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ON THE WEAK INVARIANCE PRINCIPLE FOR RANGES OF RECURRENT RANDOM WALKS WITH INFINITE VARIANCE

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1. Introduction

Let $\{ X_k : k = 1, 2, \cdots \}$ be a sequence of independent, identically distributed integer-valued random variables with common distribution function F. Throughout this paper we assume that

(A1) F belongs to the domain of attraction of a strictly α-stable distribution with $1 < \alpha \leq 2$,
(A2) $EX_1 = 0$,
(A3) $E\exp(iuX_1) = 1$ if and only if u is a multiple of 2π.

We note that $\{S_n\}$ is an aperiodic recurrent random walk, where $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k$. Let $\varphi(u) = E\exp(iuX_1)$. Then it is well-known that

$$|\varphi(u)| = \exp\{-|u|^\alpha l(1/|u|)\} \quad \text{for} \quad |u| \leq \pi,$$

where $l(x)$ is a slowly varying function at infinity. Furthermore if we choose a_n so that

$$\frac{l(a_n)}{a_n^\alpha} = \frac{1}{n}$$

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for each n, then $Y^{(n)}(t) = S_{[nt]}/a_n$ converges weakly to a strictly α-stable process $Y(t)$, where $[x]$ denotes the greatest integer not exceeding x (e.g. see page 345 of [1]).

The range R_n of random walk $\{S_k\}$ and the range $\Lambda(t)$ of stable process Y are defined as follows ;

$$R_n = \text{the cardinality of } \{S_0, S_1, \cdots, S_n\}$$

and

$$\Lambda(t) = m\{Y(s) : 0 \leq s \leq t\},$$

where " m " denotes the Lebesgue measure on $\Re^1$. Set $\Lambda^{(n)}(t) = R_{[nt]}/a_n$. The aim of the present work is to prove weak convergence of $\Lambda^{(n)}$ to Λ. In fact, we obtain the existence of $\tilde{\Lambda}^{(n)}$ and $\tilde{\Lambda}$, versions of $\Lambda^{(n)}$ and Λ, respectively, such that $\tilde{\Lambda}^{(n)}(t)$ converges to $\tilde{\Lambda}(t)$ uniformly on $[0,T]$ in L^2-sense for any $T > 0$.

Le Gall and Rosen [6] obtained various limit theorems for the range of d-dimensional random walk in the domain of attraction of a stable distribution of index α. Their results depend on the value of the ratio α/d. That is, for the case $\alpha/d \leq 1$, strong law of large numbers and central limit theorems hold and for the case $\alpha > d = 1$ which we are concerned with in this work, R_n/a_n converges in distribution to $\Lambda(1)$. In this work, we extend their result and prove the weak convergence of $\Lambda^{(n)}$ to Λ. Borodin [3] obtained a weaker result for the similar question for recurrent random walks with finite variance.

Now we state our main result, whose proof is given in Section 2.

THEOREM. *Under the assumptions (A1), (A2) and (A3), there exist processes $\tilde{Y}^{(n)}$ and $\tilde{Y}$ in $D[0,\infty)$ equipped with Skorokhod metric satisfying the following conditions ;*
(i) $\tilde{Y}^{(n)} =^{\mathcal{D}} Y^{(n)}, \quad \tilde{Y} =^{\mathcal{D}} Y,$
(ii) *$\tilde{Y}^{(n)}$ converges to $\tilde{Y}$ a.s. in $D[0,\infty)$, and*
(iii) *for each $T > 0$ and positive integer m,*

$$E\left[\sup_{0\leq t\leq T} \left|\tilde{\Lambda}^{(n)}(t) - \tilde{\Lambda}(t)\right|^{2m}\right] \longrightarrow 0 \quad \text{as } n \to \infty,$$

where $\tilde{\Lambda}^{(n)}$ and $\tilde{\Lambda}$ are defined with respect to $\tilde{Y}^{(n)}$ and $\tilde{Y}$, respectively and " $=^{\mathcal{D}}$ " means that two processes have the same finite dimensional distributions.

