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ON ASYMPTOTIC BEHAVIOR OF A RANDOM EVOLUTION

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1. Introduction

In this paper, we study the asymptotic behavior of a random evolution. Some examples of random evolution can be found in Chapter 12 of [2].

In [4][5], Kurtz and Protter worked also on an approximation of solutions of SDE applying their Theorem 5.4 in the same paper. Motivated by theorems by Kurtz and Protter, we now consider a sequence of stochastic differential equations. This study dates back at least to Khasminskii [3], who studies the behavior of trajectory of stochastic process defined by the differential equation with a rapidly varying components,

$$\frac{dx}{dt} = \epsilon F(x,t,\omega), \quad x(0) = x_0,$$

over a lenth of time of order $O(\frac{1}{\epsilon})$ as $\epsilon \to 0$.

Let E be a separable metric space, Z be an $E-$valued ergodic Markov process with stationary distribution μ. We assume that $F : R \times E \longrightarrow R$ is bounded and has bounded and continuous first order partial derivatives such that $\int F(x,y)\mu(dy) = 0$.

Let $X_n, \quad n = 1, 2, \cdots$, satisfy;

$$dX_n(t) = nF(X_n(t), Z(n^2 t))dt \tag{1.1}$$

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We shall consider the limit behavior of solution processes, X_n in an extension of the results of Wong and Zakai [6]: that certain naive approximations of semimartingale differentials lead to a lack of continuity of the corresponding solutions of stochastic differential equations. You may refer this kind of results to [4], [5].

We assume the following hypotheses:

There exists an operator A which is the generator of Z such that (letting $\mathcal{R}(A)$ be the range of A and $\mathcal{D}(A)$ be the domain of A) $L^2_E(\mu)$ is genetrated by 1 and $\mathcal{R}(A)$, and $\mathcal{D}(A)$ is an algebra.

A has the eigenvectors $\{f_k\}$ with eigenvalues $\{\lambda_k\}$ which satisfy;

CONDITION 1.1.

1) For each $T > 0$ there exists a $M_0 > 0$ such that $\sup_{0\le s\le T} E[f_k(Z(s))] < M_0$ for every k.

2) $f_0 = 1$, $\int f_i \cdot f_j d\mu = 0$ if $i \neq j$, and $\int f_i^2(z)d\mu(z) = 1$ if $i = 1, 2, \cdots$

3) $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty$

4) $R(A) = \langle 1, f_1, f_2, \cdots, f_k, \cdots \rangle = L^2_E(\mu)$,

where $\langle 1, f_1, f_2, \cdots, \rangle$ is the smallest space generated by $1, f_1, f_2, \cdots$.

Now we expand $F(x,\cdot)$ in $L^2_E(\mu)$ with f_k. Let

$$
\begin{aligned}
g_k(x) &= \int F(x,y) f_k(y)\mu(dy) = \langle F(x,y), f_k(y)\rangle_\mu \quad k = 1, 2, \cdots \\
g_0(x) &= \int F(x,y) 1 \mu(dy) = 0
\end{aligned} \tag{1.2}
$$

Then

$$
F(x,y) = \sum_{k=0}^{\infty} g_k(x) f_k(y)
$$

By Bessel's inequality,

$$
\sum_{k=0}^{\infty} |g_k(x)|^2 \le \int F(x,y)^2 \mu(dy) < \infty.
$$

Then, (1.1) can be rewritten as

$$
X_n(t) = X_n(0) + n \cdot \int_0^t (\sum_{k=0}^{\infty} g_k(X_n(s)) \cdot f_k(Z(n^2 s))ds, \tag{1.3}
$$

where the stochastic integral is just a Stieltjes integral and consequently needs no special definition. Finally, we need to assume that there exist $\eta_k, \quad k = 1, 2, \cdots$ such that

$$(1.4) \qquad \sup_x |g_k(x)| \le \frac{1}{\eta_k}, \qquad \sum_{k=0}^{\infty} \frac{1}{|\eta_k|^2} < \infty.$$

EXAMPLE 1.1. Let $Z(s)$ be Brownian Motion with state space $[0, \pi]$, which reflects at both end points. Then

$$A = \{(f, \frac{1}{2}f'')|f \in C^2[0, \pi], f'(0) = f'(\pi) = 0\}$$

is the generator of $Z(s)$. The eigenvectors of A are $f_k(x) = \sqrt{\frac{2}{\pi}} \cos kx, k = 1, 2 \cdots$ and the eigenvalues $\lambda_k = -k^2$. Then our $\{f_k\}$ and $\{\lambda_k\}$ satisfies the assumptions.

