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ON REAL HYPERSURFACES OF A COMPLEX HYPERBOLIC SPACE

EUN-HEE KANG AND U-HANG KI

0. Introduction

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature c. As is well known, complete and simply connected complex space forms are a complex projective space P_nC, a complex Euclidean space C_n or a complex hyperbolic space H_nC according as $c > 0, c = 0$ or $c < 0$.

Let M be a real hypersurfaces of $M_n(c), c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kaehlerian metric and complex structure J of $M_n(c)$. The structure vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal C and $\alpha = \eta(A\xi)$. We denote by ∇ and S, the Levi-Civita connection with respect to the Riemannian metric tensor g and the Ricci tensor of type $(1, 1)$ on M respectively. Takagi ([12]) classified all homogeneous real hypersurfaces of P_nC as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan ([3]) and Kimura ([6]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds. Also Berndt ([1],[2]) showed that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space H_nC are realized as the tubes of constant radius over certain submanifolds when the structure vector ξ is principal. Nowadays in H_nC they are said to be of type A_0, A_1, A_2 and B.

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Under certain conditions for the Ricci tensor of M, real hypersurfaces of a complex space form were studied by many geometers [4], [5], [7], [8], [9], [10] etc. In the present paper, we study real hypersurfaces of a complex space form $M_n(c), c \neq 0$ which satisfy $L_\xi S = 0$, where L_ξ is the Lie derivative in the direction of the structure vector ξ. It is remarkable that the condition $L_\xi S = 0$ in a real hypersurface of $M_n(c), c \neq 0$ implies the following equation :

$$\|S\phi - \phi S\|^2 + \frac{3}{2}c\|\nabla_\xi \xi\|^2 = 0$$

(See (2.4) in section 2 or [8]). Furthermore, Kimura and Maeda ([8]) proved a local classification theorem for real hypersurfaces of $P_n C$ which satisfy $L_\xi S = 0$. On the other hand, for real hypersurfaces of $H_n C$ we proved ([4]) that

THEOREM A. *Let M be a $(2n-1)$-dimensional real hypersurface of $H_n C, n \geq 3$. If the structure vector ξ is principal and M satisfies $L_\xi S = 0$, then M is congruent to one of the following spaces :*

(A_0) *a horosphere in $H_n C$, i.e., a Montiel tube,*

(A_1) *a tube of a totally geodesic hyperplane $H_k C (k = 0$ or $n-1)$,*

(A_2) *a tube of a totally geodesic $H_k C (1 \leq k \leq n-2)$.*

The main purpose of the present paper is to improve the above theorem. More specifically we prove

THEOREM. *Let M be a real hypersurface of $H_n C$. If it satisfies $L_\xi S = 0$ and $S\xi = \sigma\xi$ for some function σ on M, then ξ is principal.*

1. Preliminaries

Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by $\bar\nabla$ and ∇ the Riemannian connection in $M_n(c)$ and in M respectively. Then by the Gauss formula, we have the relationship between $\bar\nabla$ and ∇ : For any vector fields X and Y on M

$$\bar\nabla_X Y = \nabla_X Y + g(AX, Y)C,$$

where g is the Riemannian metric tensor of M induced from that of $M_n(c)$ and A denotes the shape operator with respect to C of M in $M_n(c)$. Furthermore, we have another equation which is called the Weingarten formula :

$$\bar{\nabla}_X C = -AX.$$

For any local vector field X on a neighborhood of x in M, the transformations of X and C under the complex structure J in $M_n(c)$ can be given by

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, where η and ξ denote a 1-form and a vector field on a neighborhood of x in M respectively. Then it is seen that $g(\xi, X) = \eta(X)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation and $\otimes$ denotes the tensor product.
Furthermore the covariant derivatives of the structure tensors are given by

$$(1.1) \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX.$$

Since the ambient space is of constant holomorphic sectional curvature c, equations of the Gauss and Codazzi are respectively given as follows ;