2. Proof of Main Result

Recall that we assume (A1), (A2) and (A3). Throughout this work, we denote P_0 and E_0 by P and E, respectively.

We present the proof of the Theorem in this section. Since the construction of $\tilde{Y}^{(n)}$ and $\tilde{Y}$ satisfying parts (i) and (ii) of the Theorem are well-known (e.g. see chapter 1 of [7]), it suffices to establish part (iii) of the Theorem. Therefore we may abuse our notation and use $Y^{(n)}$, Y, $\Lambda^{(n)}$ and Λ for $\tilde{Y}^{(n)}$, $\tilde{Y}$, $\tilde{\Lambda}^{(n)}$ and $\tilde{\Lambda}$, respectively throughout the remainder of the work. Essentially, the proof of our assertion amounts to estimating

$$E\left[\left(\Lambda^{(n)}(t)-\Lambda(t)\right)^{2m}\right], \tag{2.1}$$

since a simple monotonicity argument using also continuity of $\Lambda(t)$ implies our assertion if (2.1) converges to zero. Le Gall and Rosen [6] showed that $\tilde{\Lambda}^{(n)}(t)$ converges to $\tilde{\Lambda}(t)$ in L^1-sense, but their technique doesn't work in general. To deal with (2.1), we express the ranges of random walks and stable processes using their local times, respectively. The local time $N(n,x)$ of random walk $\{S_k\}$ is defined by

$$N(n,x) = \text{the number of } \{\, 0 \le k \le n \ :\ S_k = x \,\}.$$

Let

$$L^{(n)}(t,x) = \frac{a_n}{n} N([nt],[xa_n])$$

and

$$W_n(t) = \{x \in \Re^1 \ :\ L^{(n)}(t,x) > 0\}.$$

Then we note that

$$\begin{aligned}\Lambda^{(n)}(t) &= \frac{1}{a_n}\sum_{k\in Z}\chi_{\{N([nt],k)>0\}}\\ &= \int_{\Re^1}\chi_{W_n(t)}(x)\,dx.\end{aligned}$$

For a stable process $Y(t)$ of index $1<\alpha\le 2$, it is well-known that there exists a version of local time $\{L(t,x)\}$ which is jointly continuous in

(t,x) and satisfies the so-called occupation time density formula, that is, for any Borel set B,

$$\int_B L(t,x)\,dx = \int_0^t \chi_B(Y(s))\,ds \qquad \text{a.s.}$$

The existence and joint continuity of $L(t,x)$ were proved by Trotter [8] for Brownian motion and by Boylan [4] for stable processes of index $\alpha > 1$. Moreover, Kang and Wee [5] proved that as $n \to \infty$,

$$\sup_{(t,x)\in[0,T]\times\Re^1} \left| L^{(n)}(t,x) - L(t,x) \right| \longrightarrow 0 \quad \text{in } L^2. \tag{2.2}$$

We provide an useful expression of $\Lambda(t)$ in terms of local time $L(t,x)$ in Lemma 2.1, and then apply the result of [5] to estimate (2.1).

LEMMA 2.1. *For each* $t \ge 0$,

$$\Lambda(t) = \int_{\Re^1} \chi_{W(t)}(x)\,dx \qquad \text{a.s.}$$

where $W(t) = \{x \in \Re^1 \ :\ L(t,x) > 0\}$.