1)$\{f_k(x)\}$ is uniformly bounded.

2)$\sum_{k=1}^{\infty} |\frac{1}{\lambda_k}| = \sum \frac{1}{k^2} < \infty$

3)$A(f_k^2) = \frac{2k^2}{\pi} \cos 2kx$, $\frac{1}{\lambda_k} A(f_k^2) = -\frac{2}{\pi} \cos 2kx$

Furthermore, let $F : R \times [0, \pi] \to R$ be a bounded and even function. Then $F(x, z)$ can be expanded

$$F(x, z) = \sum_{k=0}^{\infty} g_k(x) \cos kz, \quad g_k(x) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} F(x, z) \cos kz dz$$

and $\|g_k\|_\infty \le \|F\|_\infty \cdot \frac{1}{k}$

Choosing $\eta_k = k$ we can see g_k satisfies the assumption (1.4). □

2. Main theorem

Define $W_n^k(t)$, $Y_n^k(t)$ and $Z_n^k(t)$ such that

$$(2.1) \qquad \begin{aligned} W_n^k(t) &= \int_0^t n f_k Z(n^2 s)\, ds = \frac{1}{n} \int_0^{n^2 t} f_k(Z(s))\, ds \\ Y_n^k(t) &= -\frac{1}{n\lambda_k} f_k(Z(n^2 t)) + \frac{1}{n} \int_0^{n^2 t} A(\frac{1}{\lambda_k} f_k)(Z(s))\, ds \\ Z_n^k(t) &= \frac{1}{n\lambda_k} f_k(Z(n^2 t)) \end{aligned}$$

Then $W_n^k(t) = Y_n^k(t) + Z_n^k(t)$, and (1.1) can be expressed;
(2.2)

$$\begin{aligned} X_n(t) &= X_n(0) + n\int_0^t F(X_n(s), Z(n^2 s))ds \\ &= X_n(0) + \sum_{k=0}^{\infty}\int_0^t g_k(X_n(s))dW_n^k(s) \\ &= X_n(0) + \sum_{k=0}^{\infty}\int_0^t g_k(X_n(s))dY_n^k(s) + \sum_{k=0}^{\infty}\int_0^t g_k(X_n(s))dZ_n^k(s) \\ &= X_n(0) + (*) + (**) \end{aligned}$$

Before we state our main theorem, we first see the limit behavior of $Y_n^k, k = 1, 2, \cdots$.

LEMMA 2.1. *Let*

$$A_n^{kj}(t) = \frac{1}{n^2}\int_0^{n^2 t} A(\frac{f_k}{\lambda_k}\cdot\frac{f_j}{\lambda_j})(Z(s)) - \frac{f_j}{\lambda_j}A(\frac{f_k}{\lambda_k})(Z(s)) - \frac{f_k}{\lambda_k}A(\frac{f_j}{\lambda_j})(Z(s))\,ds$$

for any $k, j = 1, 2, \cdots$ *and let*

$$\begin{aligned} C_{kj} &= \int -\frac{2}{\lambda_k} f_k^2(z)d\mu(z) = \frac{2}{|\lambda_k|} \quad \text{if } k = j \\ &= 0 \quad \text{if } k \neq j \end{aligned}$$

Then

$$A_n^{kj}(t) \to t \cdot C_{kj}, \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} E[Y_n^k]_t = \frac{2t}{|\lambda_k|} \tag{2.3}$$

