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CONNECTEDNESS IM KLEINEN AND COMPONENTS IN $C(X)$

Joo Ran Moon, Kul Hur and Choon Jai Rhee

0. Introduction

In 1970's Goodykoontz gave characterizations of connectedness im kleinen and locally arcwise connectedness of 2^X only at singleton set $\{x\} \in 2^X$ [5,6,7]. In [7], we gave necessary conditions for $C(X)$ to be arcwise connected im kleinen at any point $A \in C(X)$.

In this paper, we introduce further necessary conditions for $C(X)$ to be arcwise connected im kleinen at a point $A \in C(X)$.

Moreover we include the following result : If the hyperspace $C(X)$ is connected im kleinen at $A \in \mathcal{A}(a)$, where $\mathcal{A}(a)$ is the admissible fiber at a, then $C(X)$ is connected im kleinen at each subcontinuum B which contains A. In case X has property k and $C(X)$ is not connected im kleinen at a point $A \in C(X)$, $C(X)$ is not connected im kleinen at each subcontinuum of A.

In [7], we gave a characterization for a point $A \in C(X)$ at which $C(X)$ is not connected im kleinen and then we distinguished between the set N of all points at which X is not connected im kleinen and the set $\mathcal{N}$ of all points at which $C(X)$ is not connected im kleinen. Furthermore, we proceed to find a one-to-one relation between the components of N and $\mathcal{N}$.

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1. Preliminary

Let 2^X be the collection of all nonempty closed subsets of X and let $C(X)$ be the collection of all subcontinua of X.

Let $A \in 2^X$ and $\epsilon > 0$. Let $N(\epsilon, A)$ be the set of all $x \in X$ such that $d(x, a) < \epsilon$ for some $a \in A$. $N(\epsilon, A)$ is called the *ϵ–neighborhood* of A. For convenience, we write $N(\epsilon, \{x\}) = N(\epsilon, x)$.

For $A, B \in 2^X$, let $H(A, B) = \inf\{\epsilon > 0 : A \subset N(\epsilon, B)$ and $B \subset N(\epsilon, A)\}$. Then H is called the *Hausdorff metric* for 2^X, and we call $(2^X, H)$ and $(C(X), H)$ the *hyperspaces* of closed sets and subcontinua respectively. Also the Hausdorff metric for 2^{2^X} is denoted by H^2.

There are two special continuous maps:

(i) [8, p.513.] $2^* : 2^X \to 2^{2^X}$ is a map defined by $2^*(A) = 2^A$, $\forall\, A \in 2^X$.

(ii) [8, p.100.] The union map $\sigma : 2^{2^X} \to 2^X$ is defined by $\sigma(\mathcal{A}) = \cup\mathcal{A}$, $\forall \mathcal{A} \in 2^{2^X}$. Furthermore σ is nonexpansive, i.e, $H(\sigma(A), \sigma(B)) \leq H^2(\mathcal{A}, \mathcal{B})$ for $\mathcal{A}, \mathcal{B} \in 2^{2^X}$.

Let D be a subset of X and let $C(D) = \{A \in C(X) : A \subset D\}$ and $2^D = \{A \in 2^X : A \subset D\}$. Let $A \in C(X)$ and $C \subset X$. Let $H(A, C(D)) = \inf\{H(A, B) : B \in C(D)\}$.

2. Connectedness im kleinen and not connectedness im kleinen

DEFINITION 2.1. Let $x \in X$. The space X is said to be *connected (arcwise connected) im kleinen* at x if for each neighborhood U of x, there is a neighborhood V of x lying in U such that if $y \in V$ then there is a connected (arcwise connected) subset of U containing both x and y.

DEFINITION 2.2. Let X be a metric continuum. For $x \in X$, let $T(x) = \{A \in C(X) : x \in A\}$. $T(x)$ is called *the total fiber* of X at x. We say that a point $a \in X$ is a *k–point* of X provided that for each $\epsilon > 0$ there is a $\delta > 0$ if $A \in T(a)$ and b is in the δ–neighborhood of a, then there is an element $B \in T(b)$ such that $H(A, B) < \epsilon$. If each point of X is a k–point, then we say that X *has property k*.

DEFINITION 2.3. Let X be a metric continuum. Let $T(x)$ be the total fiber at x. An element $A \in T(x)$ is said to be *admissible* at x in X if for each $\epsilon > 0$, there is $\delta > 0$ such that each point y in the δ-neighborhood of x has an element $B \in T(y)$ such that $H(A, B) < \epsilon$. The collection $\mathcal{A}(x)$ of all $A \in T(x)$ which are admissible at x in X is called the *admissible fiber* of X at x.

Let N be the set of all points $x \in X$ at which X is not connected im kleinen. We call it the $\mathcal{N}$*-set* of X. Then N is the union of nondegenerate connected sets [9, (5.12)].

