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ON δ-SEMICLASSICAL ORTHOGONAL POLYNOMIALS

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1. Introduction

Consider an operator equation of the form :

$$H[y](x) = \alpha(x)\delta^2 y(x) + \beta(x)\delta y(x) = \lambda_n y(x), \tag{1.1}$$

where $\alpha(x)$ and $\beta(x)$ are polynomials of degree at most two and one respectively, λ_n is the eigenvalue parameter, and δ is Hahn's operator defined by

$$\delta f(x) = \frac{f(qx+w) - f(x)}{(q-1)x+w} \tag{1.2}$$

for real constants $q\,(\neq \pm 1)$ and w. Hahn [4] showed that for an orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$ the followings are all equivalent (see also [7]):

(1) $\{\delta P_n(x)\}_{n=1}^{\infty}$ is also an orthogonal polynomial system.
(2) For $n \geq 0$, $\{P_n(x)\}_{n=0}^{\infty}$ satisfies an operator equation of the form (1.1).
(3) There is a polynomial $\alpha_0(x)$ and a function $w(x)$ such that

$$P_n(x) = [w(x)]^{-1}\delta^n[\alpha_0(x)\alpha_1(x)\cdots\alpha_n(x)w(x)], \quad n \geq 0, \tag{1.3}$$

where $\alpha_i(x) = \alpha_{i+1}(qx+w)$ for $i = 0, 1, \cdots, n-1$.

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(4) There is a rational function $Q(x,y)$ such that

$$\frac{a_{n,i}}{a_{n,i-1}} = Q\left(\frac{q^n-1}{q-1}, \frac{q^i-1}{q-1}\right), \quad n \geq 0 \quad \text{and} \quad 0 \leq i \leq n, \tag{1.4}$$

where $P_n(x) = \sum\limits_{i=0}^{n} a_{n,i}x^i$.

(5) The moments $\{\sigma_n\}_{n=0}^{\infty}$ with respect to which $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal satisfy a recurrence relation of the form

$$\sigma_n = \frac{a+bq^n}{c+dq^n}\sigma_{n-1}, \quad n \geq 1, \tag{1.5}$$

where a, b, c, d are constants with $ad - bc \neq 0$.

In [7], it is shown that the condition (1) can be relaxed as :

(6) For any fixed integer $r \geq 1$, $\{\delta^r P_n(x)\}_{n=r}^{\infty}$ is a weak orthogonal polynomial system (see Definition 2.1).

We call any orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$ a Hahn class orthogonal polynomial system if $\{P_n(x)\}_{n=0}^{\infty}$ satisfy any one of the above six equivalent conditions.

In this work, we study the so called δ-semiclassical orthogonal polynomials (see Definition 3.1), which was first introduced in [10]. In particular, we give new characterizations of δ-semiclassical orthogonal polynomials using higher order structure relations. These generalize some of previous results for $\delta = d/dx$ ([1,8,9]) or for $\delta = \Delta$, the forward difference operator ([3,5,6,14]).

2. Preliminaries

All polynomials in this work are assumed to be real polynomials in one variable and we let $\mathcal{P}$ be the space of all real polynomials. We denote the degree of a polynomial $\pi(x)$ by $\deg(\pi)$ with the convention that $\deg(0) = -1$. By a polynomial system(PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n) = n$, $n \geq 0$. Note that a PS forms a basis of $\mathcal{P}$.

We call any linear functional σ on $\mathcal{P}$ a moment functional and denote its action on a polynomial $\pi(x)$ by $\langle\sigma, \pi\rangle$. For a moment functional σ, we call

$$\sigma_n := \langle\sigma, x^n\rangle, \quad n = 0, 1, \cdots$$

the moments of σ. We say that a moment functional σ is quasi-definite if its moments $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0 \tag{2.1}$$

for every $n \geq 0$. Any PS $\{\phi_n(x)\}_{n=0}^{\infty}$ determines a moment functional σ (uniquely up to a non-zero constant multiple), called a canonical moment functional of $\{\phi_n(x)\}_{n=0}^{\infty}$, by the conditions

$$\langle\sigma, \phi_0\rangle \neq 0 \quad \text{and} \quad \langle\sigma, \phi_n\rangle = 0, \quad n \geq 1.$$

DEFINITION 2.1. We call a PS $\{P_n(x)\}_{n=0}^{\infty}$ a weak orthogonal polynomial system (WOPS) (respectively, an orthogonal polynomial system (OPS)) if there is a non-zero moment functional σ such that

$$\langle\sigma, P_m P_n\rangle = K_n \delta_{mn} \quad (m \text{ and } n \geq 0), \tag{2.2}$$

where K_n are real (respectively, non-zero real) constants. In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a WOPS or an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

It is immediate from (2.2) that for any WOPS $\{P_n(x)\}_{n=0}^{\infty}$, its orthogonalizing moment functional σ must be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. It is well known (see Chapter 1 in Chihara [2]) that a moment functional σ is quasi-definite if and only if there is an OPS relative to σ.

