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ON THE EXISTENCE OF MANDATORY REPRESENTATION DESIGNS

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1. Block Designs

Let X be a finite set of elements that we shall call *points*. Let I be a set called an indexing set. A mapping $\mathcal{B} : I \longrightarrow \mathcal{P}(X)$ is called a family of *blocks* on X. For each $i \in I$, $\mathcal{B}(i)$ is also written as B_i. We always assume $|B_i| \geq 2$ for each $i \in I$.

DEFINITION. Let $v \in \mathbb{N}$, $K \subseteq \mathbb{N}$ and $\lambda \in \mathbb{N}$ be given. An ordered pair $\mathcal{D} = (X, \mathcal{B})$ consisting of a finite set X together with a family $\mathcal{B} = (B_i : i \in I)$ of blocks of X is said to be a (v, K, λ)-*PBD* (a Pairwise Balanced Design on v points with block sizes from K and index of pairwise balance λ) iff

(1) $|X| = v$
(2) $|B_i| \in K$ for every $i \in I$
(3) For every pairset $\{x, y\} \subseteq X$ there exist exactly λ indices $i \in I$ such that B_i contains the pairset $\{x, y\}$.

The integer v is called the *order* of the design. In the case that K consists of only one integer k, a $(v, \{k\}, \lambda)$-*PBD* is known as a (v, k, λ)-*BIBD* (Balanced Incomplete Block Design), and is also written as $S_\lambda(2, k, v)$ or in case of $\lambda = 1$ simply $S(2, k, v)$. The letter S is an abbreviation for "Steiner system".

Given $K \subseteq \mathbb{N}$ and $\lambda \in \mathbb{N}$, we use $B(K, \lambda)$ to denote the set of positive integers v for which a (v, K, λ)-*PBD* exists. If K consists of only one integer $k \in \mathbb{N}$, we simplify the notation by writing $B(k, \lambda)$

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instead of $B(\{k\}, \lambda)$. In the case $\lambda = 1$, we use the simpler notation $B(K)$ for $B(K, 1)$ and $B(k)$ for $B(k, 1)$. For example, it is well known that $B(3) = 6\mathbb{N}_0 + \{1, 3\}$ and $B(\{3, 5\}) = 2\mathbb{N} - 1$.

DEFINITION. For a given set K of positive integers define parameters

$$\alpha(K) = \gcd\{k - 1 \mid k \in K\}$$
$$\beta(K) = \gcd\{k(k - 1) \mid k \in K\}$$

1.1. PROPOSITION. *If $v \in B(K, \lambda)$, then*

$$\lambda(v - 1) \equiv 0 \pmod{\alpha(K)} \tag{1}$$
$$\lambda v(v - 1) \equiv 0 \pmod{\beta(K)} \tag{2}$$

Generally above conditions are not sufficient, but R. M. Wilson (Wilson 1975) proved the following fundamental theorem.

1.2. THEOREM. *Let $K \subseteq \mathbb{N}$ and $\lambda \in \mathbb{N}$ be given. Then there exists a constant $C = C(K)$ such that for all integers $v \geq C$ satisfying conditions (1) and (2), $v \in B(K, \lambda)$.*

DEFINITION. A *partial design* (with pairwise balance λ) is a pair $(X, \mathcal{B})$ consisting of a point set X and a family of blocks $\mathcal{B}$ so that any pairset occurs in at most λ times in $\mathcal{B}$.

DEFINITION. Let K be a finite set of integers. A *mandatory representation design* (v, K, λ)-*MRD* is a (v, K, λ)-*PBD* with the additional property that for each $k \in K$ there is a block of size k.

Mendelsohn and Rees (Mendelsohn and Rees 1988) introduced mandatory representation designs and examined the existence of such designs in the case $K = \{3, k\}$. They pointed out that the necessary conditions for the existence of a (v, K, λ)-*MRD* are those for the existence of a (v, K, λ)-*PBD* with the additional requirement that $v \geq P(K, \lambda)$, where $P(K, \lambda)$ denote the smallest number of points required to construct a partial design which contains every block size at least once. It is very difficult to determine $P(K, \lambda)$ for every K and λ. So we make no attempt to determine these constants. Suppose $2 \in K$. Then (v, K, λ)-*MRD* exists for all admissible v. (Construct a partial design which contains every block size and add all the missing pairsets.) Therefore we assume that $k \geq 3$ for all $k \in K$. While (v, K, λ)-PBD can be defined for an infinite set $K \subseteq \mathbb{N}$, we do not define (v, K, λ)-MRD for an infinite $K \subseteq \mathbb{N}$ because we need infinitely many points to construct such a design. Therefore K in (v, K, λ)-MRD is a finite set of integers throughout this paper. $M(K, \lambda)$ will denote the set of positive integers v for which a (v, K, λ)-MRD exists. Again, we use simpler notations $M(K)$ and $M(k)$ whenever possible.

