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# ASYMPTOTIC DIRICHLET PROBLEM FOR SCHRÖDINGER OPERATOR AND ROUGH ISOMETRY

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## 1. Introduction

The asymptotic Dirichlet problem for harmonic functions on a noncompact complete Riemannian manifold has a long history. It is to find the harmonic function satisfying the given Dirichlet boundary condition at infinity. By now, it is well understood [A, AS, Ch, S], when $M$ is a Cartan-Hadamard manifold with sectional curvature $-b^2 \leq K_M \leq -a^2 < 0$. (By a Cartan-Hadamard manifold, we mean a complete simply connected manifold of non-positive sectional curvature.) The essence of all of their works is that the curvature assumption gives tools to control the angle via the Toponogov comparison theorem and the convexity property near the boundary at infinity $M(\infty)$.

There have been many attempts to generalize the above results. The typical approach is to relax the curvature assumption, interesting generalization that does not directly involve the curvature bound is achieved by Schoen and Yau [SY]. Their result is as follows: Suppose that $(M, ds^2)$ is complete, simply connected manifold with sectional curvature $K_M$ satisfying $-b^2 \leq K_M \leq -a^2 < 0$. Let $d\tilde{s}^2$ be a new Riemannian metric on $M$ which is uniformly equivalent to $ds^2$. When $(M, d\tilde{s}^2)$ has the bounded sectional curvature and the positive injectivity radius, they were able to solve the asymptotic Dirichlet problem on $(M, d\tilde{s}^2)$, where $M(\infty)$ is defined with respect to the old metric $ds^2$. Cheng [C] removed the assumptions on the sectional curvature and the

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injectivity radius of $(M, d\tilde{s}^2)$ from Schoen-Yau's result. Recently, Choi, Kim and Lee developed a new approach for the larger class of complete Riemannian manifolds. In [CKL], they proposed and solved a new asymptotic Dirichlet problem via the rough isometry. It is stated as follows: Suppose $\varphi : M \to N$ is a rough isometry. They defined a function class $\mathcal{F}_\varphi$ on $N$, where $\varphi : M \to N$ is a rough isometry. For each $f \in \mathcal{F}_\varphi$, they found a solution $\tilde{u} \in C^\infty(N)$ such that $\Delta\tilde{u} = 0$ on $N$ and $(\tilde{u} - f)(\varphi(x)) \to 0$ as $x \to \infty$. Note the curious way of stating the continuity of the solution at infinity in this formulation.

THEOREM (CHOI, KIM AND LEE). *Let $M$ be a Cartan-Hadamard manifold and let $N$ be a complete manifold. Suppose the Dirichlet eigenvalue $\lambda_1(M) > 0$, and suppose there exist a point $o \in M$ and a constant $C \geq 1$ such that at any $x \in M$, we have $|K(\sigma)| \leq C|K(\sigma')|$, where $\sigma$, $\sigma'$ are plane sections at $x$ containing the tangent vector of the geodesic joining from $o$ to $x$ and $K(\sigma)$, $K(\sigma')$ are the sectional curvatures of plane sections $\sigma$, $\sigma'$, respectively. Let $\varphi : M \to N$ be a rough isometry. Then for any $f \in \mathcal{F}_\varphi$, there exists a solution $\tilde{u} \in C^\infty(N)$ such that $\Delta\tilde{u} = 0$ on $N$ and $(\tilde{u} - f)(\varphi(x)) \to 0$ as $x \to \infty$ for some $s \geq 2$.*

In this paper, we will solve the Dirichlet problem for the Schrödinger operator $\Delta - V$ where $V$ is a nonnegative function. This is a direct generalization of the result of Choi, Kim and Lee.

The basic concept that is needed in this paper is the so-called rough isometry. It was originally defined by Kanai, but was later slightly modified by Coulhon and Saloff-Coste. So in order to fix our terminology, we briefly recount relevant facts which was also used by Choi, Kim and Lee [CKL].

A (not necessarily continuous) map $\varphi : X \to Y$ between two metric spaces $X$ and $Y$ is called a rough isometry, if the following conditions hold:

(1) there exists a constant $\tau > 0$ such that

$$Y = B_\tau(\varphi(X))$$

in which case $\varphi$ is called $\tau$-full;

(2) there exist constants $a \geq 1$ and $b \geq 0$ such that

$$\frac{1}{a}d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq ad(x_1, x_2) + b$$

for all $x_1, x_2$ in $X$, where $d$ denotes the distances of $X$ and $Y$ induced from their metrics, respectively.