(1.2)
$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}/4 + g(AY,Z)AX - g(AX,Z)AY,$$

$$(1.3) \quad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is a tensor of type $(0,2)$ given by $S'(X,Y) = tr\{Z \to R(Z,X)Y\}$. Also it may be regarded as the tensor of type (1,1) and denoted by $S : TM \to TM$; it satisfies $S'(X,Y) = g(SX,Y)$. From (1.3) we see that the Ricci tensor S of M is given by

$$S = c\{(2n+1)I - 3\eta \otimes \xi\}/4 + hA - A^2, \tag{1.4}$$

where we have put $h = trA$. Moreover, using (1.2) we get

(1.5)
$$\begin{aligned}(\nabla_X S)Y =& -3c\{g(\phi AX,Y)\xi + \eta(Y)\phi AX\}/4 \\ &+ dh(X)AY + (hI - A)(\nabla_X A)Y - (\nabla_X A)AY,\end{aligned}$$

where d denotes the exterior differential.

In what follows, to write our formulas in convention forms, we denote $\alpha = g(A\xi,\xi)$, $\beta = g(A^2\xi,\xi)$ and ∇f by the gradient vector field of a function f . If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector ξ. Because of properties of the almost contact metric structure and the second equation of (1.1), we can get

$$\phi U = -A\xi + \alpha\xi, \tag{1.6}$$

which shows that $g(U,U) = \beta - \alpha^2$. By the definition of U and the second equation of (1.1), we easily see that

$$g(\nabla_X \xi, U) = g(A^2\xi, X) - \alpha g(A\xi, X). \tag{1.7}$$

On the other hand, differentiating (1.6) covariantly and making use of (1.1), we find

(1.8)
$$\begin{aligned}\eta(X)g(AU,Y) + g(\phi X, \nabla_Y U) =& g((\nabla_Y A)X,\xi) - g(A\phi AX,Y) \\ &- \eta(X)g(\nabla\alpha,Y) + \alpha g(A\phi X,Y),\end{aligned}$$

which enable us to obtain

$$g((\nabla_X A)\xi,\xi) = 2g(AX,U) + g(\nabla\alpha,X). \tag{1.9}$$

By the definition of U, (1.1), (1.8) and (1.9) it is verified that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{1.10}$$

2. Real hypersurfaces of H_nC satisfying $L_\xi S = 0$

In the sequel we assume that the Ricci tensor S satisfies

$$L_\xi S = 0,$$

where L_ξ denotes the Lie derivative with respect to the structure vector ξ. By definition we have

$$L_\xi S(X) = L_\xi(SX) - SL_\xi X$$

for any vector field X on M and hence using (1.1) we obtain

$$\nabla_\xi S = \phi AS - S\phi A. \tag{2.1}$$

Thus it follows that we get

$$(A\phi - \phi A)S = S(A\phi - \phi A). \tag{2.2}$$

From (1.4) we have

$$S\phi - \phi S = h(A\phi - \phi A) - A^2\phi + \phi A^2. \tag{2.3}$$

Using the last two equations, it is seen that

$$(A\phi - \phi A)(S\phi - \phi S) = 0.$$

Thus, by applying $A\phi$ to (2.2), then we have

$$\|S\phi - \phi S\|^2 + \frac{3}{2}c\|\nabla_\xi\xi\|^2 = 0. \tag{2.4}$$

Therefore, if $c > 0$, then we have $S\phi = \phi S$ and $A\xi = \alpha\xi$ (cf [8]).