DEFINITION. Let $v \in \mathbb{N}$, $K \subseteq \mathbb{N}$ and $G \subseteq \mathbb{N}$. A *group divisible design* (GDD), $GD_\lambda[K, G; v]$ is a triple $(X, \mathcal{G}, \mathcal{B})$, where

(1) $|X| = v$
(2) $\mathcal{G}$ is a class of non-empty subsets of X (called group) with sizes in G and which partition X.
(3) $\mathcal{B}$ is a family of subsets of X which are called blocks, each with size at least two in K.
(4) No block intersects a group at more than one point.
(5) Each pairset $\{x, y\} \subset X$ not contained in a group is contained in exactly λ blocks.

We use $GD_\lambda(K, G)$ to denote the set of all $v \in \mathbb{N}$ for which a $GD_\lambda[K, G; v]$ exists. If G or K are singleton set, for the sake of brevity we delete the braces.

DEFINITION. A *transversal design* $TD_\lambda[k, g]$ is a $GD_\lambda[k, g; v]$ with $v = kg$, k groups of size g, where each block intersects every group in exactly one point, that is, each block is a transversal of the class of groups.

We use $TD_\lambda(k)$ to denote the set of all $n \in \mathbb{N}$ for which $TD_\lambda[k, n]$ exists. Here the paremeter λ will be omitted for $\lambda = 1$. It is well known that the existence of a $TD[k, g]$ is equivalent to the existence of $k - 2$ mutually orthogonal latin squares of order g.

1.3. THEOREM. *(Chowla, Erdös and Strauss 1960) $TD[k, g]$ exists for every fixed k whenever g is sufficiently large.*

2. Existence of PBD

In this section we briefly summarize Wilson's work without proofs (Wilson 1972b, 1972c,1975) for later use.

DEFINITION. By a *closure operation* on the subsets A of a set X, we mean a map $A \longrightarrow \overline{A}$ from the class $\mathcal{P}(X)$ into $\mathcal{P}(X)$ satisfying

(1) $A \subseteq \overline{A}$ (extensive)
(2) $\overline{\overline{A}} = \overline{A}$ (idempotent)
(3) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$ (isotone).

A subset $A \subseteq X$ is said to be *closed*(with respect to a given closure operation) iff A is equal to its closure $\overline{A}$.

The map $B : K \longrightarrow B(K)$ is easily seen to be a closure operation on the subsets of the positive integers. We say that a set K is closed if it is closed under the B-operation, i.e. if $K = B(K)$. Wilson proved a more general result.

2.1. THEOREM. *$B(K, \lambda)$ is a closed set.*

DEFINITION. Let $J \subseteq \mathbb{N}_0$ and $\pi \in \mathbb{N}$. A *π-fiber* of J is a residue class

$$M_{f,\pi} = \{v \in J \mid v \equiv f \bmod \pi\}.$$

A π-fiber f of J is said to be *complete* iff there exist a constant C such that

$$\{v \mid v \geq C, v \equiv f \pmod{\pi}\} \subseteq J.$$

We say that J is *eventually periodic with period* π iff all non-empty π-fibers of J are complete

We can consider an eventually periodic set J as the union of arithmetic sequences to the modulus π where each sequence that has been "started" somewhere in J is completed. Note that every eventually periodic set must be infinite and if J is eventually periodic with period π then it is eventually periodic with period $n\pi$ for each $n \in \mathbb{N}$.

2.2. THEOREM. *Every closed set K (under B-operation) is eventually periodic with period $\beta(K)$.*

2.3. THEOREM. *For any set* $K \subseteq \mathbb{N}, \alpha(B(K)) = \alpha(K), \beta(B(K)) = \beta(K)$.

2.4. THEOREM. $B(K, \lambda)$ *is eventually periodic with period* $\beta(K)/(\lambda, \beta(K))$.

Theorem 1.2 for $\lambda = 1$ is equivalent to the following

2.5. THEOREM. *For every* $K \subseteq \mathbb{N}, B(K)$ *is eventually periodic with period* $\beta(K)$ *and every residue class* f *modulo* $\beta(K)$ *satisfying*

$$f - 1 \equiv 0 \pmod{\alpha(K)}$$
$$f(f-1) \equiv 0 \pmod{\beta(K)}$$

is a fiber of $B(K)$.