It is easy to check that being roughly isometric is an equivalence relation. But it is also important to note that two roughly isometric metric spaces may have completely different topology, since $\varphi$ is not assumed to be continuous. For example, an infinite cylinder is roughly isometric to an infinite cylinder with infinitely many identical handles attached at equal distance going off to the infinity.

They also assumed that $M$ and $N$ satisfy the following additional conditions:

(3) there exists a constant $C \geq 1$ such that

$$\frac{1}{C}volB_1(x) \leq volB_1(\varphi(x)) \leq CvolB_1(x)$$

for each $x \in M$.

(4) for all $r > 0$, there exists a constant $C_r > 0$ depending only on $r$ such that

$$volB_{2r}(x) \leq C_r volB_r(x)$$

for all $x$ in $M$ (in $N$, respectively).

But these assumptions are nothing but technical improvement of Kanai's assumptions. ¿From now on, when we say $\varphi$ is a rough isometry between Riemannian manifolds, it means that $\varphi$ satisfies the conditions (1) and (2), and the Riemannian manifolds satisfy the conditions (3) and (4); and $\tau$ always denotes the constant that which appears in (1).

One of key tools in combinatorially approximating a Riemannian manifold $M$ is the concept of the net. Let $d$ be the distance function on $M$. A subset $P$ of $M$ is called $\tau$-separated for some $\tau > 0$ if $d(p, p') \geq \tau$ for any distinct $p$ and $p'$ of $P$. Let $P$ be a maximal(in the sense of the inclusion) $\tau$-separated subset of $M$. Then we say that this $P$ is a $\tau$-net of $M$. This net plays the crucial role to estimate of the gradient of the data. (See Choi, Kim and Lee [CKL].)

## 2. The Asymptotic Dirichlet Problem for the Schrödinger Operator

Let $N$ be a complete Riemannian manifold which is roughly isometric to a Cartan-Hadamard manifold $M$, and let $\varphi : M \to N$ be a rough isometry. The problem is to solve an asymptotic Dirichlet problem for the Schrödinger operator on $N$. Let $M(\infty)$ be the boundary at infinity of $M$ which is the set of asymptotic classes of unit speed geodesic rays, and let $\varphi : M \to N$ be a rough isometry. Suppose $f$ is a function on $N$. We say that $f$ is an element of the class $\mathcal{F}_\varphi$ if $f$ satisfies the following conditions:

(i) $f \circ \varphi$ is continuous on $M(\infty)$;
(ii) for given $\epsilon > 0$, there exists $T > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $y \in B_T(x)$ and $d(o', x) \geq T$ for some fixed point $o' \in N$.

In the above, the statement (i) means that $f \circ \varphi(x)$ converges to a number $A$ as $x$ converges to any $v \in M(\infty)$, i.e., for any $\epsilon > 0$, there exists a neighborhood $K$ of $v$ such that $|f(x) - A| < \epsilon$ whenever $x \in K$.

In [CKL], they obtained the following lemma:

LEMMA 2.1. *Let $\varphi$, $\psi$ : $M \to N$ be rough isometries such that $\varphi^{-1} \circ \psi$ extends to a continuous map $\varphi^{-1} \circ \psi : M(\infty) \to M(\infty)$. Then for each $f \in \mathcal{F}_\varphi$, $f \circ \psi$ is also continuous on $M(\infty)$.*

Let $\varphi : M \to N$ be a rough isometry of a Cartan-Hadamard manifold $M$ into a complete manifold $N$. Let $f$ be a function on $N$ such that $f \in \mathcal{F}_\varphi$. The asymptotic Dirichlet problem for Schrödinger operator is to find a solution $\tilde{u}$ on $N$ such that $\tilde{u} \circ \varphi$ has the same boundary value as $f \circ \varphi$ on $M(\infty)$.