Proof. Since $Z(s)$ is ergodic with stationary distribution μ

$$A_n^{kj}(t) \longrightarrow t \cdot C_{kj} \qquad \text{a.s.}\ .$$

Note that $Y_n^k(t)Y_n^j(t) - A_n^{kj}(t)$ is a martingale and hence,

$$E[Y_n^k, Y_n^j]_t = E[A_n^{kj}(t)]$$

$$E[Y_n^k]_t = \frac{1}{n^2}\int_0^{n^2 t} E[\frac{1}{{\lambda_k}^2}A({f_k}^2)(Z(s)) - \frac{2{f_k}^2}{\lambda_k}(Z(s))]ds$$
$$\longrightarrow \frac{2t}{|\lambda_k|},$$

as $n \to \infty$. □

LEMMA 2.2. *For every d, $d = 1, 2, \cdots$ there exists a process $Y = (Y^1, \cdots, Y^d)$ with sample paths in $C_{R^d}[0,\infty)$ such that $(Y_n^1, \cdots, Y_n^d) \Rightarrow (Y^1, \cdots, Y^d)$ and Y^i, $Y^iY^j - C_{ij}$, $i, j = 1, 2, \cdots, d$ are martingales with respect to $\{\mathcal{F}_t^Y\}$. The process Y has independent Gaussian increments.*

Proof. For each $i, j = 1, 2 \cdots$ $Y_n^i Y_n^j - A_n^{ij}(t)$ is an $\mathcal{F}_t^n$-martingale and $A_n^{ij}(t) \longrightarrow C_{ij}(t)$. So, by the martingale central limit theorem (Th.7.1.4 [2]) we get the conclusion. □

We shall show that the sequence of solution to equation (1.5), $\{X_n\}$ is relatively compact and get a possible limit. In fact, in (2.2) we show that $(*)$ and $(**)$ are relatively compact in $D_R[0,\infty)$. Then $\{X_n\}$ is also relatively compact, since the limits are continuous. If we apply Theorem 2.2 [4] we can see the limit of $(*)$ and using the ergodic theorem, we will see the limit of $(**)$.

THEOREM 2.1. *Let $Z(s)$ be an ergodic process with generator A and stationary distribution μ satisfying the above hypotheses. If $X_n(0) \Rightarrow X(0)$, then $\{X_n\}$ is relatively compact and any limit point X satisfies*

$$X(t) = X(0) + \sum_{k=0}^{\infty}\int_0^t g_k(X(s))dY^k(s)$$
$$+ \sum_{k=0}^{\infty}\int_0^t\int_E \frac{1}{\lambda_k} g_k'(X(s))f_k(z)F(X(s), z)\mu(dz)ds$$

where Y^k, $k = 1, 2\cdots$ are martingale processes with Gaussian independent increments.

Proof. First, for convenience, let's denote

$$
\begin{aligned}
\bar{X}_n(t) &= \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dY_n^k(s) \\
\tilde{X}_n(t) &= \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dZ_n^k(s) \\
X_n(t) &= X_n(0) + \bar{X}_n(t) + \tilde{X}_n(t),
\end{aligned} \tag{2.4}
$$

where $Y_n^k(t) = -\frac{1}{n\lambda_k} f_k(Z(n^2(t)) + \frac{1}{n}\int_0^{n^2 t} A(\frac{f_k}{\lambda_k})(Z(s))ds$. We shall show the relative compactness for $X_n(t)$ in (2.4). Step1 To show the relative compactness of $\{\bar{X}_n\}$ according to (1.4), choose $\eta_k > 0$ such that

$$
\sup_{0\le s\le t} |g_k(X_n(s))| \le \frac{1}{\eta_k} \quad \text{for every} \quad n \quad \text{and} \quad \sum \frac{1}{\eta_k^2} < \infty
$$