Let M be a real hypersurface of H_nC of constant holomorphic sectional curvature -4. Now, suppose that

$$S\xi = \sigma\xi \tag{2.5}$$

for some function σ. Then by (1.4) we have

$$A^2\xi = hA\xi + (\beta - h\alpha)\xi, \tag{2.6}$$

where we put

$$\beta - h\alpha = -\sigma - 2(n-1). \tag{2.7}$$

Differentiating (2.5) covariantly along M, we find

$$(\nabla_X S)\xi + S\nabla_X\xi = (X\sigma)\xi + \sigma\nabla_X\xi.$$

Since we can, using (2.1) and (2.5), see that $(\nabla_\xi S)\xi = 0$, if we replace X by ξ, then we obtain

$$SU = d\sigma(\xi)\xi + \sigma U.$$

On the other hand, applying ξ to the both sides of (2.2), and making use of the second equation of (1.1) and (2.5), we obtain $SU = \sigma U$ and hence

$$d\sigma(\xi) = 0, i.e., d(\beta - h\alpha)(\xi) = 0 \tag{2.8}$$

Thus it follows that we have

$$hAU - A^2U = (h\alpha - \beta + 3)U, \tag{2.9}$$

where we have used (1.4) and (2.7).

We put $A\xi = \alpha\xi + \mu W$, where W is a unit vector field orthogonal to ξ. Then from (1.6) we see that $U = \mu\phi W$, and W is also orthogonal to U. We assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector and we put $\Omega = \{p \in M | \mu(p) \neq 0\}$. Then Ω is an open subset of M and from now on we discuss our arguments on Ω. Making use of (2.6), we find

$$\mu AW = (h - \alpha)A\xi + (\beta - \alpha h)\xi \tag{2.10}$$

and hence

$$A^2W - hAW = (\beta - \alpha h)W$$

because of $\mu \neq 0$. From this and (1.4), it follows that we get

$$SW = -\{2n + 1 + \beta - \alpha h\}W. \tag{2.11}$$

If we apply W to the both sides of (2.2) and take account of (1.1), (2.10) and (2.11), then we obtain

$$(2n + 1 + \beta - \alpha h)\{AU - (h - \alpha)U\} = (h - \alpha)SU - SAU,$$

which together with (1.4) and (2.9) implies that

$$AU = (h - \alpha)U. \tag{2.12}$$

Accordingly (2.9) means that

$$g(U, U) = \beta - \alpha^2 = 3. \tag{2.13}$$

Therefore, (2.6) turns out to be

$$A^2\xi = hA\xi + (\alpha^2 - h\alpha + 3)\xi.$$

Differentiating this covariantly along Ω and using the second equation of (1.1), we find

(2.14)

$$\begin{aligned}&(\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - hA\phi AX\\ &= dh(X)A\xi + h(\nabla_X A)\xi + d(\alpha^2 - h\alpha)(X)\xi + (\alpha^2 - h\alpha + 3)\phi A.\end{aligned}$$

From (2.14), using (1.9) and (2.12) we obtain

$$(\nabla_\xi A)A\xi = U + \alpha\nabla\alpha + h(h - \alpha)U.$$

Replacing X by ξ, we also have from (2.14)

$$(\nabla_\xi A)A\xi - 3\alpha(h - \alpha)U + A\nabla\alpha = dh(\xi)A\xi + h\nabla\alpha + (\alpha^2 - h\alpha + 3)U,$$

where we have used (1.3), (2.7), (2.8) and (2.12). Combining the last two equations, it follows that

$$dh(\xi)A\xi = A\nabla\alpha - (h-\alpha)\nabla\alpha + (h^2 - 3\alpha h + 2\alpha^2 - 2)U, \tag{2.15}$$

which enable us to obtain

$$h^2 - 3\alpha h + 2\alpha^2 = 2 \tag{2.16}$$

because of $g(A\xi, U) = 0$ and (2.12). Thus it is seen that

$$(2h - 3\alpha)\nabla h + (4\alpha - 3h)\nabla\alpha = 0,$$

which shows that

$$(2h - 3\alpha)dh(\xi) + (4\alpha - 3h)d\alpha(\xi) = 0.$$

On the other hand, because of (2.8) and (2.13), we obtain

$$(2\alpha - h)d\alpha(\xi) - \alpha dh(\xi) = 0.$$

From the last two equations, we have $(h-\alpha)d\alpha(\xi) = 0$ and hence $d\alpha(\xi) = 0$ by virtue of (2.16). Therefore we have

$$dh(\xi) = 0. \tag{2.17}$$

In facts, suppose that Ω_1 be the set of points at which $dh(\xi) \neq 0$ in Ω and Ω_1 is not empty. Then we have $\alpha = 0$ and consequently $A\xi = 0$ in Ω_1 because of (2.15) and (2.16). This is impossible in Ω.