Since $M$ and $N$ are roughly isometric, there exist nets $P$ and $Q$ of $M$ and $N$ respectively such that $P$ and $Q$ are roughly isometric. Since $M$(respectively $N$) and $P$(respectively $Q$) are roughly isometric, we can choose a rough isometry $\psi : M \to N$ such that $\varphi^{-1} \circ \psi$ extends to identity map of $M(\infty)$. (See Choi, Kim Lee [CKL].) Thus by Lemma 2.1, $f \circ \psi$ is again continuous on $M(\infty)$. For this reason, we may redefine $\varphi$ by $\psi$. Then the restriction of $\varphi$ on $P$ is also a rough isometry from $P$

into $Q$. It is easy to check that solving the asymptotic Dirichlet problem for this newly defined $\varphi$ is enough to solve this original asymptotic Dirichlet problem.

We need to add the following condition on $M$:

(5) there exists a constant $C > 0$ depending only on $r > 0$ such that

$$\int_{B_r(x)} |\nabla f| \geq C \int_{B_r(x)} |f - \overline{f}|$$

for all $x \in M$ and for all $f \in C^\infty(B_r(x))$, where $\overline{f} = \frac{1}{volB_r(x)} \int_{B_r(x)} f$.

Note that if the Ricci curvature is bounded below, then we have a constant $C > 0$ satisfying condition (5). (See Buser [B].)

THEOREM 2.1. *Let $M$ be a Cartan-Hadamard manifold and let $N$ be a complete manifold. Suppose the Dirichlet eigenvalue $\lambda_1(M) > 0$, and suppose there exist a point $o \in M$ and a constant $C \geq 1$ such that at any $x \in M$, we have $|K(\sigma)| \leq C|K(\sigma')|$, where $\sigma$, $\sigma'$ are plane sections at x containing the tangent vector of the geodesic joining from $o$ to $x$ and $K(\sigma)$, $K(\sigma')$ are the sectional curvatures of plane sections $\sigma$, $\sigma'$, respectively. Let $\varphi : M \to N$ be a rough isometry. Then for any $f \in \mathcal{F}_\varphi$, there exists a solution $\tilde{u} \in C^\infty(N)$ such that $(\Delta - V)\tilde{u} = 0$ on $N$ and $(\tilde{u} - f)(\varphi(x)) \to 0$ as $x \to \infty$, where $V$ is a nonnegative and bounded function and its $L^s$-norm is finite for some $s \geq 2$.*

Fix $f \in \mathcal{F}_\varphi$. Define an extension $h$ of $f \circ \varphi$ such that $h \in C^\infty(M) \cap C^0(M \cup M(\infty))$ and $h|_{M(\infty)} = f \circ \varphi|_{M(\infty)}$. We may define $h$ to be a radially constant outside some compact subset of $M$. The local pinching condition for curvatures of the above theorem imposes that $|\nabla h| \in L^s(M)$ for sufficiently large $s \geq 2$ [C, Theorem 3.1]. Without loss of generality, we may assume that $h$ is positive and bounded.

Now we introduce several functions which control the gradient of the data function. (See Choi, Kim and Lee [CKL].) Let $P$ and $Q$ be $\tau$-nets of $M$ and $N$, respectively. Define a function $h_\tau$ on $P$ by

$$h_\tau(p) = \left( \frac{1}{volB_{4\tau}(p)} \int_{B_{4\tau}(p)} h^s \right)^{\frac{1}{s}}$$

for $p \in P$. Define a function $k$ on $Q$ by

$$k(q) = h_\tau \circ \varphi^{-1}(q)$$

for $q \in Q$, where $\varphi^{-1} : Q \to P$ is a rough isometry satisfying $d(\varphi \circ \varphi^{-1}(q), q) \leq \tau$ for each $q \in Q$. Define a new function $g : N \to \mathbb{R}$ by

$$g(x) = \sum_{q \in Q} k(q)\eta_q(x),$$

where $\eta_q(x)$ is a partition of unity defined as follows:

Let $\xi_q$ be a Lipschitz function given by

$$\xi_q(x) = \begin{cases} 1 - \frac{2}{3\tau} d(x,q), & x \in B_{\frac{3\tau}{2}}(q) \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$\eta_q(x) = \frac{\xi_q(x)}{\sum_{q' \in Q} \xi_{q'}(x)}.$$

Then it is easy to check that there exist $C_1 > 0$ and $C_2 > 0$ such that $\sum_{q' \in Q} \xi_{q'}(x) \geq C_1 > 0$ and $|\nabla \eta_q|(x) \leq C_2$ for all $x \in N$ and for all $q \in Q$, where $\nabla \eta_q$ is a weak derivative.