Wilson managed to reduce the above theorem into a simpler form through a series of theorems, and obtained the following theorem.

2.6. THEOREM. *Given positive integer* k, $(v, k, 1)$*-BIBD's exist for all sufficiently large integers* v *for which the following congruences are valid. And this fact implies theorem 1.2.*

$$\lambda(v-1) \equiv 0 \pmod{k-1}$$
$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$$

One must note that eventual periodicity of $B(K, \lambda)$ is not enough for the proof of theorem 2.6. To illustrate the point, cosider $B(6)$. Here $\alpha(B(6)) = 5$ and $\beta(B(6)) = 30$. Solutions of the necessary conditions are $v \equiv 1, 6, 16, 21 \pmod{30}$. To complete the proof of the theorem for $k = 6$, we will need to find examples of $(v, 6, 1)$-*BIBD*'s with $v \equiv 1, 6, 16, 21$ mod 30. Wilson's success was due to his construction method of a design for each fiber, although his construction does not yield specific examples.

3. Existence of (v, K, λ)-*MRD*

3.1. MAIN THEOREM. *Let* $K \subset \mathbb{N}$, $\lambda \in \mathbb{N}$ *be given. Then there exists a constant* $C = C(K)$ *such that for all integers* $v \geq C$ *satisfying*

$$\lambda(v-1) \equiv \pmod{\alpha(K)}$$
$$\lambda v(v-1) \equiv \pmod{\beta(K)}$$

there exist a (v, K, λ)*-MRD.*

Proof of this theorem requires several steps. First, we show that $M(K, \lambda)$ is a closed set under the B-operation(Theorem 3.2) thus it is eventually periodic by theorem 2.2. Second, we shall prove that the theorem is true for any K and $\lambda = 1$.(Teorem 3.3) Finally we show that the theorem is valid for any K and $\lambda > 0$.(Theorem 3.7)

3.2. THEOREM. *For every finite set* $K \subseteq \mathbb{N}$ *and* $\lambda \in \mathbb{N}$, $M(K, \lambda)$ *is closed with respect to the* B*-operator i.e.* $B(M(K, \lambda)) = M(K, \lambda)$.

Proof. Clearly $M(K, \lambda) \subseteq B(M(K, \lambda))$. Therefore we only need to show that $B(M(K, \lambda)) \subseteq M(K, \lambda)$. Let $v \in B(M(K, \lambda))$ and $(X,\ \mathcal{B} = \{B_1 \dots B_l\})$ be a $(v, M(K, \lambda), 1)$-PBD. Since $|B_i| \in M(K, \lambda)$ for each i, we have a $(|B_i|, K, \lambda)$-MRD say, $(B_i,\ \mathcal{B}_i = \{B_{i_1} \dots B_{i_j}\})$. Then $(X, \bigcup \mathcal{B}_i)$ is a (v, K, λ)-MRD. To see this, note that a given pairset $\{x, y\} \subseteq X$ can occur in a block of $\mathcal{B}_i$ only if $\{x, y\} \subseteq B_i$ and there is a unique block of $\mathcal{B}$ that contains $\{x, y\}$. If $\{x, y\} \subseteq B_i$, then there are λ blocks B_{i_r}'s that contain $\{x, y\}$. Thus $(X, \bigcup \mathcal{B}_i)$ is a (v, K, λ)-PBD. From the construction of $(B_i, \mathcal{B}_i)$ each block size $k \in K$ occurs at least once in $(X, \bigcup \mathcal{B}_i)$. This shows that $v \in M(K, \lambda)$ and $M(K, \lambda)$ is closed.

By Theorem (2.2) $M(K, \lambda)$ is eventually periodic with period $\beta(M(K, \lambda))$. Thus we need to calculate this period $\beta(M(K, \lambda))$ for any given K and λ. But calculating the period directly is difficult, so we prove the main theorem without direct calculation of the periods. Note that $M(K, \lambda) \subseteq B(K, \lambda)$ for any K and λ, so the period of $M(K, \lambda)$ can not be smaller than that of $B(K, \lambda)$. Moreover if the main theorem is true $M(K, \lambda)$ and $B(K, \lambda)$ coincide for sufficiently large v.

