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UNIQUENESS OF BASES FOR ALMOST LINEAR SPACES

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O. Mayer[9] introduced an almost linear space (*als*), a generalization of a linear space. The notion of a basis for an *als* was introduced by G. Godini[3]. Later, many properties of an *als* established by a number of authors. In this paper, we prove that the cardinality of bases for an *als* is unique. All spaces involved in this paper are over the real field $\mathbb{R}$. Let us denote by $\mathbb{R}_+$ the set $\{\lambda \in \mathbb{R} : \lambda \geq 0\}$. We recall some definitions used in this paper.

An *almost linear space* (*als*) is a set X together with two mappings $s : X \times X \to X$ and $m : \mathbb{R} \times X \to X$ satisfying the conditions $(L_1)-(L_8)$ given below. For $x, y \in X$ and $\lambda \in \mathbb{R}$ we denote $s(x, y)$ by $x + y$ and $m(\lambda, x)$ by λx, when these will not lead to misunderstandings. Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$. (L_1) $x+(y+z) = (x+y)+z$; (L_2) $x+y = y+x$; (L_3) There exists an element $0 \in X$ such that $x+0 = x$ for each $x \in X$; (L_4) $1x = x$; (L_5) $\lambda(x+y) = \lambda x+\lambda y$; (L_6) $0x = 0$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda + \mu)x = \lambda x + \mu x$ for $\lambda \geq 0$, $\mu \geq 0$. We denote $-1x$ by $-x$, and $x - y$ means $x + (-y)$. For an *als* X we introduce the following two sets:

$$V_X = \{x \in X : x - x = 0\}$$

$$W_X = \{x \in X : x = -x\}.$$

V_X and W_X are almost linear subspaces of X(i.e., closed under addition and multiplication by scalars) and, in fact, V_X is a linear space. Clearly an *als* X is a linear space iff $V_X = X$ iff $W_X = \{0\}$. Note that $V_X \cap W_X = \{0\}$ and $W_X = \{x - x : x \in X\}$.

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A *norm* on an *als* X is a functional $||\cdot|| : X \to \mathbb{R}$ satisfying the conditions $(N_1)-(N_3)$ below. Let $x, y, z \in X$ and $\lambda \in \mathbb{R}$. (N_1) $||x-z|| \leq ||x-y|| + ||y-z||$; (N_2) $||\lambda x|| = |\lambda|\ ||x||$; (N_3) $||x|| = 0$ iff $x = 0$. An *als* X together with $||\cdot|| : X \to \mathbb{R}$ satisfying $(N_1) - (N_3)$ is called a *normed almost linear space* (*nals*).

A subset B of an *als* X is called a *basis* for X if for each $x \in X \setminus \{0\}$ there exist unique sets $\{b_1, b_2, ..., b_n\} \subset B$, $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset \mathbb{R} \setminus \{0\}$ (n depending on x) such that $x = \sum_{i=1}^{n} \lambda_i b_i$, where $\lambda_i > 0$ for $b_i \notin V_X$. Clearly, if B is a basis for X then $0 \notin B$.

In contrast to the case of a linear space, there is an *als* which has no basis.

EXAMPLES 1. (1) Let $A_1 = \{(\alpha, 0) : \alpha \in \mathbb{R}\}$, $A_2 = \{(0, \beta) : \beta \in \mathbb{R}_+\}$ be subsets of $\mathbb{R}^2$ and let $X = A_1 \cup A_2$. Define $s(x, y) = x + y$ if both $x, y \in A_i$, $i = 1, 2$, and $s(x, y) = s(y, x) = y$ if $x \in A_1$, $y \in A_2 \setminus \{(0, 0)\}$. And define $m(\lambda, x) = \lambda x$ if $x \in A_1$, $m(\lambda, y) = |\lambda| y$ if $y \in A_2$. Let $0 \in X$ be the zero element $(0, 0) \in \mathbb{R}^2$. Then X is an *als*. We have $V_X = A_1$ and $W_X = A_2$. Also, X has no basis.