LEMMA 2.2. *For sufficiently large $s \geq 2$, we have the following:*

$$\int_N |\nabla g|^s < \infty,$$

*where $g$ is defined as above.*

This fact is useful to show that the solution of Theorem 2.1 converges to the boundary data at infinity. Choi, Kim and Lee obtained this result using Kanai's program in [K1,K2,K3].

Now solve the following Dirichlet problem, in the weak sense, on $B_R(o)$

$$\begin{cases} \Delta u_R - V u_R = -\Delta g + V g & \text{on } B_R(o) \\ u_R = 0 & \text{on } \partial B_R(o), \end{cases} \tag{2.1}$$

where $V$ is a nonnegative and bounded function and $V \in L^s(N)$. Define a functional $E$ by

$$E(v) = \int_{B_R(o)} \frac{1}{2}|\nabla v|^2 + \nabla v \cdot \nabla g + \frac{1}{2}v^2 + Vgv$$

for $v \in H_0^{1,2}(B_R(o))$. Since for some $s \geq 2$,

$$\begin{aligned} E(v) &\geq \int_{B_R(o)} -\frac{1}{2}|\nabla g|^2 - \frac{1}{2}Vg^2 \\ &\geq -\frac{1}{2}(volB_R(o))^{1-\frac{2}{s}}\{(\int_{B_R(o)} |\nabla g|^s)^{\frac{2}{s}} + \sup_N |V|(\int_{B_R(o)} g^s)^{\frac{2}{s}}\}, \end{aligned}$$

we can take a minimizer of this functional $E(v)$. For this solution, we have the following result.

LEMMA 2.3. *Let $N$ be a complete Riemannian manifold with the Dirichlet eigenvalue $\lambda_1(N) > 0$. Then there exists a constant $C$ such that for each solution $u_R$ of equation (2.1)*

$$\int_{B_R(o)} |u_R|^s \leq C$$

*for any $s \geq 2$*

*Proof.* Set $u = u_R$. Since $u|_{\partial B_R(o)} = 0$, we have

$$\begin{aligned} &\int_{B_R(o)} \nabla((sgnu)|u|^{s-1}) \cdot \nabla u + \int_{B_R(o)} (sgnu)|u|^{s-1}Vu \\ &= -\int_{B_R(o)} \nabla((sgnu)|u|^{s-1}) \cdot \nabla g - \int_{B_R(o)} (sgnu)|u|^{s-1}Vg. \end{aligned}$$

From this equation, we have

$$
\begin{aligned}
(s-1)\int_{B_R(o)} |u|^{s-2}|\nabla|u||^2 &+ \int_{B_R(o)} V|u|^s \\
&\leq (s-1)\int_{B_R(o)} |u|^{s-2}|\nabla|u|||\nabla g| + \int_{B_R(o)} |u|^{s-1}Vg \\
&\leq \frac{1}{2}(s-1)\int_{B_R(o)} |u|^{s-2}|\nabla|u||^2 + \frac{1}{2}(s-1)\int_{B_R(o)} |u|^{s-2}|\nabla g|^2 \\
&\quad + \sup_N |g| \int_{B_R(o)} |u|^{s-1}V.
\end{aligned}
$$

Since $\left|\nabla|u|^{\frac{s}{2}}\right|^2 = \frac{s^2}{4}|u|^{s-2}|\nabla|u||^2$, we have

$$
\begin{aligned}
&\frac{2(s-1)}{s^2}\int_{B_R(o)} \left|\nabla|u|^{\frac{s}{2}}\right|^2 \\
&\leq \frac{1}{2}(s-1)\int_{B_R(o)} |u|^{s-2}|\nabla g|^2 + \sup_N |g| \int_{B_R(o)} |u|^{s-1}V \\
&\leq \frac{1}{2}(s-1)\left(\int_{B_R(o)} |u|^s\right)^{\frac{s-2}{s}}\left(\int_{B_R(o)} |\nabla g|^s\right)^{\frac{2}{s}} \\
&\quad + \sup_N |g| \left(\int_{B_R(o)} |u|^s\right)^{\frac{s-1}{s}}\left(\int_{B_R(o)} V^s\right)^{\frac{1}{s}}.
\end{aligned}
$$