DEFINITION. We say that two sets $S, T \subseteq \mathbb{N}$ eventually coincide iff there exist a constant M such that

$$\{s \in S \mid s \geq M\} = \{t \in T \mid t \geq M\}.$$

To prove the main theorem we need to show that $M(K, \lambda)$ contains all the fibers of $B(K, \lambda)$ modulo the period of $B(K, \lambda)$. Then the period of $M(K, \lambda)$ must be the same as the period of $B(K, \lambda)$. Then $M(K, \lambda)$ and $B(K, \lambda)$ eventually coincide with each other. Now, we prove the main theorem for $\lambda = 1$.

3.3. THEOREM. *For any finite set $K \subset \mathbb{N}$ there exist a constant $C = C(K)$ such that for all integers $v \geq C$ satisfying*

$$(v-1) \equiv 0 \pmod{\alpha(K)}$$
$$v(v-1) \equiv 0 \pmod{\beta(K)}$$

there exists a $(v, K, 1)$-MRD.

Proof. By theorem (2.4) $B(K)$ is eventually periodic with period $\beta(K)$. In order to prove that $M(K)$ is eventually coincide with $B(K)$ we need to show that there exists at least one $(v, K, 1)$-MRD for each fiber f modulo $\beta(K)$. We use induction on the number of elements in K. The case when $K = \{k\}$ is covered by theorem 1.2. Assume that the theorem is true for any K with $|K| = n-1$. Let $K = \{k_1 \ldots k_n\}, K_i = K \backslash \{k_i\}$ $i = 1, \ldots, n$. Choose a fiber f modulo $\beta(K)$. If we have a v satisfying $v \equiv f \pmod{\beta(K)}$ and $v \in M(K)$ we have nothing to prove. Suppose not, i.e. for any $v \equiv f \pmod{\beta(K)}, v \notin M(K)$. By theorem (1.2) there exist a costant $C = C(K)$ such that for every v satisfying $v > C(K) \equiv f \pmod{\beta(K)}$, v is in $B(K)$. Especially $v \in B(K_i)$ for some i.(If there is no such i then every block size must be used, which is contrary to our assumption.) Since $B(K_i)$ and $M(K_i)$ eventually coincide with each other by the induction hypothesis, every sufficiently large $v \equiv f \pmod{\beta(K)}$ is in $M(K_i)$. Now choose w such that $w \equiv 1 \pmod{\beta(K)}$ and $w \in B(k_i)$ to obtain $wv \equiv f \pmod{\beta(K)}$. (such a w always exists since $\beta(K)|k_i(k_i-1)$ and by theorem 1.2.) It only remains to show that $wv \in M(K)$. By the existence of transversal designs $TD[k, g]$ for every k whenever g is sufficiently large(Theorem 1.3), we can choose v so that $TD[w, v]$ exist. Break up each group of size v to make a $(v, K_i, 1)$-MRD. Since $w \in B(k_i)$ we also have blocks of size k_i therefore $wv \in M(K)$. Since our choice of f is arbitrary $M(K)$ eventually coincide with $B(K)$.

DEFINITION. Given $K \subseteq \mathbb{N}$, we define

$$\gamma(K) = \begin{cases} \beta(K)/\alpha(K) & \text{if} \quad \alpha(K) \neq 0 \\ 1 & \text{if} \quad \alpha(K) = 0. \end{cases}$$

3.4.LEMMA. *If* $\lambda = a_1\lambda_1 + \cdots + a_n\lambda_n$ *for* $a_i \geq 0, \lambda_i \geq 1$ *then*

$$\bigcap_{i=1}^{n} M(K, \lambda_i) \subseteq M(K, \lambda)$$

3.5.LEMMA. *Let* a *and* c *be relatively prime integers. If* $\lambda(f-1) \equiv 0 \pmod{a}$ *and* $\lambda f(f-1) \equiv 0 \pmod{ac}$, *then there exists an integer* d *such that*

$$\begin{aligned} \lambda(d-f) &\equiv 0 \pmod{ac} \\ d-1 &\equiv 0 \pmod{a} \\ d(d-1) &\equiv 0 \pmod{ac} \end{aligned}$$

3.6.LEMMA. $\alpha(K)$ *and* $\gamma(K)$ *are relatively prime.*

Proofs of these lemmas are similar to the proofs of Wilson's lemmas in (Wilson 1972c)

3.7. THEOREM. *If the theorem 3.1 is valid for a given set* $K \subseteq \mathbb{N}$ *and* $\lambda = 1$, *then it is valid for all* $\lambda \geq 1$.