Throughout the paper, we use the following notations :

$$[0] := 0, \qquad \text{and} \qquad [n] := \begin{cases} \frac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases}$$

for any real number q and any integer $n \geq 0$. Furthermore, we use factorial notations as

$$[0]! = 1 \qquad \text{and} \qquad [n]! := [n][n-1]\cdots[1], \quad n \geq 1.$$

We call systems $\{\phi_n(x)\}_{n=0}^{\infty}$ and $\{\tilde{\phi}_n(x)\}_{n=0}^{\infty}$ defined inductively by

$$\phi_0(x) = 1, \quad \phi_n(x) = \phi_{n-1}(x)(x - [n-1]w), \quad n \geq 1,$$

and

$$\tilde{\phi}_0(x) = 1, \quad \tilde{\phi}_n(x) = \tilde{\phi}_{n-1}(x)(x - [n]w), \quad n \geq 1,$$

the factorial polynomials.

LEMMA 2.1. *We have*

(i) $\delta^n x^n = [n]!, \quad n \geq 1.$

(ii) $\delta\phi_n(x) = [n]\phi_{n-1}(x), \qquad \delta\tilde{\phi}_n(x) = [n]\tilde{\phi}_{n-1}(x), \qquad n \geq 1.$

(iii) $\phi_n(\frac{1}{q}(x-w)) = \frac{1}{q^n}\tilde{\phi}_n(x), \quad \phi_{n+1}(x) = x\tilde{\phi}_n(x), \qquad n \geq 0.$

(iv) $x\phi_n(x) = \phi_{n+1}(x) + w[n]\phi_n(x), \quad n \geq 0.$

(v) $\tilde{\phi}_n(x) = \phi_n(x) + \sum\limits_{k=0}^{n-1}(-1)^{n-k}\frac{[n]!}{[k]!}w^{n-k}\phi_k(x), \; n \geq 0.$

Proof. The proofs are straightforward. □

For a moment functional σ and a polynomial $\phi(x)$, we let $^+\delta\,\sigma$, $\phi\sigma$, $T_{q,w}\sigma$ and $T_{q,w}^{-1}\sigma$ be the moment functionals defined by

$$\begin{aligned}
\langle\phi\sigma, \psi\rangle &= \langle\sigma, \phi\psi\rangle, \quad \langle{}^+\delta\,\sigma, \psi\rangle = -\langle\sigma, \delta\psi\rangle,\\
\langle T_{q,w}^{-1}\sigma, \psi\rangle &= \langle\sigma, T_{q,w}(\psi)\rangle = \langle\sigma, \psi(qx+w)\rangle,\\
\langle T_{q,w}\sigma, \psi\rangle &= \langle\sigma, T_{q,w}^{-1}(\psi)\rangle = \langle\sigma, \psi(\frac{1}{q}(x-w))\rangle \quad (\psi \in \mathcal{P}).
\end{aligned}$$

Then the followings are easy consequences of definitions.

LEMMA 2.2. *Let σ be a quasi-definite moment functional and $\{P_n(x)\}_{n=0}^{\infty}$ an OPS relative to σ. Then we have*

(i) *for any polynomial $\phi(x)$, $\phi(x)\sigma = 0$ if and only if $\phi(x) \equiv 0$.*

(ii) *for any moment functional τ and any integer $k \geq 0$, $\langle\tau, P_n\rangle = 0$ for $n > k$ if and only if $\tau = \psi(x)\sigma$ for some polynomial $\psi(x)$ of degree $\leq k$.*

PROPOSITION 2.3. *Let σ be a moment functional. Then we have*

(i) ${}^{+}\delta\,\sigma = 0$ *if and only if* $\sigma = 0$ *;*

(ii) *(Leibniz's rule) for any polynomial* $\phi(x)$ *and* $\psi(x)$*,*

$$\delta(\phi(x)\psi(x)) = \phi(qx+w)\delta\psi(x) + \psi(x)\delta\phi(x)\,; \tag{2.3}$$

(iii) *for any polynomial* $\phi(x)$*,*

$$T_{q,w}^{-1}\delta\phi = q\delta(T_{q,w}^{-1}\phi), \qquad \delta(T_{q,w}\phi) = qT_{q,w}(\delta\phi)\,; \tag{2.4}$$

(iv) *for any polynomial* $\phi(x)$*,*

$${}^{+}\delta(\phi\sigma) = T_{q,w}^{-1}\phi\,{}^{+}\delta\,\sigma + \delta(T_{q,w}^{-1}\phi)\sigma = \phi\,{}^{+}\delta\,\sigma + T_{q,w}(\delta\phi\sigma). \tag{2.5}$$