Proof. We write

$$\Lambda(t) = m\,(\,G(t)\,) + m(\,W(t)\,),$$

where

$$G(t) = \{\, x \in \Re^1 \ :\ Y(s) = x \text{ for some } s \in [0,t]\,,\ \ L(t,x) = 0\,\}\,.$$

Let

$$\tau_x = \inf\{s \ge 0 \ :\ Y(s) = x\}\,,$$
$$\hat{Y}(s) = Y(s+\tau_x) - x\,,$$

and $\hat{L}(s,y)$ be the local time of $\hat{Y}$. The strong Markov property implies that

$$\begin{aligned} E\,[\,m(\,G(t)\,)\,] &= \int_{\Re^1} P\Big(\tau_x \le t\,,\ \ \hat{L}(t-\tau_x\,,0) = 0\Big)\ dx \\ &= \int_{\Re^1}\int_0^t P\Big(\,\hat{L}(t-s\,,0) = 0\Big)\ P(\tau_x \in ds)\,dx \\ &= 0, \end{aligned} \tag{2.3}$$

where the last equality follows from the definition of $\hat{L}(t,0)$ as a continuous additive functional with support $\{0\}$ (see page 216 of [2]).□

LEMMA 2.2. *For each $t \geq 0$ and positive integer m,*

$$E\left[\left(\Lambda^{(n)}(t) - \Lambda(t)\right)^{2m}\right] \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Proof. Recall that

$$W_n(t) = \{x \in \Re^1 \ : \ L^{(n)}(t,x) > 0\}$$

and

$$W(t) = \{x \in \Re^1 \ : \ L(t,x) > 0\}.$$

For each $K > 0$, let

$$\Lambda_K^{(n)}(t) = \int_{-K}^{K} \chi_{W_n(t)}(x)\, dx$$

and

$$\Lambda_K(t) = \int_{-K}^{K} \chi_{W(t)}(x)\, dx.$$

Then

$$\begin{aligned}
&E\left[\left(\Lambda^{(n)}(t) - \Lambda_K^{(n)}(t)\right)^{2m}\right] \\
(2.4)\quad &\leq E\left[\Lambda^{(n)}(t)^{2m} \cdot \chi_{\{\sup_{0\leq l\leq [nt]} |S_l| \, > \, Ka_n\}}\right] \\
&\leq \left\{E\left[\left(\Lambda^{(n)}(t)\right)^{4m}\right]\right\}^{1/2} \cdot \left\{P\left(\sup_{0\leq l\leq [nt]} |S_l| \ > \ Ka_n\right)\right\}^{1/2}
\end{aligned}$$

Weak convergence of $Y^{(n)}$ to Y implies that for any $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that

$$\sup_n P\left(\sup_{0\leq l\leq [nt]} |S_l| \ > \ Ka_n\right) \ < \ \varepsilon. \tag{2.5}$$

It follows from [6] that any finite moment of $\Lambda^{(n)}(t)$ is bounded uniformly in n. Thus by (2.5), we may choose K large so that (2.4) is sufficiently small for all n large. For $1 < \alpha < 2$, it is easy to see that for any $u > 0$,

$$E\left[\exp(u\Lambda(t))\right] < \infty$$

without assuming the symmetry of $Y(t)$, by modifying the argument in Lemma 4.1 of [9]. Thus

$$E\left[\Lambda(t)^{2m}\right] < \infty,$$

which is obvious for $\alpha = 2$. This enables us to have that for any $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that

$$E\left[(\Lambda(t) - \Lambda_K(t))^{2m}\right] < \varepsilon.$$

Now we fix K large enough, and observe that

(2.6)

$$\begin{aligned}
&E\left[\left(\Lambda_K^{(n)}(t) - \Lambda_K(t)\right)^{2m}\right] \\
&= E\left[\left(\int_{-K}^{K} \chi_{W_n(t)\cap W(t)^c}(x)\,dx - \int_{-K}^{K} \chi_{W_n(t)^c\cap W(t)}(x)\,dx\right)^{2m}\right] \\
&\le 2^{4m-2}K^{2m-1}\,E\left[\int_{-K}^{K} \chi_{W_n(t)\cap W(t)^c}(x)\,dx\right] \\
&\quad + 2^{4m-2}K^{2m-1}\,E\left[\int_{-K}^{K} \chi_{W_n(t)^c\cap W(t)}(x)\,dx\right].
\end{aligned}$$