3. Proof of Theorem

Let M be a real hypersurface of H_nC satisfying $L_\xi S = 0$ and $S\xi = \sigma\xi$. Then we have $SU = \sigma U$ and hence

$$g((S\phi - \phi S)U, W) = 3\mu,$$

where we have used (2.7) and (2.11). Therefore we see, using (2.4) with $c = -4$, that

$$\|S\phi - \phi S - \mu(u \otimes W + w \otimes U)\|^2 = 0,$$

where we have defined $u(X) = g(X, U)$ and $w(X) = g(X, W)$. Accordingly we have

$$(S\phi - \phi S)X = g(\phi X, U)U - g(X, U)\phi U. \tag{3.1}$$

Using (1.4), it can be rewritten as

$$(hA - A^2)\phi X - \phi(hA - A^2)X = TX, \tag{3.2}$$

where we have put

$$TX = g(\phi X, U)U - g(X, U)\phi U. \tag{3.3}$$

Because of (1.7) and (2.13), we have

$$g(\nabla_X \xi, U) = (h - \alpha)g(A\xi, X) + (\alpha^2 + 3 - h\alpha)\eta(X).$$

Thus, if we take account of (1.6), (2.12) and (3.1), then we obtain

$$S\phi A = \phi SA + (h - \alpha)T + 3\eta \otimes U.$$

Thus (2.1) turns out to be

$$\nabla_\xi S + (h - \alpha)T = 0. \tag{3.4}$$

On the other hand, by (1.5) and (2.17) we have

$$(\nabla_\xi S)X = 3u(X)\xi + \eta(X)U + h(\nabla_\xi A)X - A(\nabla_\xi A)X - (\nabla_\xi A)AX.$$

Hence (3.4) becomes

$$h(\nabla_\xi A)X - A(\nabla_\xi A)X = (\nabla_\xi A)AX - (h - \alpha)TX - 3\{\eta(X)U + u(X)\xi\}, \tag{3.5}$$

or using the Codazzi equation (1.3),

$$h(\nabla_X A)\xi - A(\nabla_X A)\xi - h\phi X + A\phi X = (\nabla_\xi A)AX - (h-\alpha)TX - 3\{\eta(X)U + u(X)\xi\}.$$

Combining this with (2.14), it follows that we obtain

$$\begin{aligned}(\nabla_X A)A\xi - (\nabla_\xi A)AX &= -A^2\phi AX + hA\phi AX + dh(X)A\xi + d(\alpha^2 - h\alpha)(X)\xi \\ &+ (\alpha^2 - h\alpha + 3)\phi AX + h\phi X - A\phi X - (h-\alpha)TX \\ &- 3\{\eta(X)U + u(X)\xi\}.\end{aligned} \tag{3.6}$$

By differentiating (3.2) covariantly along Ω and using (1.1) and (1.3), we find

$$\begin{aligned}(\nabla_X T)Y - (\nabla_Y T)X &= dh(Y)\phi AX - dh(X)\phi AY - h\{\eta(X)(Y - \eta(Y)\xi) \\ &- \eta(Y)(X - \eta(X)\xi)\} - \eta(X)\phi A\phi Y + \eta(Y)\phi A\phi X \\ &+ 2g(\phi X, Y)U + \phi(\nabla_X A)AY - \phi(\nabla_Y A)AX \\ &+ dh(X)A\phi Y + h(\nabla_X A)\phi Y - (\nabla_X A)A\phi Y - A(\nabla_X A)\phi Y \\ &- dh(Y)A\phi X - h(\nabla_Y A)\phi X + (\nabla_Y A)A\phi X + A(\nabla_Y A)\phi X \\ &+ (h\alpha - \beta)\{\eta(X)AY - \eta(Y)AX\} + \eta(Y)(hA^2X - A^3X) \\ &- \eta(X)(hA^2Y - A^3Y).\end{aligned}$$