Proof. (i) It immediately follows from the relations

$$\langle\sigma,\phi_n(x)\rangle = \langle\sigma,\frac{1}{[n+1]}\delta(\phi_{n+1}(x))\rangle = -\frac{1}{[n+1]}\langle{}^{+}\delta\,\sigma,\phi_{n+1}(x)\rangle = 0,\ n\geq 0.$$

(ii) For any polynomials $\phi(x)$ and $\psi(x)$, we have

$$\begin{aligned}\delta(\psi(x)\phi(x)) &= \frac{\phi(qx+w)\psi(qx+w)-\phi(x)\psi(x)}{(q-1)x+w}\\ &= \frac{\phi(qx+w)(\psi(qx+w)-\psi(x))}{(q-1)x+w} + \frac{\psi(x)(\phi(qx+w)-\phi(x))}{(q-1)x+w}\\ &= \phi(qx+w)\delta\psi(x) + \psi(x)\delta\phi(x).\end{aligned}$$

(iii) It comes easily from the definition of Hahn's operator.

(iv) For any polynomial $\psi(x)$, we have by (ii)

$$\begin{aligned}\langle{}^{+}\delta(\phi\sigma),\psi\rangle &= -\langle\sigma,\phi(x)\delta\psi(x)\rangle\\ &= -\langle\sigma,\delta(\phi(\frac{1}{q}(x-w))\psi(x)) - \psi(x)\delta\phi(\frac{1}{q}(x-w))\rangle\\ &= \langle\phi(\frac{1}{q}(x-w))\,{}^{+}\delta\,\sigma + \delta\phi(\frac{1}{q}(x-w))\sigma,\psi\rangle.\end{aligned}$$

On the other hand, we have via (ii) for any polynomial $\psi(x)$,

$$\begin{aligned}\langle{}^{+}\delta(\phi\sigma),\psi\rangle &= -\langle\sigma,\delta(\phi(x)\psi(x)) - \psi(qx+w)\delta\phi(x)\rangle\\ &= \langle\phi(x)\,{}^{+}\delta\,\sigma + T_{q,w}^{-1}(\delta\phi(x)\sigma),\psi(x)\rangle.\end{aligned}$$

Hence, we obtain (2.5). □

3. Main results

The theory of semiclassical OPS's is well developed by many authors [9,11,12,14] when δ is the differential operator d/dx or the difference operator Δ. We now consider δ-semiclassical OPS's, which are first introduced by Maroni [10].

DEFINITION 3.1. ([10]) A quasi-definite moment functional σ is called δ-semiclassical if there is a pair of polynomials $(\alpha(x), \beta(x)) \neq (0,0)$ such that

$$
{}^{+}\delta(\alpha\sigma) = \beta\sigma. \tag{3.1}
$$

For any δ-semiclassical moment functional σ, we call

$$
s := \min\{\max(\deg(\alpha) - 2, \deg(\beta) - 1)\}
$$

the class number of σ, where the minimum is taken over all pairs of polynomials $(\alpha, \beta) \neq (0,0)$ satisfying the equation (3.1). In this case, we call σ a δ-semiclassical moment functional of class s and its corresponding OPS $\{P_n(x)\}_{n=0}^{\infty}$ is called a δ-semiclassical OPS of class s.

PROPOSITION 3.1. *If σ is a δ-semiclassical moment functional satisfying the equation* (3.1), *then* $\deg(\alpha) \geq 0$ *and* $\deg(\beta) \geq 1$ *so that the class number s is non-negative.*

Proof. Suppose that $\alpha(x) \equiv 0$. Then $\beta\sigma = {}^{+}\delta(\alpha\sigma) = 0$ so that $\beta(x) \equiv 0$ since σ is quasi-definite. It contradicts to $(\alpha, \beta) \neq (0,0)$. Assume that $\beta(x) \equiv 0$. Then ${}^{+}\delta(\alpha\sigma) = 0$ so that $\alpha\sigma = 0$ and so $\alpha(x) \equiv 0$ which contradicts to $(\alpha, \beta) \neq (0,0)$. If $\beta(x) \equiv c (\neq 0)$, then $c\langle \sigma, 1\rangle = \langle \beta\sigma, 1\rangle = \langle {}^{+}\delta(\alpha\sigma), 1\rangle = -\langle \alpha\sigma, \delta(1)\rangle = 0$, which also contradicts to quasi-definiteness of σ. □

LEMMA 3.2. (cf.[5,8,13]) *Let σ be a δ-semiclassical moment functional satisfying*

$$
\begin{aligned}
{}^{+}\delta(\phi_1\sigma) &= \psi_1\sigma \qquad (s_1 := \max(t_1 - 2, p_1 - 1)\ ) \\
{}^{+}\delta(\phi_2\sigma) &= \psi_2\sigma \qquad (s_2 := \max(t_2 - 2, p_2 - 1)\ ),
\end{aligned} \tag{3.2}
$$

where $t_j = \deg(\phi_j)$ and $p_j = \deg(\psi_j)$, $j = 1, 2$. Let $\phi(x)$ be a common factor of $\phi_1(x)$ and $\phi_2(x)$ of the highest degree. Then, there is a polynomial $\psi(x)$ such that

$$^+\delta(\phi\sigma) \;=\; \psi\sigma,$$

where $s := \max(\deg(\phi) - 2, \deg(\psi) - 1) = s_1 - t_1 + \deg(\phi) = s_2 - t_2 + \deg(\phi)$.