(2) Let $X = \{[a, b] \subset \mathbb{R} : a \leq b\}$. Define $s(A, B) = \{a+b : a \in A,\ b \in B\}$ and $m(\lambda, A) = \{\lambda a : a \in A\}$ for $A, B \in X$, $\lambda \in \mathbb{R}$. The element $0 \in X$ is $\{0\} \subset \mathbb{R}$. Then X is an *als*. We have $V_X = \{\{a\} \in X : a \in \mathbb{R}\}$ and $W_X = \{[-a, a] \in X : a \geq 0\}$. And $B = \{[-1, 1], \{1\}\}$ is a basis for X. $B_1 = \{\{1\}\}$ is a basis for V_X. Also, $Y = \{[a, b] \in X : a \leq 0,\ b \geq 0\}$ is an almost linear subspace of X. And $B_2 = \{[-1, 0], [0, 1]\}$ is a basis for Y.

(3) Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ in $\mathbb{R}^2$, and let $X = \{\alpha e_1 + \beta e_2 : \alpha \in \mathbb{R},\ \beta \in \mathbb{R}_+\}$. Define $s(x, y) = x + y$ for $x, y \in X$, and $m(\lambda, x) = (\lambda\alpha)e_1 + (|\lambda|\beta)e_2$ for $x = \alpha e_1 + \beta e_2 \in X$, $\lambda \in \mathbb{R}$. The element zero of X is $(0, 0) \in \mathbb{R}^2$. Then X is an *als* and $\{e_1, e_2\}$ is a basis for X. Let $Y = \{\alpha e_1 + \beta e_2 \in X : \alpha, \beta \in \mathbb{R},\ \beta > 0\} \cup \{(0, 0)\}$. Then Y is an almost linear subspace of X. But Y has no basis.

Now, we give some propositions needed in the sequel.

PROPOSITION 2 ([3]). *Let X be an als with a basis B. Then,*

(a) *The relations $x + y = x + z$, $x, y, z \in X$ imply that $y = z$.*

(b) *For each* $x \in X \setminus V_X$, *there exist unique* $b_1, b_2, ..., b_n \in B \setminus V_X$, $v \in V_X$, *and* $\lambda_1, \lambda_2, ..., \lambda_n > 0$ *such that* $x = \sum_{i=1}^{n} \lambda_i b_i + v$.
(c) *There exists a basis* B' *of* X *with the property that for each* $b' \in B' \setminus V_X$ *we have* $-b' \in B' \setminus V_X$. *Moreover* $\text{card}(B \setminus V_X) = \text{card}(B' \setminus V_X)$. *We shall call such a basis* B' *a* **symmetric** *basis.*
(d) *For* $x, y \in X$, $x + y \in V_X$ *implies* $x, y \in V_X$.
(e) $B \cap V_X$ *is a basis for* V_X.
(f) *The relations* $w_1 + v_1 = w_2 + v_2$, $w_i \in W_X$, $v_i \in V_X$, $i = 1, 2$ *imply that* $w_1 = w_2$ *and* $v_1 = v_2$.

An almost linear subspace Y of an *als* X with a basis does not have a basis in general (see, Example 1(3)). But we have the following:

PROPOSITION 3. *If an als* X *has a basis, then* W_X *has a basis.*

Proof. Let B be a symmetric basis for X. Let $B_1 = \{b - b : b \in B \setminus V_X\} \subset W_X$. We show that B_1 is a basis for W_X. Let $w \in W_X \setminus \{0\}$. By Proposition 2(b), $w = \sum_{i=1}^{n} \lambda_i b_i + v$, where $b_i \in B \setminus V_X$, $b_i \neq b_j$ for $i \neq j$, $\lambda_i > 0$, $1 \leq i \leq n$, $v \in V_X$. Then $-w = \sum_{i=1}^{n} \lambda_i(-b_i) - v$ and so $w = (1/2)(w - w) = \sum_{i=1}^{n} (\lambda_i/2)(b_i - b_i)$. To show the uniqueness of this representation, suppose

$$w = \sum_{i=1}^{k} \lambda_i (b_i - b_i) = \sum_{i=1}^{k} \mu_i (b_i - b_i),$$

where $b_i \in B \setminus V_X$, $b_i - b_i \neq b_j - b_j$ for $i \neq j$, and $\lambda_i, \mu_i \geq 0$, $1 \leq i \leq k$. Then $b_i \neq \pm b_j$ for $i \neq j$, and since for each $b \in B \setminus V_X$, $-b \in B \setminus V_X$ we must have $\lambda_i = \mu_i$, $1 \leq i \leq k$. □

The converse to Proposition 3 is not true. Indeed, in Example 1(3) $\{e_2\}$ is a basis for W_Y. But Y has no basis. In general, we have the following result.