Then for all n

$$
\begin{aligned}
&E[|\bar{X}_n(t)|^2] \\
&\le E[\sum_{k=0}^{\infty} \int_0^t g_k^2(X_n(s)d[Y_n^k]_s + E[\sum_{k\ne j} \int_0^t g_k(X_n(s))g_j(X_n(s))d[Y_n^k, Y_n^j]_s] \\
&\le \sum_{k=0}^{\infty} \frac{1}{\eta_k^2} E[Y_n^k]_t + \sum_{k\ne j} (\int_0^t g_k^2(X_n(s))d[Y_n^k]_s)^{\frac{1}{2}} (\int_0^t g_j^2(X_n(s))d[Y_n^j]_s)^{\frac{1}{2}} \\
&\qquad \text{by Kunita-Watanabe inequality} \\
&\to \sum_{k=0}^{\infty} \frac{1}{\eta_k^2}\frac{2t}{|\lambda_k|} + \sum_{k\ne j} \frac{1}{\eta_k}(\frac{2t}{|\lambda_k|})^{\frac{1}{2}} \frac{1}{\eta_j}(\frac{2t}{\lambda_j|})^{\frac{1}{2}} \quad \text{by (2.3)} \\
&= C_0 \cdot t, \quad C_0 = 2(\sum_{k=0}^{\infty} \frac{1}{\eta_k^2 |\lambda_k|} + \sum_{k\ne j} \frac{1}{\eta_k |\lambda_k|^{\frac{1}{2}}} \frac{1}{\eta_j |\lambda_j|^{\frac{1}{2}}})
\end{aligned} \tag{2.5}
$$

Hence, for each $\eta > 0$,

$$
\varlimsup_{n\to\infty} \mathbf{P}\{|\bar{X}_n(t)| > (\frac{C_0 t}{\eta})^{\frac{1}{2}}\} \le \lim_{n\to\infty} \frac{E[|\bar{X}_n(t)|^2]}{C_0 t}\eta \le \eta
$$

for every n. Choose $\Gamma_{\eta,t} = \bar{B}(0, (\frac{Ct}{\eta})^{\frac{1}{2}})$, then

$$\varliminf_{n\to\infty} \mathbf{P}\{\bar{X}_n(t) \in \Gamma_{\eta,t}\} \geq 1-\eta.$$

To see the other criteria for relative compactness for $\{\bar{X}_n(t)\}$,

$$\begin{aligned}
&E[|\bar{X}_n(t+u) - \bar{X}_n(t)|^2\,|\mathcal{F}_t] \\
&\leq \sum_{k=0}^{\infty} E[| \int_t^{t+u} (g_k(X_n(s))dY_n^k(s))^2 \\
&+ \sum_{k\neq j} (\int_t^{t+u} g_k^2(X_n(s))d[Y_n^k]_s)^{\frac{1}{2}} (\int_t^{t+u} g_j^2(X_n(s))d[Y_n^j]_s)^{\frac{1}{2}}\,|\,|\mathcal{F}_t] \\
&\leq E[\sum_{k=0}^{\infty} \frac{1}{\eta_k^2}([Y_n^k]_{t+u} - [Y_n^k]_t)|\mathcal{F}_t] \\
&+ E[\sum_{k\neq j} \frac{1}{\eta_k}([Y_n^k]_{t+u} - [Y_n^k]_t)^{\frac{1}{2}} \frac{1}{\eta_j}([Y_j^n]_{t+u} - [Y_j^n]_t)^{\frac{1}{2}}|\mathcal{F}_t],
\end{aligned}$$

Let

$$\begin{aligned}
\gamma_n(\delta) = &\sum_{k=o}^{\infty} \frac{1}{\eta_k}([Y_n^k]_{t+\delta} - [Y_n^k]_\delta) \\
&+ \sum_{k\neq j} \frac{1}{\eta_k\eta_j}([Y_n^k]_{t+\delta} - [Y_n^k]_t)^{\frac{1}{2}}([Y_j^n]_{t+\delta} - [Y_j^n]_t)^{\frac{1}{2}}
\end{aligned}$$

Then, we have for $0 \leq t \leq T,\quad 0 \leq u \leq \delta$,

$$E[|\bar{X}_n(t+u) - \bar{X}_n(t)|^2\,|\mathcal{F}_t] \leq E[\gamma_n(\delta)\,|\mathcal{F}_t]$$

and since $Y_n^k(t)Y_n^j(t) - A_n^{kj}(t)$ is a martingale and $Y_n^k(t)Y_n^j(t) - [Y_n^k, Y_n^j]_t$ is also a martingale.