Proof. We may assume that $\phi_1 = \tilde{\phi}_1\phi$ and $\phi_2 = \tilde{\phi}_2\phi$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ have no common factor except real constants. From the equation (3.2), we have

$$(T_{q,w}^{-1}\tilde{\phi}_1)\,^+\delta(\phi\sigma) = (\psi_1 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_1)\,\sigma, \tag{3.3}$$

$$(T_{q,w}^{-1}\tilde{\phi}_2)\,^+\delta(\phi\sigma) = (\psi_2 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_2)\,\sigma. \tag{3.4}$$

Multiplying (3.3) by $T_{q,w}^{-1}\tilde{\phi}_2$ and (3.4) by $T_{q,w}^{-1}\tilde{\phi}_1$ and then substracting the two equations, we have

$$(T_{q,w}^{-1}\tilde{\phi}_2)[\psi_1 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_1)] = [\psi_2 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_2)](T_{q,w}^{-1}\tilde{\phi}_1).$$

Since $\tilde{\phi}_1$ and $\tilde{\phi}_2$ have no common factor, $T_{q,w}^{-1}\tilde{\phi}_1$ and $T_{q,w}^{-1}\tilde{\phi}_2$ also have no non-constant common factor. Hence, $\psi_2 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_2)$ and $\psi_1 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_1)$ are divisible by $T_{q,w}^{-1}\tilde{\phi}_2$ and $T_{q,w}^{-1}\tilde{\phi}_1$ respectively so that there exists a polynomial ψ such that

$$\psi_2 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_2) = \psi(T_{q,w}^{-1}\tilde{\phi}_2) \text{ and } \;\psi_1 - \phi\delta(T_{q,w}^{-1}\tilde{\phi}_1) = \psi(T_{q,w}^{-1}\tilde{\phi}_1).$$

From the equations (3.3) and (3.4), we have

$$(T_{q,w}^{-1}\tilde{\phi}_2)(^+\delta(\phi\sigma) - \psi\sigma) = 0 \text{ and } \;(T_{q,w}^{-1}\tilde{\phi}_1)(^+\delta(\phi\sigma) - \psi\sigma) = 0.$$

Since $T_{q,w}^{-1}\tilde{\phi}_1$ and $T_{q,w}^{-1}\tilde{\phi}_2$ have no common factor, we have

$$^+\delta(\phi\sigma) - \psi\sigma = 0.$$

Finally, the formula for s follows just by counting degrees of $\phi(x)$ and $\psi(x)$. □

PROPOSITION 3.3. (cf.[5,8]) *Let σ be a δ-semiclassical moment functional of class s satisfying the equation* (3.1) *with* $s = \max(\deg(\phi) - 2, \deg(\psi) - 1)$. *If σ satisfies the equation* (3.1) *with another pair of polynomials* $(\phi_1, \psi_1) \neq (0,0)$, *then $\phi_1(x)$ is divisible by $\phi(x)$.*

Proof. Let $\alpha(x)$ be a common factor of $\phi(x)$ and $\phi_1(x)$ of the highest degree. Then by Lemma 3.2, there is a polynomial $\beta(x)$ such that

$$^{+}\delta(\alpha\sigma) \;=\; \beta\sigma$$

and $s_0 := \max(\deg(\alpha) - 2, \deg(\beta) - 1) = s - \deg(\phi) + \deg(\alpha)$. Since $s_0 \geq s$, $\deg(\alpha) \geq \deg(\phi)$ so that $\alpha(x) = c\phi(x)$ for some non-zero constant c. Hence, $\phi(x)$ must divide $\phi_1(x)$. □

COROLLARY 3.4. *For any δ-semiclassical moment functional σ, the pair of polynomials $(\alpha, \beta) \neq (0,0)$ which realizes the class number of σ is unique up to a non-zero-constant multiple.*

Proof. Assume that two pairs of polynomials $(\alpha, \beta) \neq (0,0)$ and $(\alpha_1, \beta_1) \neq (0,0)$ realize the class number of σ. Then, by Proposition 3.3, $\alpha_1(x)$ is divisible by $\alpha(x)$ and vice versa. Hence, $\alpha(x) = c\alpha_1(x)$ for some non-zero constant c. □

Maroni [10] have found several necessary and sufficient conditions for a moment functional σ to be δ-semiclassical.