By the hypothesis $\lambda_1(N) > 0$, we have

$$
\begin{aligned}
&\frac{2(s-1)}{s^2}\lambda_1(N)\int_{B_R(o)} |u|^s \\
&\leq \frac{1}{2}(s-1)\left(\int_{B_R(o)} |u|^s\right)^{\frac{s-2}{s}} \\
&\left(\int_{B_R(o)} |\nabla g|^s\right)^{\frac{2}{s}} + \sup_N |g| \left(\int_{B_R(o)} |u|^s\right)^{\frac{s-1}{s}}\left(\int_{B_R(o)} V^s\right)^{\frac{1}{s}}.
\end{aligned}
$$

Since the above inequality is quadratic in $\int |u|^s$ with the positive coefficients, $\int |u|^s$ is bounded by a constant $C$ depending only on these coefficients. □

For any compact subset $\Omega$ in $N$, we have

$$\sup_{\Omega} |u_R + g| \leq \sup_{N} |g| \quad \text{and} \quad \sup_{N} |g| \leq \sup_{N} |h|.$$

By the standard Schauder estimates, we can choose a subsequence $\{u_{R_k} + g\}$ of $\{u_R + g\}$ converging uniformly to $\tilde{u}$ on any compact subset of $N$ satisfying $(\Delta - V)\tilde{u} = 0$ on $N$. Set $u = \tilde{u} - g$, then by the standard Moser iteration, we have the following lemma.

LEMMA 2.4. *Let $N$ be given in Lemma 2.3 with the Sobolev constant $S_1(N) > 0$. Then $u(z) \to 0$ as $z \to \infty$.*

*Proof.* By Lemma 2.3, we have $\int_N |u|^s < \infty$. Thus for any $\epsilon > 0$, there exists a sufficiently large $\tilde{R} > 0$ such that $\int_{B_{R_0}(z)} |u|^s < \epsilon$ if $d(o', z) \geq \tilde{R}$ for some fixed $R_0 > 0$. By the standard Moser iteration, we have for any $z \in N$ and $R_0 > 0$, there exist constants $\delta > 0$ and $C > 0$ such that

$$\sup_{B_{\frac{R_0}{2}}(z)} |u| \leq C \left( \frac{1 + R_0^2}{R_0} \right)^{\frac{n}{s}} \left( \int_{B_{R_0}(z)} |u|^s \right)^{\frac{\delta}{s}},$$

where $C$ depends on $\sup_N |\nabla g|, \sup_N |g|, \sup_N V, \int_N |\nabla g|^s$ and $\int_N V^s$. Thus we have for any $\epsilon > 0$, there exists a sufficiently large $R > 0$ such that $|u(z)| < \epsilon$ if $d(o', z) \geq R$. □

In [K1,K2], Kanai proved that $\lambda_1 > 0$ and $S_1 > 0$ are preserved under the rough isometry. In [CS], Coulhon and Saloff-Coste also proved these fact under the rough isometry with conditions (1), (2), (3), (4) and (5). Since $\lambda_1(M) > 0$ and $S_1(M) > 0$, we have $\lambda_1(N) > 0$ and $S_1(N) > 0$.

*Proof of Theorem 2.1.* We have only to show that

$$(g \circ \varphi(x) - f \circ \varphi(x)) \to 0 \quad \text{as} \quad x \to \infty.$$

Note that

$$\begin{aligned}
&|g \circ \varphi(x) - f \circ \varphi(x)| \\
&= \left| \sum_{q \in Q} \eta_q(\varphi(x)) \left( \frac{1}{volB_{4\tau}(\varphi^{-1}(q))} \int_{B_{4\tau}(\varphi^{-1}(q))} h^s \right)^{\frac{1}{s}} - f(\varphi(x)) \right| \\
&\le \sum_{q \in Q} \eta_q(\varphi(x)) \left( \frac{1}{volB_{4\tau}(\varphi^{-1}(q))} \int_{B_{4\tau}(\varphi^{-1}(q))} |h(y) - f(\varphi(x))|^s dy \right)^{\frac{1}{s}}
\end{aligned}$$