$$\lim_{\delta\to 0}\sup_n E[\gamma_n(\delta)] = \lim_{\delta\to 0}(\sum_{k=0}^{\infty} \frac{1}{\eta_k}\frac{2\delta}{|\lambda_k|} + \sum_{k\neq j}\frac{1}{\eta_k\eta_j}\frac{2\delta}{|\lambda_k\lambda_j|^{\frac{1}{2}}}) = 0$$

<u>*Step*2</u> To show $\{\sum_{k=0}^{\infty}\int_0^{\cdot} g_k(X_n(s))dZ_n^k(s)\}$ is relatively compact, fix $T>0$. Since $[X_n,X_n]_t=0$ and $[g_k(X_n),Z_n^k]_t=0$, by integration parts

$$\begin{aligned}\int_0^t g_k(X_n(s))dZ_n^k(s) &= g_k(X_n(t))Z_n^k(t)-g_k(X_n(0))Z_n^k(0)\\ &\quad -\int_0^t g_k'(X_n(s))Z_n^k(s)dX_n(s)\\ &\quad -\int_0^t g_k(X_n(s))dZ_n^k(s)\end{aligned}$$

Since

$$\|g_k\|_\infty \le \|F\|_\infty, \|g_k'\|_\infty \le \|\frac{\partial F}{\partial x}\|_\infty,$$

we have for $0\le t\le T$,
(2.7)

$$\begin{aligned}&E[|\int_0^t g_k(X_n(s))dZ_n^k(s)|]\\ &=E[|g_k(X_n(t))Z_n^k(t)-g_k(X_n(0)Z_n^k(0)-\int_0^t g_k'(X_n(s))Z_n^k(s)dX_n(s)|]\\ &\le \|g_k\|_\infty E[|\frac{f_k(Z(n^2t))}{n\lambda_k}|+|\frac{f_k(Z(0))}{n\lambda_k}|]+\|g_k'\cdot F\|_\infty E[\int_0^t \frac{f_k(Z(n^2s))}{\lambda_k}ds]\\ &\le \frac{1}{\eta_k}\frac{2M_0}{n\lambda_k}+\frac{1}{\lambda_k}\|g_k'\cdot F\|_\infty M_0 t\,(\le \frac{2M_0}{n\eta_k\lambda_k}+\|\frac{\partial F}{\partial x}\cdot F\|_\infty\frac{M_0t}{\lambda_k}),\end{aligned}$$

where $\sup_{0\le s\le T}E[f_k(Z(s))]<M_0$ for all k by Condition 1.1.

Hence for every $\eta>0$, let $\Gamma_{\eta,t}=B(0,\frac{1}{\eta}\sum_{k=0}^{\infty}\frac{2M_0}{\lambda_k\eta_k}+\frac{1}{\lambda_k}\|g_k'\cdot F\|_\infty M_0t)$. Then,

$$\lim_{n\to\infty}\mathbf{P}\{\sum_{k=0}^{\infty}\int_0^t g_k(X_n(s))dZ_n^k(s)\in\Gamma_{\eta,t}\}\ge 1-\eta$$

Also,

$$
\begin{aligned}
&|\tilde{X}_n(t+u) - \tilde{X}_n(t)| \\
&\leq \sum_{k=0}^{\infty} |g_k(X_n(t+u))Z_n^k(t+u) - g_k(X_n(t))Z_n^k(t)| \\
&+ |\int_t^{t+u} g_k'(X_n(s))Z_n^k(s)dX_n(s)| \\
&\leq \sum_{k=0}^{\infty} \frac{2M_0}{n\lambda_k} + \|g_k' F\|_\infty \frac{M_0 u}{\lambda_k}
\end{aligned}
$$

Let

$$\gamma_n(\delta) = \sum_{k=0}^{\infty} \frac{2M_0}{n\lambda_k} + \|g_k' F\|_\infty \frac{M_0 \delta}{\lambda_k}$$

