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STABILITY IN DISTRIBUTION FOR A CLASS OF DIFFUSIONS WITH JUMP[1]

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1. Introduction

We consider a diffusion $\{X^x(t); t \geq 0\}$ on R^1 satisfying the following stochastic differential equation.

$$X^x(t) = x + \int_0^t \sigma(X^x(s))dB(s) + \int_0^t b(X^x(s))ds + \int_0^t \int c(X^x(s), u)\tilde{\nu}(du, ds)$$

where σ and b are Lipschitz continuous functions on R^1, c is a measurable function on R^2, $\{B(t); t \geq 0\}$ is a standard 1-dimensional Brownian motion and $\tilde{\nu}$ is a compensated Poisson random measure on $R_+ \times R$. That is, there is a σ-finite measure π on $R^1 \setminus \{0\}$ such that $\tilde{\nu}([0,t) \times A) = \nu([0,t) \times A) - t\pi(A)$ where ν is a Poisson random measure on $R_+ \times R$ with $E[\nu([0,t) \times A)] = t\pi(A)$ for any Borel set A of R^1. Let $p(t, x, dy)$ denote the transition probability of the diffusion. First, we introduce the following definitions applying to general diffusions.

DEFINITION 1. A diffusion is stable in distribution if its transition probability $p(t, x, dy)$ converges weakly to some probability measure $\pi(dy)$ as $t \to \infty$, for every x.

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DEFINITION 2. The stochastic flow $\{X^x(t); t \geq 0, x \in R\}$ is asymptotically flat (in probability) uniformly on compacts if

$$\sup_{x,y \in K} P(|X^x(t) - X^y(t)| > \epsilon) \to 0 \tag{2}$$

as $t \to \infty$ for every $\epsilon > 0$ and every compact set K.

It is simple to check that stability in distribution follows from the following (ref. [3]):

(i) tightness of $\{p(t,x,dy); 0 \leq t < \infty\}$ and

(ii) asymptotic flatness (definition 2).

Now (2) can be derived by the following property.

DEFINITION 3. The stochastic flow $\{X^x(t); t \geq 0, x \in R\}$ is a asymptotically flat in the second mean if every compact set K in R,

$$\lim_{t \to \infty} \sup_{x,y \in K} E[X^x(t) - X^y(t)]^2 = 0. \tag{3}$$

REMARK 1. The exponent 2 in (3) can be changed to any $\delta > 0$ to imply the Definition 2.

In this paper, we consider the question:under what conditions on σ, b, c and ν, is the diffusion tight or stable in distribution? In the next section, we have some sufficient conditions for this even though it is very special. If $c \equiv 0$, then $X^x(t)$ is continuous a.s. and in that case, there are lots of literature including the above question ([1], [5], [2]). But with nonzero $c(x,u)$, $X^x(t)$ is right continuous with left limit. Sufficient conditions that $X^x(t)$ exists uniquely are well known ([4]) and those are given in the following section. We primarily follow the idea in [1] to have some conditions that x^2 is a Liapunov function for the generator of the process (1).

2. Main results

Consider the following conditions for σ, b, and c.

There is a constant M such that for all $x \in R$,

$$\sigma^2(x) + b^2(x) + \int c(x,u)^2 \pi(du) \leq L(1+x^2). \tag{4}$$

Lipschitz conditions; there exist positive constants λ_0, λ_1 and λ_2 such that for all x, y

$$|\sigma(x) - \sigma(y)| \leq \lambda_0 |x-y| \tag{5}$$

$$|b(x) - b(y)| \leq \lambda_1 |x-y|, \quad \frac{d}{dx} b \leq -\lambda_1 < 0, \tag{6}$$

and

$$\int |c(x,u) - c(y,u)|^2 \pi(du) \leq \lambda_2 |x-y|^2. \tag{7}$$

With above conditions (4)-(7), the diffusion exists uniquely and $E|X_t^x|^2 < \infty$ for all t and x. (cf. [4] part II, ch.2). And without (5) and (6), but if σ,b are continuous, X_t in (1) exists. (cf: [6])