Putting $Y = \xi$ in above equation, we obtain

$$\begin{aligned}(\nabla_X T)\xi - (\nabla_\xi T)X &= -dh(X)U + h(X - \eta(X)\xi) + hA^2X - A^3X \\ &+ (\alpha^2 + 3 - h\alpha)AX + \phi A\phi X + \phi((\nabla_X A)A\xi - (\nabla_\xi A)AX) \\ &- \{h(\nabla_\xi A)\phi X - (\nabla_\xi A)A\phi X - A(\nabla_\xi A)\phi X\},\end{aligned}$$

where we have used (2.6), (2.13) and (2.17). If we substitute (3.5) and (3.6) into the last equation, then we find

$$\begin{aligned}(\nabla_X T)\xi - (\nabla_\xi T)X &= hA^2X - A^3X - \phi A^2\phi AX + h\phi A\phi AX \\ &+ (\alpha^2 + 3 - h\alpha)\eta(AX)\xi - 2(h-\alpha)\{u(X)U + g(\phi X, U)\phi U\}\end{aligned} \tag{3.7}$$

because of (3.3). Putting $X = U$ in (3.7) and making use of (2.6) and (2.12), we get

$$(3.8) \qquad (\nabla_U T)\xi - (\nabla_\xi T)U = -3(h-\alpha)U.$$

Using the same method as that used to derive (3.8) from (3.2), we can derive from (3.3) the following :

$$(\nabla_U T)\xi - (\nabla_\xi T)U = 3(h-\alpha)U + d\alpha(U)U - 3\nabla\alpha,$$

where we have used (1.1), (1.6) and (1.10). From this and (3.8) it follows that we obtain

$$\nabla\alpha = \frac{1}{3}d\alpha(U)U + 2(h-\alpha)U,$$

which together with (2.13) gives $h = \alpha$ in Ω. Thus, by (2.16) it is contradictory. Hence we conclude that Ω is empty. It completes the proof of main theorem.

References

[1] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132–141.

[2] ———, *Real hypersurfaces with constant principal curvatures in complex space forms*, Geometry and Topology of Submanifolds II, Avignon (1988), 10–19; (1990), World Scientific.

[3] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[4] U-H. Ki and N.-G. Kim and S.-B. Lee, *On certain real hypersurfaces of a complex space form*, J. Korean Math. Soc. **29** (1992), 63–77.

[5] U-H. Ki and Y. J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama **32** (1990), 207–221.

[6] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.

[7] ———, *Some real hypersurfaces of a complex projective space*, Saitama Math. J. **5** (1987), 1–5.

[8] M. Kimura and S. Maeda, *Lie derivatives on real hypersurfaces in a complex projective space*, Czechoslovak Math. J. **45** (1995), 1229–1235.

[9] J. H. Kwon and H. Nakagawa, *Real hypersurfaces with cyclic-parallel Ricci tensor of a complex projective space*, Hokkaido Math. J. **17** (1988), 355–371.

[10] Y. J. Suh, *On real hypersurfaces of a complex space form*, Tsukuba J. Math. **14** (1990), 27–37.

[11] R. Takagi, *On homogeneous real hypersurfaces of a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

[12] ———, *Real hypersurfaces in a complex projective space with constant principal curvatures* I, II, J. Math. Soc. Japan **27** (1975), 43–53; 507–516.

[13] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser, (1983).

DEPARTMENT OF MATHEMATICS, KYUNGPOOK UNIVERSITY, TAEGU 702-701, KOREA