DEFINITION 3.2. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called to be quasi-orthogonal of order k, $k \geq 0$ an integer, if there is a moment functional σ such that

$$\begin{aligned} &\langle \sigma, P_m P_n \rangle = 0, \quad 0 \leq m \leq n-k \\ &\langle \sigma, P_{r-k} P_r \rangle \neq 0, \quad \text{for some } r \geq k. \end{aligned} \tag{3.5}$$

In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal of order k relative to σ.

We see that any PS $\{P_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal of order 0 if and only if it is a WOPS.

THEOREM 3.5. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ and $\{Q_n(x)\}_{n=0}^{\infty} := \{\delta(P_{n+1}(x))\}_{n=0}^{\infty}$. Then the followings are equivalent.*

(i) *σ is δ-semiclassical ;*

(ii) *$\{Q_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal ;*

(iii) *There are integers s and t with $0 \le t \le s+2$ and a polynomial $\alpha(x)$ of degree t such that for $n \ge s$,*

(3.6)
$$\alpha(x)Q_n(x) = \sum_{j=n-s}^{n+t} \theta_{n,j}P_j(x), \quad \text{and} \quad \theta_{n,n-s} \ne 0 \quad \text{for some} \quad n \ge s.$$

Proof. See Theorem 3.1 in [10]. □

We call (3.6) a structure relation of order one for a δ-semiclassical OPS $\{P_n(x)\}_{n=0}^{\infty}$.

PROPOSITION 3.6. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ.*

(i) *If σ is a δ-semiclassical moment functional satisfying (3.1), then for any integer $r \ge 1$, $\{\delta^r P_n(x)\}_{n=r}^{\infty}$ is quasi-orthogonal of order $\le rs$ relative to $\alpha^r(x)\sigma$.*

(ii) *If $\{\delta^r P_n(x)\}_{n=r}^{\infty}$ is quasi-orthogonal of order k for some integer $r \ge 1$, then $\{P_n(x)\}_{n=0}^{\infty}$ is a δ-semiclassical OPS of class $\le k+2r-2$.*

Proof. See Theorem 4.2 and Theorem 4.4 in [10]. □

In particular, when $r = 1$, we obtain : $\{P_n(x)\}_{n=0}^{\infty}$ is a δ-semiclassical OPS of class 0 if and only if $\{\delta P_n(x)\}_{n=0}^{\infty}$ is a WOPS. Therefore, $\{P_n(x)\}_{n=0}^{\infty}$ is a Hahn-class OPS if and only if $\{P_n(x)\}_{n=0}^{\infty}$ is a δ-semiclassical OPS of class 0 (see Section one).

We now give some new characterizations of δ-semiclassical OPS's. First, we have:

THEOREM 3.7. *Let σ be a quasi-definite moment functional. Then σ is a δ-semiclassical moment functional satisfying (3.1) if and only if*

$$(3.7) \qquad \langle \sigma, L[\phi]\psi \rangle = \langle \sigma, \phi L[\psi] \rangle, \quad (\phi(x) \text{ and } \psi(x) \in \mathcal{P})$$

where $L[\cdot] := \alpha(x)\delta^2 T_{q,w}^{-1} + \beta(x)\delta T_{q,w}^{-1}$.

Proof. $\Rightarrow$) It can be easily shown that

$$L[\phi]\sigma = \frac{1}{q}\,{}^{+}\delta[(\delta\phi)\alpha\sigma], \quad \phi \in \mathcal{P}.$$

Hence, for any polynomials $\phi(x)$ and $\psi(x)$ we have

$$\begin{aligned}\langle \sigma, L[\phi]\psi\rangle &= \frac{1}{q}\langle {}^{+}\delta[(\delta\phi)\alpha\sigma], \psi\rangle = \frac{1}{q}\langle \alpha\sigma, \delta\phi\delta\psi\rangle \\ &= \frac{1}{q}\langle {}^{+}\delta[(\delta\psi)\alpha\sigma], \phi\rangle = \langle \sigma, L[\psi]\phi\rangle.\end{aligned}$$

$\Leftarrow$) By choosing $\phi(x) \equiv 1$, we have for any polynomial $\psi(x) \in \mathcal{P}$,

$$\begin{aligned}0 &= \langle \sigma, L[1]\psi\rangle = \langle \sigma, L[\psi]\rangle = \langle \sigma, \alpha\delta^2(T_{q,w}^{-1}\psi) + \beta\delta(T_{q,w}^{-1}\psi)\rangle \\ &= -\langle {}^{+}\delta(\alpha\sigma) - \beta\sigma, \delta(T_{q,w}^{-1}\psi\rangle.\end{aligned}$$

Hence ${}^{+}\delta(\alpha\sigma) - \beta\sigma = 0$ since any polynomial can be written in the form $\delta(T_{q,w}^{-1}\psi)(x)$. $\square$

The equation (3.7) means that the operator $\sigma L[\cdot]$ is formally symmetric on polynomials. We can also generalize Theorem 3.5 as :

THEOREM 3.8. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ. Then for any integer $r \geq 1$, the followings are equivalent.*