Then, for $0 \leq t \leq T$, $0 \leq u \leq \delta$

$$E[|\tilde{X}_n(t+u) - \tilde{X}_n(t)|\,|\mathcal{F}_t] \leq E[\gamma_n(\delta)\,|\mathcal{F}_t]$$

and

$$\lim_{\delta\to 0} \limsup_n E[\gamma_n(\delta)] = \lim_{\delta \to 0} \sum_{k=0}^{\infty} \|g_k' F\|_\infty \frac{M_0\delta}{\lambda_k} = 0$$

Hence $\{\tilde{X}_n = \sum_{k=0}^{\infty} \int_0^{\cdot} g_k(X_n(s))dZ_n^k(s)\}$ is relatively compact. So far, we have seen that $\{X_n(t)\}$ is relatively compact.

Since for all n

$$\sum_{k=0}^{\infty} \int_0^t |g_k(X_n(s))dZ_n^k(s)| < \infty$$

according to (2.7), the limit of the series,

$$\sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dZ_n^k(s)$$

is the same as the sum of limits of each term. From (2.7)

$$\begin{aligned}
&\sum_{k=0}^{\infty}\int_0^t g_k(X_n(s))dZ_n^k(s)\\
(2.8)\qquad &=\sum_{k=0}^{\infty}(g_k(X_n(t))Z_n^k(t)-g_k(X_n(0))Z_n^k(0))\\
&\quad-\sum_{k=0}^{\infty}\int_0^t g_k'(X_n(s))Z_n^k(s)dX_n(s)
\end{aligned}$$

It is obvious as $n\to\infty$,

$$g_k(X_n(t))Z_n^k(t)-g_k(X(0))Z_n^k(s)\to 0.$$

□

The following lemma is to get a limit of the second series of (2.8).

LEMMA 2.3. *Let X be a limit point of X_n. Then along the appropriate subsequence*

$$\begin{aligned}
&\int_0^t g_k'(X_n(s))Z_n^k(s)dX_n(s)\\
&\Rightarrow \int_0^t\int_E \frac{1}{\lambda_k}g_k'(X(s))f_k(z)F(X(s),z)\mu(dz)ds
\end{aligned}$$

Proof. For $B\subset E$, let

$$\begin{aligned}
\Gamma_n([0,t]\times B)&\equiv\int_0^t I_B(Z(n^2s))ds\\
\Gamma([0,t]\times B)&\equiv\int_0^t I_B(Z(s))d\mu(z)\cdot t
\end{aligned}$$

By the ergodicity of $Z(s)$, $\Gamma_n\to\Gamma$ a.s. as $n\to\infty$. Let

$$\begin{aligned}
U_n(t)&=\int_0^t g_k'(X_n(s))\frac{f_k(Z(n^2s))}{n\lambda_k}nF(X_n(s),Z(n^2s))ds\\
&=\int_0^t\int_E\frac{1}{\lambda_k}g_k'(X_n(s))f_k(z)F(X_n(s),z)\Gamma_n(ds\times dz),
\end{aligned}$$

and let $X(t)$ be a weak limit of $X_n(t)$. Since X is continuous on $[0,t]$ and $(X_n, \Gamma_n) \Rightarrow (X, \Gamma)$, we get $U_n(t) \Rightarrow U(t)$, where

$$U(t) = \int_0^t \int_E \frac{1}{\lambda_k} g_k'(X(s)) f_k(z) F(X(s), z) d\mu(z) ds.$$

□

Finally, we shall show the limit of $(*)$ in (2.2)

LEMMA 2.4. *Let X be a limit point of X_n. Then along the approproate subsequence*

$$\sum_{k=0}^{\infty} \int_0^{\cdot} g_k(X_n(s)) dY_n^k(s) \Rightarrow \sum_{k=0}^{\infty} \int_0^{\cdot} g_k(X(s)) dY^k(s)$$

