H$ is the set of diagonal matrices in $G$; $B$ is the set of upper-triangular matrices in $G$; $U$ is the set of unipotent matrices in $B$; $W$ is isomorphic to the symmetric group $S_n$ and generated by the matrices $w_{ij} = I - e_{ii} - e_{jj} + e_{ij} - e_{ji}$, where $i<j$.

LEMMA 4. *Let $b = (b_{ij}) \in B$ and $1 \neq w \in W$. Then some $\pm b_{kl}$, with $k < l$, appears on the diagonal fo $bw$.*

*Proof.* Note that $bw_{ij}$, with $i < j$ is obtained by permuting (up to signs) the $i$-th and the $j$-th column of $b$. Therefore, after a non-trivial permutation (up to signs) of columns of $b$, we clearly have some $\pm b_{kl}$, with $k < l$, on the diagonal. □

LEMMA 5. *If $1 \neq w$, then*

$$\sum_{b \in B} \chi(\det(bw)) \cdot (\lambda(\mathrm{tr}(bw)) = 0.$$

*Proof.* Since $\det(W) = 1$, the above sum equals

$$\sum_{b=(b_{ij}) \in B} (\chi(b_{11}) \cdots \chi(b_{nn})) \cdot (\lambda(c_{11}) \cdots \lambda(c_{nn})),$$

where $bw = (c_{ij})$. But $c_{ii} = \pm b_{kl}$ for some $i$ and $k < l$ by Lemma 4. Therefore, the above sum is equal to the product of $\sum_{b_{kl} \in \mathbb{F}_q} \lambda(b_{kl})$ and some other term, and hence equal to zero by Lemma 3.

Thus, we have

$$\begin{aligned}
&\sum_{g \in \mathbf{GL}_n(q)} \chi(\det(g)) \cdot \lambda(\mathrm{tr}(g)) \\
&= \sum_{b \in B} \chi(\det(b)) \cdot \lambda(\mathrm{tr}(b)) && \text{(by Lemma 2 and Lemma 5)} \\
&= \sum_{h \in H, u \in U} \chi(\det(hu)) \cdot \lambda(\mathrm{tr}(hu)) \\
&= |U| \sum_{h \in H} \chi(\det(h)) \cdot \lambda(\mathrm{tr}(h)) && (\det(u) = 1, \quad \mathrm{tr}(hu) = \mathrm{tr}(h)) \\
&= |U| \sum_{u_1, \cdots, u_n \in \mathbb{F}_q^\times} (\chi(u_1) \cdots \chi(u_n)) \cdot (\lambda(u_1) \cdots \lambda(u_n)) \\
&= q^{\binom{n}{2}} \cdot \mathcal{G}(\chi, \lambda)^n
\end{aligned}$$

and, by the same arguement, we have □

$$\begin{aligned}
&\sum_{g\in \mathbf{SL}_n(q)} \lambda(\mathrm{tr}(g)) \\
&= |U| \sum_{h\in H} \lambda(\mathrm{tr}(h)) \\
&= |U| \sum_{u_1,\cdots,u_{n-1}\in \mathbb{F}_q^\times} \lambda(u_1)\cdots\lambda(u_{n-1})\lambda(u_1^{-1}\cdots u_{n-1}^{-1}) \\
&= g^{\binom{n}{2}} \cdot \mathcal{K}_{n-1}(\lambda;1,\cdots,1;1).
\end{aligned}$$

This proves the following theorem.

THEOREM C. (See [3].) *if $\phi$ is the natural $n$-dimensional representation, then*

$$\mathcal{G}(\mathbf{GL}_n(q),\phi,\chi,\lambda) = q^{\binom{n}{2}} \cdot \mathcal{G}(\chi,\lambda)^n$$

*and*

$$\mathcal{G}(\mathbf{SL}_n(q),\phi,\chi,\lambda) = q^{\binom{n}{2}} \cdot \mathcal{K}_{n-1}(\lambda;1,\cdots,1;1).$$

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