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is a δ-semiclassical OPS.*
(ii) *There is an integer $u(\geq r)$ and a polynomial $\pi(x)$ of degree $t \geq 0$ such that*

$$\pi(x)\delta^r(P_n(x)) = \sum_{j=n-u}^{n-r+t} \theta_{n,j}P_j(x), \quad n > u. \tag{3.8}$$

Proof. (i)$\Rightarrow$ (ii) : Let σ satisfy ${}^{+}\delta(\alpha\sigma) = \beta\sigma$ for some polynomials $(\alpha, \beta) \neq (0, 0)$ with $s := \max(\deg \alpha - 2, \deg \beta - 1)$ and $\deg \alpha = k$. Then, we may write

$$\alpha^r(x)\delta^r(P_n(x)) = \sum_{j=0}^{n-r+rk} \theta_{n,j} P_j(x). \tag{3.9}$$

Multiplying both sides of (3.9) by $P_m(x)$, $m = 0, 1, \cdots, n - r - rs$ and then applying σ, we have

$$\theta_{n,m}\langle \sigma, P_m^2 \rangle = \sum_{j=0}^{n-r+rk} \theta_{n,j}\langle \sigma, P_m P_j \rangle = \langle \alpha^r(x)\sigma, P_m \delta^r(P_n) \rangle = 0,$$

since $\{\delta^r P_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal of order $\leq rs$ relative to $\alpha^r(x)\sigma$ by Proposition 3.6 (i). Hence we have (3.8) with $\pi(x) = \alpha^r(x)$, $u = r + rs$ and $t = rk$.
(ii)$\Rightarrow$ (i) : Assume that (3.8) holds for some integer $r \geq 1$. Then for $0 \leq m < n - u$,

$$\langle \pi\sigma, P_m \delta^r(P_n) \rangle = \sum_{j=n-u}^{n-r+t} \theta_{n,j}\langle \sigma, P_m P_j \rangle = 0$$

and so

$$\langle \pi\sigma, (\delta^r P_m)(\delta^r P_n) \rangle = 0 \quad \text{for } 0 \leq m \leq n - u + r.$$

Hence $\{\delta^r P_n(x)\}_{n=r}^{\infty}$ is quasi-orthogonal of order $\leq u - r$ relative to $\pi(x)\sigma$ so that $\{P_n(x)\}_{n=0}^{\infty}$ is a δ-semiclassical OPS of class $\leq u + 2r - 2$ by Proposition 2.6 (ii). $\square$

We may call (3.8) a structure relation of order r for a δ-semiclassical OPS $\{P_n(x)\}_{n=0}^{\infty}$. When $\delta = d/dx$ (that is, when $q \to 1$ and $w = 0$), Theorem 3.8 is proved in [9].

On the other hand, Al-Salam and Chihara [1] proved : an OPS $\{P_n(x)\}_{n=0}^{\infty}$ is classical, i.e., a semiclassical OPS of class 0 if and only if there is a polynomial $\pi(x)$ of degree ≥ 0 such that

$$\pi(x)P_n'(x) = r_n P_{n+1}(x) + s_n P_n(x) + t_n P_{n-1}(x), \quad n \geq 1,$$

where r_n, s_n, t_n are constants.

LEMMA 3.9. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ. Then for any integer $k \geq 0$ and $n \geq 0$, we have*

$$x^k P_n(x) = \sum_{j=n-k}^{n+k} C_{n,j} P_j(x), \tag{3.10}$$

where $C_{n,n+k} \neq 0$ and $C_{n,j} = 0$ for $j < 0$ and

$$P_{n+k}(x) = \pi_k(x;n) P_n(x) + \sum_{j=1}^{k} C_{n+k,j} P_{n-j}(x), \tag{3.11}$$

where $\pi_k(x;n)$ is a polynomial of degree k with coefficients depending on n and $C_{n+k,j} = 0$ for $j > n$.

Proof. We may write $x^k P_n(x)$ as $x^k P_n(x) = \sum_{j=0}^{n-k} C_{n,j} P_j(x)$. Then

$$C_{n,l}\langle \sigma, P_l^2 \rangle = \langle \sigma, P_l \sum_{j=0}^{n+k} C_{n,j} P_j \rangle = \langle \sigma, x^k P_l P_n \rangle = 0, \quad k + l < n.$$

Hence $C_{n,l} = 0$ if $l < n - k$ and (3.10) follows. (3.11) can be proved easily for any fixed $n \geq 0$ by induction on $k \geq 0$ using the three term recurrence relation satisfied by $\{P_n(x)\}_{n=0}^{\infty}$. □

THEOREM 3.10. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ. Then for any integer $r \geq 1$, the followings are equivalent.*

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is a Hahn class OPS (i.e., a δ-semiclassical OPS of class 0).*

(ii) *There is a polynomial $\pi(x)$ of degree $t \geq 0$ such that*

$$\pi(x)\delta^r(P_n(x)) = \pi_{t-r}(x;n) P_n(x) + \sum_{j=1}^{r} \theta_{n,j} P_{n-j}(x), \quad n > r, \tag{3.12}$$

where $\pi_{t-r}(x;n)$ is a polynomial of degree $t - r$ with coefficients depending on n.