Proof. Let X be a limit of X_n and we have in Lemma 1.2

$$Y_n^k \Rightarrow Y^k \qquad \text{for } k = 1, 2, \cdots$$

Applying the Skorohod representation theorem again, we can assume that
$(X_n, Y_n) \to (X, Y)$ a.s. Note that we have

$$E[\sum_{k=0}^{\infty} | \int_0^t g_k(X_n(s)) dY_n^k(s)|] \le \sum_{k=0}^{\infty} \frac{1}{\eta_k} (\frac{1}{\lambda_k} M_0 t)^{\frac{1}{2}} < \infty,$$

uniformly in n, so $\sum_{k=0}^{\infty} \int_0^t g_k(X_n(s)) dY_n^k(s)$ converges with probability 1 uniformly in n, by the generalized Borel-Cantelli lemma. Since for any ϵ, we can choose N s.t.

$$E[| \sum_{k=N}^{\infty} \int_0^t g_k(X_n(s)) dY_n^k(s) - \sum_{k=N}^{\infty} \int_0^t g_k(X(s)) dY^k(s)|] \le \epsilon^2,$$

we have

$$\mathbf{P}(\sum_{k=N}^{\infty} | \int_0^t g_k(X_n(s)) dY_n^k(s) - \sum_{k=N}^{\infty} \int_0^t g_k(X(s)) dY^k(s)| \ge \epsilon) \le \epsilon$$

Now, for each k, $(*)$ implies that Y_n^k satisfies the Condition 2.2(1) [4] and hence, $(X_n, Y_n^k) \Rightarrow (X, Y^k)$ implies that

$$\sum_{k=0}^{N} \int_0^t g_k(X_n(s))dY_n^k(s) \Rightarrow \sum_{k=0}^{N} \int_0^t g_k(X(s))dY^k(s)$$

by Theorem 2.2 [4]. It impiles that

$$\sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dY_n^k(s) \Rightarrow \sum_{k=0}^{\infty} \int_0^t g_k(X(s))dY^k(s).$$

□

EXAMPLE(continued). Let $Z(s)$ be Brownian motion with state space $[0,\pi]$, which reflects at both end points. Then $A = \{(f, \frac{1}{2}f'') \,|\, f \in C^2[0,\pi], f'(0) = f'(\pi) = 0\}$ is the generator of $Z(s)$. The eigenfunctions of A are $f_k(x) = 2\cos k(x), k = 1, 2, \cdots$ and the eigenvalues $\lambda_k = -k^2$. Then our $\{f_k(x)\}$ and $\{\lambda_k\}$ satisfys the assumptions. Let $F : R \times [-\pi,\pi] \to R$ be a bounded function. Assume for each fixed $x \in R$, $F(x,\cdot) \in C^1[0,\pi]$ and is even function. Since $\int_0^\pi F(x,z)dz = 0$, $F(x,z)$ can be expanded

$$F(x,z) = \sum_{k=0}^{\infty} \sqrt{\frac{2}{\pi}} g_k(x) \cos kz, \quad g_k(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x,z) \sqrt{\frac{2}{\pi}} \cos kz dz$$

And the Feller semigroup $\{S(t)\}$ on $C^1(R)$ generated by A has a unique stationary distribution μ, which is $\frac{1}{\pi}dx$,dx is the Lebesgue measure.

Consider an equation,

$$dX_n(t) = nF(X_n(t), Z(n^2t))dt$$

Then $\{X_n(t)\}$ is relatively compact and any limit point $X(t)$ satisfies

$$\begin{aligned} X(t) =& \sum_{k=0}^{\infty} \int_0^t g_k(X(s))dY^k(s) \\ & - \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \int_0^t \int_0^\pi \frac{1}{k^2} g_k'(X(s)) g_j(X(s)) f_k(x) f_j(x) dx ds \\ =& \sum_{k=0}^{\infty} \int_0^t g_k(X(s))dY_k(s) - \sum_{k=0}^{\infty} \frac{1}{k^2} \int_0^t g_k'(X(s)) g_k(X(s)) ds \end{aligned}$$

since $\frac{1}{\pi}\int_0^\pi \cos kx \cos jx dx = \delta_{k,j}$ Here, $Y_k, k = 1, 2, \cdots$ are Brownian motions with covariance C_{kj},

$$\begin{aligned} C_{kj} &= 0 \qquad \text{if } k \neq j \\ &= \frac{2}{k^2} \qquad \text{if } k = j \end{aligned}$$

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