THEOREM 1.
(1) *Assume conditions (4)-(7). If there exists a constant $\beta > 0$ such that*

$$-2\lambda_1 + \frac{\sigma^2(x)}{x^2} + \lambda_2 \leq -\beta$$

for all sufficiently large $|x|$, there exists an invariant probability.
(2) *Assume the conditions (4)-(7). If there exists a constnat $\alpha > 0$ such that $-2\lambda_1 + \lambda_0^2 + \lambda_2 < -\alpha$, then the diffusion (1) is stable in distribution.*

REMARK 1. Note that

$$\begin{aligned}\sigma^2(x) &= (\sigma(x) - \sigma(0) + \sigma(0))^2 \\ &= (\sigma(x) - \sigma(0))^2 + \sigma(0)^2 + 2\sigma(0)(\sigma(x) - \sigma(0)) \\ &\leq \lambda_0^2 x^2 + \sigma(0)^2 + 2|\sigma(0)|\lambda_0|x|.\end{aligned}$$

Hence

$$-2\lambda_1 + \frac{\sigma^2(x)}{x^2} + \lambda_2 \leq (-2\lambda_1 + \lambda_0^2 + \lambda_2) + \frac{\sigma(0)^2}{x^2} + 2|\sigma(0)|\frac{\lambda_0|x|}{x^2}$$
$$= (-2\lambda_1 + \lambda_0^2 + \lambda_2) + O(|x|)$$

as $|x| \to \infty$ and we see that (2) in Theorem 1 implies (1).

Proof of Theorem 1. (1) Since $p(t, x, dy)$ is Feller continuous (ref: Theorem 1 p.276 and Lemma 2 p.284 of [4]), it is sufficient to show that $\sup_{t\geq 0} E|X_t^x|^2 < \infty$ and for this, by the following Lemma 1, it is enough to show that for some constant $\beta' > 0$, $L\phi(y) \leq -\beta' y^2$ for large enough $|y|$ where $\phi(y) = y^2$ and $L\phi(y) = 2yb(y) + \sigma^2(y) + \int c^2(y,u)\pi(du)$. By Ito's formular for X_t^x in (1),

$$\begin{aligned}
(X_t^x)^2 &= x^2 + \int_0^t 2X_s\sigma(X_s^x)dB_s \\
&+ \int_0^t \int ((X_s^x + c(X_s, u))^2 - (X_s^x)^2)\tilde{\nu}(du, ds) \\
&+ \int_0^t \int (2X_s^x b(X_s^x) + \sigma^2(X_s^x))ds \\
&+ \int_0^t \int ((X_s^x + c(X_s^x, u))^2 - (X_s^x)^2 - 2c(X_s^x, u)X_s^x)\pi(du)ds \\
&= x^2 + \int_0^t 2X_s^x\sigma(X_s^x)dB_s \\
&+ \int_0^t \int (2c(X_s^x, u)X_s^x + c^2(X_s^x, u))\tilde{\nu}(du, ds) \\
&+ \int_0^t (2X_s^x b(X_s^x) + \sigma^2(X_s^x) + \int c^2(X_s^x, u)\pi(du))ds.
\end{aligned}$$

Then by the conditions (4), (6) and (7),

$$
\begin{aligned}
&2yb(y) + \sigma^2(y) + \int c(y,u)^2\pi(du)\\
&= 2y(b(y) - b(0)) + 2yb(0) + \sigma^2(y) + \int (c(y,u) - c(0,u) + c(0,u))^2\pi(du)\\
&\le -2\lambda_1 y^2 + 2yb(0) + \sigma^2(y) + \lambda_2 y^2 + M + 2\int (c(y,u) - c(0,u))c(0,u)\pi(du)\\
&\le (-2\lambda_1 + \frac{\sigma^2(y)}{y^2} + \lambda_2)y^2 + 2yb(0) + M + 2(M\lambda_2 y^2)^{1/2}\\
&\le (-2\lambda_1 + \frac{\sigma^2(y)}{y^2} + \lambda_2)y^2 + O(|y|) \le -\beta' y^2
\end{aligned}
$$

for large enough $|y|$. Hence by the following Lemma 1, $\sup_{t\ge 0} E|X_t^x|^2 < \infty$. Therefore there exists an invariant probability.