(iii) *There is a polynomial $\pi(x)$ of degree t with $0 \leq t \leq 2r$ such that*

$$\pi(x)\delta^r(P_n(x)) = \sum_{j=n-r}^{n-r+t} \theta_{n,j} P_j(x), \quad n > r. \tag{3.13}$$

Proof. The equivalence of (ii) and (iii) follows immediately from Lemma 3.9.
(i)$\Rightarrow$(iii) : It is a special case of Theorem 3.8.
(iii)$\Rightarrow$(i) : Assume that there is a polynomial $\pi(x)$ of degree $t \geq 0$ satisfying (3.13). Then $\{\delta^r P_n(x)\}_{n=r}^{\infty}$ is quasi-orthogonal of order 0, that is, $\{\delta^r P_n(x)\}_{n=r}^{\infty}$ is a WOPS (see the proof of Theorem 3.8). Hence, $\{P_n(x)\}_{n=0}^{\infty}$ must be a Hahn class OPS (see Section one). $\square$

If σ is a δ-semiclassical moment functional satisfying (3.1), then for any polynomial $\phi(x)$, σ also satisfies

$$^+\delta\{(\phi\alpha + \delta(\phi)[(q-1)x + w]\alpha)\sigma\} = (\phi\beta - \delta(\phi)\alpha)\sigma$$

so that σ satisfies infinitely many distinct equations of the form (3.1). It is so natural to ask : How can we see whether the pair (α, β), with which σ satisfies (3.1), gives the class number of σ or not ?

LEMMA 3.11. *Let σ be a quasi-definite moment functional and satisfy*

$$^+\delta(\alpha\sigma) - \beta\sigma = \pi(x)[^+\delta(\alpha_1\sigma) - \beta_1\sigma], \quad \pi \in \mathcal{P}. \tag{3.14}$$

Then, $\max(\deg(\alpha)-2, \deg(\beta)-1) = \deg(\pi)+\max(\deg(\alpha_1)-2 \deg(\beta_1)-1)$.

Proof. By the direct calculation,
we have $^+\delta(\alpha\sigma) - \beta\sigma = \pi(x)[^+\delta(\alpha_1\sigma) - \beta_1\sigma]$ if and only if

$$\alpha(x) = \pi(qx+w)\alpha_1(x), \quad \beta(x) = \alpha_1(x)\delta\pi(x) + \pi(x)\beta_1(x).$$

So we have $\deg(\alpha) = \deg(\pi)+ \deg(\alpha_1)$. If $\pi(x) = c\ (\neq 0)$, then $\alpha(x) = c\alpha_1(x)$ and $\beta(x) = c\beta_1(x)$ so that the conclusion is trivial. Assume that $\deg(\pi) \geq 1$. Then there are three cases ;

(a) $\deg(\alpha) - 2 > \deg(\beta) - 1$;
(b) $\deg(\alpha) - 2 = \deg(\beta) - 1$;
(c) $\deg(\alpha) - 2 < \deg(\beta) - 1$.

(a): Since $\deg(\alpha)-2>\deg(\beta)-1$, we have $\max(\deg(\alpha)-2, \deg(\beta)-1)=\deg(\alpha)-2$. On the other hand, by counting the degree of $\beta(x)-\delta(\pi)\alpha_1(x)=\pi(x)\beta_1(x)$ we have

$$\begin{aligned}\deg(\pi\beta_1)&=\deg[\beta(x)-\delta(\pi)\alpha_1]\le\max(\deg(\beta),\deg(\delta(\pi))+\deg(\alpha_1))\\&=\max(\deg(\beta),\deg(\alpha)-1)=\deg(\alpha)-1\end{aligned}$$

so that $\deg(\beta_1)\le\deg(\alpha_1)-1$. Hence, we obtain

$$\begin{aligned}\deg(\pi)+\max(\deg(\alpha_1)-2,\ \deg(\beta_1)-1)&=\deg(\pi)+\deg(\alpha_1)-2\\&=\deg(\alpha)-2,\end{aligned}$$

which is the required result. The proof for cases (b) and (c) is similar to the above. □

LEMMA 3.12. ([12]) *Let σ and τ be moment functionals and c be an arbitrary constant. Then $(x-c)\tau=\sigma$ if and only if*

$$\tau=\tau_0\delta_c+(x-c)^{-1}\sigma \tag{3.15}$$

where $\tau_0=\langle\tau,1\rangle$ and δ_c is the dirac delta function at c.