PROPOSITION 4. *Let* X *be an als with a basis. If* W_X *has a basis, then* $W_X + V_X$ *has a basis.*

Proof. Let B_1 be a basis for W_X and B_2 a basis for the linear space V_X. By Proposition 2(f), $B = B_1 \cup B_2$ is a basis for $W_X + V_X$. $\square$

COROLLARY 5. *For a split als $X = W_X + V_X$ with a basis B,*

$$\mathrm{card}(B) = \mathrm{card}(B_1) + \mathrm{card}(B_2),$$

where B_1 and B_2 are bases for W_X and V_X, respectively.

REMARK. For a linear space X with a basis B, let Y, Z be subspaces of X with $X = Y + Z$ and $Y \cap Z = \{0\}$. If B_1 and B_2 are bases for Y and Z respectively, then $\mathrm{card}(B) = \mathrm{card}(B_1) + \mathrm{card}(B_2)$. But it is not true when X is an *als*. Indeed, in Example 1(2) $X = V_X + Y$ and $V_X \cap Y = \{0\}$. However, $\mathrm{card}(B) < \mathrm{card}(B_1) + \mathrm{card}(B_2)$.

For an *als* X, we introduce the following set

$$U_X = \{x \in X : x \notin V_X\} \cup \{0\}.$$

Then U_X is an almost linear subspace of X by Proposition 2(d). In Example 1(2), $U_X = \{[a, b] \subset \mathbb{R} : a < b\}$ has no basis. In general, we have the following theorem.

THEOREM 6. *Let X be a nals. If U_X has a basis, then $U_X = X$.*

Proof. Let B be a basis for U_X and $b_1 \in B$. For any $v \in V_X$, $v + b_1 \in U_X$ and $-v + b_1 \in U_X$. Let

$$v + b_1 = \sum_{i=1}^{n} \lambda_i b_i, \quad -v + b_1 = \sum_{i=1}^{n} \mu_i b_i,$$

where $\lambda_i, \mu_i \in \mathbb{R}_+$, $b_i \in B$. Then $2b_1 = \sum_{i=1}^{n} (\lambda_i + \mu_i) b_i$. By the uniqueness of expression by a basis, we have $\lambda_1 + \mu_1 = 2$, $\lambda_i + \mu_i = 0$ if $i > 1$. However, since each $\lambda_i, \mu_i \in \mathbb{R}_+$, we find $\lambda_i = \mu_i = 0$ for all $i > 1$. This implies

$$v + b_1 = \lambda_1 b_1.$$

If $\lambda_1 > 1$, then $b_1 = \lambda_1 b_1 - v = (\lambda_1 - 1 + 1)b_1 - v = (\lambda_1 - 1)b_1 - v + b_1$. Since $(\lambda_1 - 1)b_1 - v \in U_X$ and U_X has a basis, we can use cancellation law. We have $(\lambda_1 - 1)b_1 - v = 0$, whence $(\lambda_1 - 1)b_1 = v \in V_X$, a contradiction. If $\lambda_1 < 1$, then $v + b_1 = v + (1 - \lambda_1 + \lambda_1)b_1 = v + (1 - \lambda_1)b_1 + \lambda_1 b_1$. We have $0 = v + (1 - \lambda_1)b_1$, whence $(1 - \lambda_1)b_1 = -v \in V_X$, a contradiction. Thus $\lambda_1 = 1$ and $v + b_1 = b_1$. Since X is a *nals*, $v = 0$(cf. [8; Proposition 1.2]). Therefore $V_X = \{0\}$, and the proof is completed. □

REMARK. The statement of Theorem 6 is false if X is an *als*. Indeed, in Example 1(1) $U_X = W_X$ and $U_X \neq X$, but $B = \{e_2\}$ is a basis for U_X.