Proof. By theorem (2.4) $\beta(B(K,\lambda)) = \beta(K)/(\lambda, \beta(K))$. Thus we shall show that every residue class f modulo $\beta(K)/(\lambda, \beta(K))$ satisfying

$$\begin{aligned} \lambda(f-1) &\equiv 0 \bmod \alpha(K) \\ \lambda f(f-1) &\equiv 0 \bmod \beta(K) \end{aligned}$$

is in fact a fiber modulo $\beta(K)/(\lambda, \beta(K))$ of the closed set $M(K,\lambda)$. That is for any such residue class f there exist $v \in M(K,\lambda)$ with $v \equiv f \bmod \beta(K)/(\lambda,\beta(K))$. Given such f, choose $a = \alpha(K), c = \gamma(K)$ and apply lemma (3.5) we have an integer d such that

$$\begin{aligned} \lambda(d-f) &\equiv 0 \pmod{\beta(K)} \\ d-1 &\equiv 0 \pmod{\alpha(K)} \\ d(d-1) &\equiv 0 \pmod{\beta(K)} \end{aligned}$$

Since d satisfies all the necessary conditions, d is a fiber of $M(K)$ by theorem (3.3), i.e. every sufficiently large v with $v \equiv d \pmod{\beta(K)}$ is in $M(K)$. Since $M(K) \subseteq M(K,\lambda)$ these v are also in $M(K,\lambda)$. Now,

$$v \equiv d \bmod \frac{\beta(K)}{(\lambda,\beta(K))} \qquad \text{since} \frac{\beta(K)}{(\lambda,\beta(K))} | \beta(K)$$

$$\text{by the choice of } d \qquad d \equiv f \bmod \frac{\beta(K)}{(\lambda,\beta(K))}$$

Therefore,

$$v \equiv f \bmod \frac{\beta(K)}{(\lambda,\beta(K))}$$

4. Application of MRD

An immediate application of mandatory representation design is a sub-design problem.

PROBLEM. Let $(Y,\mathcal{A})$ be a $S(2,k,u)$, find a $S(2,k,v)$ $(X,\mathcal{B})$ which contains $(Y,\mathcal{A})$ as a sub-design.

In terms of MRD this problem is equivalent to determining $M(\{k,u\})$. To see this, note that for any $v \in M(\{k,u\})$ each block of size u may be considered as a sub-design. Conversely, a sub-design of order u can be written as one block of size u thus $v \in M(\{k,u\})$. Sub-design problems can be easily generalized into pairwise balanced designs or mandatory representation designs. Although a complete solution for such a problem is known only for $K = \{3\}$ we can expect that sufficiently large v might work. And the following theorem confirms our expectation.

4.1. THEOREM. *Let $(Y,\mathcal{A})$ be a $(u,K,1)$-MRD, then there exists a constant $C = C(K,u)$ such that for all integers $v \geq C$ satisfying*

$$v - 1 \equiv 0 \pmod{\alpha(K)}$$
$$v(v-1) \equiv 0 \pmod{\beta(K)}$$

we have $(v,K,1)$-MRD $(X,\mathcal{B})$ which contains $(Y,\mathcal{A})$ as a sub-design.

Proof. It is enough to show that for every such v, there is a $(v, K \cup \{u\}, 1)$-MRD. Note that $\alpha(K \cup \{u\}) = \alpha(K)$ and $\beta(K \cup \{u\}) = \beta(K)$. By the main theorem there exists a constant $C(K, u)$ such that for all $v \geq C(K, u)$ satisfying necessary conditions, a $(v, K \cup \{v\}, 1)$-MRD exists. thus we have $(v, K, 1)$-MRD which contains a mandatory sub-design of order u.

4.2. COROLLARY. *For any $S(2, k, u)$ there exists a constant $C(k, u)$ such that for all $v \geq C(k, u)$ satifying necessary conditions*

$$v - 1 \equiv 0 \pmod{k-1}, v(v-1) \equiv 0 \pmod{k(k-1)}$$

there is an $S(2, k, v)$ which contains $S(2, k, u)$ as a sub-design.

REMARK. Finding the smallest such $C(k, u)$ is still a very difficult problem. In general, existence of (v, K, λ)-MRD for small v is very much open.

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