By the compactness of $M \cup M(\infty)$, for given $\epsilon > 0$, there exist $R > 0$, $v_1, \cdots, v_i \in M(\infty)$ and positive numbers $\delta_1, \cdots, \delta_i$ such that $M(\infty) \subset \bigcup_{j=1}^{i} K(v_j, \delta_j, R)$ and

$$|f \circ \varphi(x) - h(v_j)| < \epsilon, \quad |h(y) - h(v_j)| < \epsilon$$

if $x, y \in K(v_j, 2\delta_j, R)$ for each $j = 1, 2, \cdots, i$.
For any $y \in \cup_{q \in B_{\frac{3}{2}\tau}(\varphi(x))} B_{4\tau}(\varphi^{-1}(q))$, $d(x, y) \le (\frac{5}{2}a + 4)\tau + b$. Thus for sufficiently large $R > (3a+4)\tau + b$, we have $d(x, y) < R$. If $d(o, x) \ge 2R$, then for some $v_j \in M(\infty)$, $x \in K(v_j, \delta_j, \frac{R}{2})$. Thus $y \in K(v_j, 2\delta_j, R)$. This implies $|f \circ \varphi(x) - h(y)| < 2\epsilon$. Thus for any $\epsilon > 0$, there exists $R > 0$ such that

$$|\tilde{u} \circ \varphi(x) - f \circ \varphi(x)| < 3\epsilon \quad \text{if} \quad d(o, x) \ge 2R,$$

where $R \ge a\tilde{R} + b$ and $\tilde{R}$ is given in the proof of Lemma 2.4. This completes the proof. □

As Corollaries, we have some interesting new results on the usual asymptotic Dirichlet problem for the Schrödinger operator.

COROLLARY 2.1. *Let $M$ be a Cartan-Hadamard manifold. Then the usual asymptotic Dirichlet problem for the Schrödinger operator $\Delta - V$ on $M$, where $V$ is the same in Theorem 2.1, is solvable, provided that $M$ is roughly isometric to another Cartan-Hadamard manifold with the sectional curvature pinched between two negative constants.*

*Proof.* Using the result of Li and Wang [LW], we can extend given rough isometry to a homeomorphism between boundaries at infinity. The rest of the proof is the same as that in the proof of Corollary 2.2. □

COROLLARY 2.2. *Let $\varphi, M$ and $N$ be as in Theorem 2.1. Suppose further that $N$ is also a Cartan-Hadamard manifold such that $\varphi : M \to N$ extends to a continuous map $\varphi : M(\infty) \to N(\infty)$. Then the usual asymptotic Dirichlet problem for the Schrödinger operator $\Delta - V$ on $N$, where $V$ is the same in Theorem 2.1, is solvable.*

*Proof.* Let $f$ be a continuous function on $N \cup N(\infty)$. Since the rough isometry $\varphi : M \to N$ extends to a continuous map $\varphi : M(\infty) \to N(\infty)$, we have a continuous function $f \circ \varphi$ on $M(\infty)$. Let $h$ be an extension of $f \circ \varphi$ such that $h|_{M(\infty)} = f \circ \varphi|_{M(\infty)}$ and it is radially constant outside some compact subset of $M$. Define a function $g$ on $N$ by

$$g(z) = \sum_{q \in Q} \eta_q(z) \left( \frac{1}{vol B_{4\tau}(\varphi^{-1}(q))} \int_{B_{4\tau}(\varphi^{-1}(q))} h^s \right)^{\frac{1}{s}}$$

where $\eta_q(x)$ is a partition of unity as defined just above the statement of Lemma 2.2. By Lemma 2.2 and Lemma 2.4, we have a solution $\tilde{u}$ on $N$ such that $|\tilde{u}(z) - g(z)| \to 0$ as $z \to \infty$. By the same way of proof of Theorem 2.1, we have $|g(z) - f(z)| \to 0$ as $z \to \infty$. □

Finally, Lemma 2.1 and Theorem 2.1 imply the invariance property of our asymptotic Dirichlet problem for the Schrödinger operator.

COROLLARY 2.3. *Let $\varphi, M$ and $N$ be as in Theorem 2.1. Let $\psi : M \to N$ be another rough isometry such that $\varphi^{-1} \circ \psi : M(\infty) \to M(\infty)$ is continuous. Then for any $f \in \mathcal{F}_\varphi$, we have a solution $\tilde{u} \in C^\infty(N)$ such that $(\Delta - V)\tilde{u} = 0$ on $N$ and $(\tilde{u} - f)(\psi(x)) \to 0$ as $x \to \infty$, where $V$ is the same in Theorem 2.1.*

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