PROPOSITION 2.4.[7]. *Let X be a metric continuum having nonempty $\mathcal{N}$-set N of X. Let N_α be a component of N. Then $\overline{N_\alpha} \in \mathcal{K}$, i.e., $C(X)$ is connected im kleinen at $\overline{N_\alpha}$, where $\mathcal{K}$ is the set of all $A \in C(X)$ at which $C(X)$ is connected im kleinen.*

PROPOSITION 2.5.[7]. *X is connected im kleinen at x if and only if $C(X)$ is connected im kleinen at $\{x\}$.*

For $A \in C(X)$ and $D \subset X$, let $H(A, C(D)) = inf\{H(A, B) : B \in C(D)\}$.

THEOREM 2.6. *Let $A \in C(X)$. If, for each open set U in X containing A and the component C_A of U containing A, there exists $\epsilon > 0$ such that $H(A, C(C_\alpha)) > \epsilon$ for each component C_α of U other than C_A, then $C(X)$ is arcwise connected im kleinen at A.*

Proof. Let $0 < \delta < \epsilon$ and $\mathcal{U}$ be the δ-neighborhood of A in $C(X)$. Let $U = N(\frac{\delta}{2}, A)$ and C_A be the component of U containing A. Let $\mathcal{V}$ be the $\frac{\delta}{2}$-neighborhood of A and $B \in \mathcal{V}$. Then $H(A, B) < \frac{\delta}{2}$ implies that $B \notin C(C_\alpha)$ for any $C_\alpha \neq C_A$. But $B \subset N(\frac{\delta}{2}, A)$ so that it must be contained C_A. Now let β be an order arc in $C(X)$ from B to $\overline{C_A}$ and α be an order arc in $C(X)$ from A to $\overline{C_A}$. Then it is easy to see that $\alpha \cup \beta \subset \mathcal{U}$. □

For $A, B \in C(X)$ such that $A \subset B$, let $C(A, B) = \{D \in C(X) : A \subset D \subset B\}$.

THEOREM 2.7. *Let X be a metric continuum. Suppose $C(X)$ is connected im kleinen at $A \in \mathcal{A}(a)$. Then $C(X)$ is connected im kleinen at each $B \in C(A, X)$.*

Proof. Let U be an open set containing B with the component C_B containing B. Let $\{B_n\}$ be a sequence of subcontinua converging to B. We may assume that $B_n \subset U$ for all n. Let V be an ϵ−neighborhood of A such that $\overline{V} \subset U$. Let C_A be the component of V which contains A and let $\{b_n\}, b_n \in B_n$, such that $\{b_n\}$ converges to a. Since $A \in \mathcal{A}(a)$ there exists a sequence $\{A_n\}$ of subcontinua, $b_n \in A_n$ for each n, which converges to A and there is an integer N such that $A_n \subset C_A$ for all $n \geq N$ by [3, Theorem 2]. Since, for each $n \geq N, b_n \in C_A \subset C_B, C_B$ being the component of U, we have that $B_n \subset C_B$ for each $n \geq N$. Therefore by [3, Therem 2] $C(X)$ is connected im kleinen at B. □

From Proposition 2.1 and Theorem 2.7, we can obtain the following resuit:

COROLLARY 2.8. *A metric continuum X is connected im kleinen at $x \in X$ if and only if $C(X)$ is connected im kleinen at each point of $C(\{x\}, X)$.*

THEOREM 2.9. *Let X be a metric continuum with property k. Suppose $C(X)$ is not connected im kleinen at $A \in C(X)$. Then $C(X)$ is not connected im kleinen at each point $B \in C(A)$.*

Proof. Let $B \in C(A)$ and $a \in B$ and let U and V be open sets containing A and B respectively such that $V \subset U$. Let C_A be the component of U containing A and let C_B be the component of V containing B. Then $C_B \subset C_A$. Since $C(X)$ is not connected im kleinen at A, there is a sequence $\{A_n\}$ of subcontinua, each A_n is contained in a distinct component of U disjoint from C_A, which converges to A. Let $\{a_n\}, a_n \in A_n$, be a sequence which converges to a. Since $B \in \mathcal{A}(a)$, there is a sequence $\{B_n\}, a_n \in B_n \subset V$, of subcontinua which converges to B. Since the components of V containing a_n are distinct, we have that B_n are not contained in C_B. Hence $C(X)$ is not connected im kleinen at B. □

3. Relationship between components in X and components in $C(X)$

Let X be a metric continuum. Let N be the set of all $x \in X$ at which X is not connected im kleinen, and let $\mathcal{N}$ be the set of all $A \in C(X)$ at which $C(X)$ is not connected im kleinen, let $\mathcal{K}$ be the set of all $A \in C(X)$ at which $C(X)$ is connected im kleinen, and finally let $\mathcal{L}$ be the set of all $A \in C(N)$ at which $C(X)$ is connected im kleinen. We may note here that if $N \neq \emptyset$ then each of the components of N is nondegenerate [9, 5.13]. Let us note that $C(X) = \mathcal{N} \cup \mathcal{K}$ and $\mathcal{L} \subset \mathcal{K}$. At the end of this section, we give an example of a space with $\mathcal{L} \neq \emptyset$.