Let $Y[0;t] = \{Y(s) \ : \ 0 \le s \le t\}$, $cl(Y[0;t])$ be it's closure, and $U_\delta(t)$ be the δ-neighborhood of $cl(Y[0;t])$. Then as $\delta \to 0$,

$$m(U_\delta(t)) \to m(cl(Y[0;t])) = m(Y[0;t]) \quad \text{a.s.}$$

and part (ii) of the Theorem implies that for fixed $\delta > 0$,

$$W_n(t) \subset U_\delta(t) \quad \text{a.s.} \tag{2.7}$$

for all sufficiently large n. Now fix $\delta > 0$ so that $E\left[m\left(U_\delta(t) \cap Y[0;t]^c\right)\right]$ is sufficiently small. Then by (2.7) and (2.3), for n large,

$$
\begin{aligned}
& E\left[\int_{-K}^{K} \chi_{W_n(t) \cap W(t)^c}(x)\, dx\right] \\
& \leq E\left[\int_{-K}^{K} \chi_{U_\delta(t) \cap W(t)^c}(x)\, dx\right] \\
& \leq E\left[\int_{-K}^{K} \chi_{Y[0;t] \cap W(t)^c}(x)\, dx\right] + E\left[m\left(U_\delta(t) \cap Y[0;t]^c\right)\right] \\
& \leq E[m(\, G(t)\,)] + E\left[m\left(U_\delta(t) \cap Y[0;t]^c\right)\right] \\
& = E\left[m\left(U_\delta(t) \cap Y[0;t]^c\right)\right].
\end{aligned}
$$

Hence the first summand of (2.6) can be made arbitrarily small for n sufficiently large. To estimate the second summand of (2.6), observe that for any $\eta > 0$,

$$
\begin{aligned}
P\Big(\, L(t,x) > 0\,,\ \ L^{(n)}(t,x) = 0\,\Big) \leq{}& P\left(\, 0 < L(t,x) < \eta\,\right) \\
& + \ P\left(\, \left|L^{(n)}(t,x) - L(t,x)\right| \geq \eta\,\right) .
\end{aligned}
$$

Use (2.2) and bounded convergence theorem to show that

$$
\begin{aligned}
& E\left[\int_{-K}^{K} \chi_{W_n(t)^c \cap W(t)}(x)\, dx\right] \\
& = \int_{-K}^{K} P\Big(L(t,x) > 0\,,\ \ L^{(n)}(t,x) = 0\Big)\, dx \\
& \to 0
\end{aligned}
$$

as n goes to infinity. □

Proof of the Theorem. Fix $h > 0$, which will be chosen later. Let $0 = t_0 < t_1 < \cdots < t_k \leq t_{k+1} = T$ be a partition of $[0,T]$ such that

$t_j - t_{j-1} = h$ for all $1 \le j \le k$ and $k = [T/h]$. Observe that by simple monotonicity of $\Lambda^{(n)}$ and Λ,

$$
\begin{aligned}
&E\left[\sup_{0\le t\le T}\left|\Lambda^{(n)}(t)-\Lambda(t)\right|^{2m}\right]\\
(2.8)\qquad &\le 3^{2m-1}2^{2m}\,E\left[\max_{0\le j\le k}\left|\Lambda(t_{j+1})-\Lambda(t_j)\right|^2\right]\\
&\quad + 3^{2m-1}(2^{2m}+1)\sum_{j=0}^{k+1}E\left[\left|\Lambda^{(n)}(t_j)-\Lambda(t_j)\right|^2\right].
\end{aligned}
$$

By the almost sure continuity of the mapping $t \mapsto \Lambda(t)$ and dominate convergence theorem, for given $\varepsilon > 0$, we can choose h so that the first term of (2.8) is less than $\varepsilon/2$. Then Lemma 2.2 implies that the second term of (2.8) can be made arbitrarily small if n is large enough, which completes the proof. □

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On the weak invariance principle

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