THEOREM 7. *Let B and B' be bases for an als X. Then, up to positive scalar times and ignoring the V_X part, the two sets are identical. More precisely, there is a bijective mapping ϕ on $B \setminus V_X$ onto $B' \setminus V_X$ such that, for each $b \in B$,*

$$\phi(b) = v(b) + \lambda(b)b,$$

where $v(b) \in V_X$, $\lambda(b) > 0$.

Proof. For simplicity, let $b = b_1 \in B \setminus V_X$. Let $b_1 = u + \sum_{i=1}^{i=k} \lambda_i b_i'$ with $u \in V_X$, $\lambda_i > 0$ and $b_i' \in B' \setminus V_X$; and let $b_i' = u_i + \sum_j \mu_{ij} b_j$, where $u_i \in V_X$, $b_j \in B \setminus V_X$ and $\mu_{ij} \geq 0$ (we need to allow some $\mu_{ij} = 0$). Then

$$\begin{aligned} b_1 &= u + \sum_{i=1}^{k} \lambda_i b_i' \\ &= u + \sum_i \lambda_i \left(u_i + \sum_j \mu_{ij} b_j \right) \\ &= u + \sum_i \lambda_i u_i + \sum_j \left(\sum_i \lambda_i \mu_{ij} \right) b_j. \end{aligned}$$

By the uniqueness of expression by a basis, we have

$$u+\sum_i \lambda_i u_i = 0,\ \sum_i \lambda_i \mu_{i1} = 1,\ \sum_i \lambda_i \mu_{ij} = 0 \text{ if } j > 1.$$

However, since each $\lambda_i > 0$ and $\mu_{ij} \geq 0$, we find $\mu_{ij} = 0$ for all $i \geq 1$ and all $j > 1$. This implies $b'_i = u_i + \sum_j \mu_{ij} b_j = u_i + \mu_{i1} b_1$, and $\mu_{i1} > 0$ since $b'_i \notin V_X$. We have

$$b_1 = \frac{1}{\mu_{i1}}(-u_i + b'_i).$$

Since $\{b'_1, b'_2, \cdots, b'_k\}$ is in a basis B', i can only be 1. Thus, $b'_1 = u_1 + \mu_{11} b_1$. Hence we can define

$$\phi : B \setminus V_X \to B' \setminus V_X$$

by $\phi(b) = v(b) + \lambda(b)b$, where $v(b) \in V_X$, $\lambda(b) > 0$. Similarly, we can define

$$\phi' : B' \setminus V_X \to B \setminus V_X$$

by $\phi'(b') = v(b') + \lambda(b')b'$, where $v(b') \in V_X$, $\lambda(b') > 0$. For each $b \in B \setminus V_X$,

$$\begin{aligned}(\phi' \circ \phi)(b) &= \phi'(\phi(b)) \\ &= v(\phi(b)) + \lambda(\phi(b))\phi(b) \\ &= v(\phi(b)) + \lambda(\phi(b))(v(b) + \lambda(b)b) \\ &= (v(\phi(b)) + \lambda(\phi(b))v(b)) + (\lambda(\phi(b))\lambda(b))b.\end{aligned}$$

Since $\{(\phi' \circ \phi)(b),\ b\}$ is in a basis $B \setminus V_X$, $(\phi' \circ \phi)(b) = b$, whence $\phi' \circ \phi = I_{(B \setminus V_X)}$. Similarly, $\phi \circ \phi' = I_{(B' \setminus V_X)}$. Therefore ϕ is bijective. □

COROLLARY 8. *Let X be an als. If U_X has a basis, then the basis for U_X is unique up to positive scalar multiple for each element.*

Now, we shall give our main theorem.

THEOREM 9. *Any two bases B and B' of an als X have the same cardinal number.*

Proof. By Proposition 2(e), $B \cap V_X$ and $B' \cap V_X$ are bases for V_X. Since any two vector bases of a linear space have the same cardinal number,

$$\mathrm{card}(B \cap V_X) = \mathrm{card}(B' \cap V_X).$$

However, by Theorem 7,

$$\mathrm{card}(B \setminus V_X) = \mathrm{card}(B' \setminus V_X).$$

Therefore $\mathrm{card}(B) = \mathrm{card}(B')$. □

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