PROPOSITION 3.1.[7]. *Suppose $C(X)$ is not connected im kleinen at $A \in C(X)$ (i.e., $A \in C(X) \backslash \mathcal{K}$). Then $A \subset N$. If $A \in \overline{N}$ then $A \in C(\overline{N})$.*

PROPOSITION 3.2.[7]. *If $N \neq \emptyset$, then $C(N) = \mathcal{N} \cup \mathcal{L}$ and $\mathcal{L} \cap \mathcal{N} = \emptyset$.*

PROPOSITION 3.3.[7]. *Let $\mathcal{N}_f$ be a component of $\mathcal{N}$. Then $\cup \mathcal{N}_f \subset N_f$, where N_f is a component of N.*

THEOREM 3.4. *Let N_α be a component of N of a metric continuum X. Then $C(N_\alpha) \backslash \mathcal{L}$ is a component of $\mathcal{N}$ of $C(X)$.*

Proof. Since N_α is connected, the set $N_\alpha{}^* = \{\{x\} : x \in N_\alpha\}$ is connected subset of $\mathcal{N}$. Let $\mathcal{N}_\alpha$ be a component of $\mathcal{N}$ such that $\cup \mathcal{N}_\alpha \subset N_\alpha$. We show that $C(N_\alpha) \backslash \mathcal{L} = \mathcal{N}_\alpha$. Clearly $\mathcal{N}_\alpha \subset C(N_\alpha) \backslash \mathcal{L}$. Let $A \in C(N_\alpha) \backslash \mathcal{L}$. Since $A \in \mathcal{N}$, let U be an open set containing A and $\{C_n\}$ be a sequence of distinct components of U and let $\{A_n\}$ be a sequence of subcontinua of X such that $A_n \subset C_n$ and $LtA_n = A$ by Theorem 3.3 [7]. Then $A \in LtC(A_n)$ and $A^* = \{\{x\} : x \in A\} \subset LtC(A_n)$. Hence $LsC(A_n)$ is connected. Furthermore $LsC(A_n) \subset \mathcal{N}$ by Theorem 3.3 [7]. Since $LsC(A_n) \cap N_\alpha^* \neq \emptyset$, we conclude that $C(N_\alpha) \backslash \mathcal{L}$ is connected. Therefore $C(N_\alpha) \backslash \mathcal{L}$ is contained in a component of $\mathcal{N}$. Hence $\mathcal{N}_\alpha = C(N_\alpha) \backslash \mathcal{L}$. □

COROLLARY 3.5. *Let N_α be a component of N and let $\mathcal{N}_\alpha$ be the component of $\mathcal{N}$ containing an element $C(N_\alpha)$. Then $C(N_\alpha) \backslash \mathcal{L}' = \mathcal{N}_\alpha$, where $\mathcal{L}'$ is the set of all $B \in C(N_\alpha)$ at which $C(X)$ is connected im kleinen.*

EXAMPLE. An example of the space for which $\mathcal{L} \neq \emptyset$ is given in the plane. Let $p = (0,0)$ $q = (0,1)$ $r = (-1,-2)$ and $s = (1,-2)$. For each positive interger n, let $p_n = (\frac{-1}{n}, 1)$ $q_n = (\frac{1+1}{n}, -2)$ $t_n = (\frac{-1}{n}, 0)$ and $s_n = (\frac{1}{n}, 0)$. Let pq, pr and ps be the segment joining respectively p and q, p and r,p and s and for each n, let p_nt_n, t_nr, qs_n and s_nq_n be segment joining respectively p_n and t_n, and t_n and r, q and s_n and s_n and q_n. Let $X = pq \cup pr \cup ps \cup (\cup_{n=1}^{\infty}(p_nt_n \cup t_nr)) \cup (\cup_{n=1}^{\infty}(qs_n \cup s_nq_n))$. Then the space X is a continuum. The set N of all point x at which X is not connected im kleinen is $pq \cup ps \cup (pr \backslash \{r\})$. Let $p' = (\frac{1}{2}, -1)$ and let pp' be the segment joining p and p'. Let $A = pq \cup pp'$. Then $A \in C(N)$. Hence by applying Theorem 2 in [3], we see that $C(X)$ is connected im kleinen at A.

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JOO RAN MOON

DEPARTMENT OF MATHEMATICS, WONKWANG UNIVERSITY, IKSAN 570-749, KOREA

KUL HUR
DEPARTMENT OF MATHEMATICS, WONKWANG UNIVERSITY, IKSAN 570-749, KOREA
E-mail: kulhur@wonnms.wonkwang.ac.kr

CHOON JAI RHEE
DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETRIOT, MICHIGAN 48202, U.S.A.
E-mail: rhee@math.wayne.edu