(2) Define for a given pair (x, y) with $x \ne y$,

$$
\begin{aligned}
Z_t^{x,y} &= X_t^x - X_t^y\\
&= x - y + \int_0^t (b(X_s^x) - b(X_s^y))ds + \int_0^t (\sigma(X_s^x) - \sigma(X_s^y))dB_s\\
&+ \int_0^t \int (c(X_s^x, u) - c(X_s^y, u))\tilde{\nu}(du, ds)\\
&= (x - y) + \int_0^t f(s)ds + \int_0^t \gamma(s)dB_s + \int_0^t \int g(s,u)\tilde{\nu}(du, ds).
\end{aligned}
$$

Let $\tau_0 := \inf\{t \ge 0 : Z^{x,y}(t) = 0\}$. By Ito formular applied to $\phi(x) = x^2$,

$$
\begin{aligned}
(Z_t^{x,y})^2 &= (x-y)^2 + \int_0^t 2Z_s^{x,y}\gamma_s dB_s\\
&+ \int_0^t \int ((Z_s^{x,y} + g(s,u))^2 - (Z_s^{x,y})^2)\tilde{\nu}(du, ds)\\
&+ \int_0^t \tilde{L}(\phi)(X_s^x, X_s^y)ds
\end{aligned}
$$

where $\tilde{L}(\phi)(X_s^x, X_s^y) = 2Z_s^{x,y} f(s) + \gamma(s)^2 + \int g(s,u)^2\pi(du)$. Now

$$
(x-y)(b(x) - b(y)) = \int_0^1 (x-y)^2 b'(y + \theta(x-y))d\theta \le -\lambda_1(x-y)^2
$$

by (6). Hence

$$\tilde{L}(\phi)(X_s^x, X_s^y) \leq (-2\lambda_1 + \lambda_0^2 + \lambda_2)(Z_s^{x,y})^2 \leq -\alpha(Z_s^{x,y})^2.$$

Hence we have

$$E(Z^{x,y}(t \wedge \tau_0))^2 \leq (x-y)^2 - \alpha E(\int_0^{t\wedge\tau_0} (Z_s^{x,y})^2 ds).$$

Notice that $Z_t^{x,y} = 0$ a.s. for all $t \geq \tau_0$. Therefore $E(Z_t^{x,y})^2 \leq e^{-\alpha t}(x-y)^2$ for all $t \geq 0$. Hence $E(Z_t^{x,y})^2 \to 0$ as $t \to \infty$ for any x, y in compact set and it implies the asymptotic flatness of the second mean. Hence by Remark 1, the process of (1) is stable in distribution. □

LEMMA 1. *If $\phi(y) = y^2$ and for some $\beta > 0$, $L\phi(y) = 2yb(y) + \sigma^2(y) + \int c(y,u)^2\pi(du) \leq -\beta y^2$ for all large $|y|$, then for the process X_t^x of (1), we have $\sup_{t\geq 0} E(X_t^x)^2 < \infty$ for any x.*

Proof. Take N large enough so that $L\phi(y) \leq -\beta y^2$ if $|y| \geq N$. Then take expectation on (8) using usual truncation (p.275 of [4]), we have

$$E(X_t^x)^2 = x^2 + \int_0^t EL\phi(X_s^x)ds$$

since with the conditions (4)-(7), we can take constant C such that $L\phi(y) \leq C(1+y^2)$ for all y, hence $\int_0^t EL\phi(X_s^x)ds < \infty$ and stochastic integrals with respect to B and $\tilde{\nu}$ are martingales. Therefore we have $\frac{d}{dt}E(X_t^x)^2 = EL\phi(X_t^x)$ and we can take positive constants M_1, M_2 such that

$$\begin{aligned}\frac{d}{dt}E(X_t^x)^2 &= E(L\phi(X_t^x)1_{(|X_t^x|<N)}) + E(L\phi(X_t^x)1_{(|X_t^x|\geq N)}) \\ &\leq M_1 - \beta E((X_t^x)^2 1_{|X_t^x|\geq N}) \\ &= M_1 - \beta E(X_t^x)^2 + \beta E((X_t^x)^2 1_{|X_t^x|<N}) \\ &\leq M_1 + M_2 - \beta E(X_t^x)^2.\end{aligned}$$

Hence by routine calculation, $\sup_t E(X_t^x)^2 < \infty$. □

REMARK 2. Without any difficulty, we can extend Theorem 1 to multidimensionl case under the same condition of the components of coefficients and $|x|^2$ instead of x^2.

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