THEOREM 3.13. (cf.[13]) *Let σ be a δ-semiclassical moment functional satisfying (3.1) with $s:=\max(\deg(\alpha)-2,\deg(\beta)-1)$. Then σ is of class s if and only if for any root c of $\alpha(x)$,*

$$|r_c|+|\langle\sigma,\beta_c\rangle|\ne 0$$

where $\alpha(x)=(x-c)\alpha_c(x)$ and $\beta(x)-\delta(\frac{x-w-cq}{q})\alpha_c(x)=(\frac{x-w-cq}{q})\beta_c(x)+r_c$.

Proof. Let c be a root of $\alpha(x)$ and so $\alpha(x)=(x-c)\alpha_c(x)$. Then

$$\begin{aligned}0&={}^{+}\delta(\alpha\sigma)-\beta\sigma={}^{+}\delta[(x-c)\alpha_c(x)\sigma]-\beta\sigma\\&=(\frac{x-w-cq}{q})\,{}^{+}\delta(\alpha_c(x)\sigma)+\delta(\frac{x-w-cq}{q})\alpha_c(x)\sigma-\beta\sigma\\&=(\frac{x-w-cq}{q})\,{}^{+}\delta(\alpha_c(x)\sigma)-[\beta(x)-\delta(\frac{x-w-cq}{q})\alpha_c(x)]\sigma.\end{aligned}$$

We set

$$\beta(x) - \delta(\frac{x-w-cq}{q})\alpha_c(x) = (\frac{x-w-cq}{q})\beta_c(x) + r_c.$$

Then $(\frac{x-w-cq}{q})[^{+}\delta(\alpha_c(x)\sigma) - \beta_c(x)\sigma] = r_c\sigma$. By Lemma 3.12,

$$\tau := {}^{+}\delta(\alpha_c(x)\sigma) - \beta_c(x)\sigma = \tau_0\delta_{w+cq} + (\frac{q}{x-w-cq})r_c\sigma$$

and

$$\tau_0 = \langle \tau, 1\rangle = \langle {}^{+}\delta(\alpha_c\sigma) - \beta_c\sigma, 1\rangle = -\langle \sigma, \beta_c\rangle.$$

Suppose that σ is of class s but there is a root c of $\alpha(x)$ such that $r_c = 0$ and $\langle \sigma, \beta_c\rangle = 0$. Then $\tau = {}^{+}\delta(\alpha_c(x)\sigma) - \beta_c\sigma = 0$, so that

$$0 = {}^{+}\delta(\alpha\sigma) - \beta\sigma = (\frac{x-w-cq}{q})[^{+}\delta(\alpha_c\sigma) - \beta_c\sigma].$$

By Lemma 3.11, $s = 1+$ max(deg$\alpha_c - 2$, deg$\beta_c - 1$), so that $s \leq s-1$. It is a contradiction.

Conversely, suppose that σ is of class $\tilde{s} < s$ but $r_c \neq 0$, $\langle \sigma, \beta_c\rangle \neq 0$ for any root c of $\alpha(x)$. Then there exists $(\tilde{\alpha}, \tilde{\beta}) \neq (0,0)$ such that ${}^{+}\delta(\tilde{\alpha}\sigma) = \tilde{\beta}\sigma$ and $\tilde{s}$=max(deg$\tilde{\alpha} - 2$, deg$\tilde{\beta} - 1$). By Proposition 3.3, there is a polynomial $\pi(x)$ with $\deg(\pi) \geq 1$ such that $\alpha(x) = \pi(x)\tilde{\alpha}(x)$. Hence, from the equation (3.1) we have

$$\begin{aligned}\beta\sigma &= {}^{+}\delta(\alpha\sigma) = {}^{+}\delta(\pi(x)\tilde{\alpha}\sigma) = (T_{q,w}^{-1}\pi)\,{}^{+}\delta(\tilde{\alpha}\sigma) + \delta(T_{q,w}^{-1}\pi)\tilde{\alpha}\sigma \\ &= ((T_{q,w}^{-1}\pi)\tilde{\beta} + \delta(T_{q,w}^{-1}\pi)\tilde{\alpha})\sigma\end{aligned}$$

so that

$$\beta(x) = (T_{q,w}^{-1}\pi)\tilde{\beta} + \delta(T_{q,w}^{-1}\pi)\tilde{\alpha}.$$

Let c be a root of $\pi(x)$ and so $(x-c) = \pi_c(x)$. Then we have $\alpha_c(x) =$

$\pi_c(x)\tilde{\alpha}(x)$ and so

$$
\begin{aligned}
&\beta(x) - \delta(\frac{x-w-cq}{q})\alpha_c(x) \\
&= \pi(\frac{x-w}{q})\tilde{\beta}(x) + \delta(\pi(\frac{x-w}{q}))\tilde{\alpha}(x) - \delta(\frac{x-w-cq}{q})\alpha_c(x) \\
&= (\frac{x-w-cq}{q})\pi_c(\frac{x-w}{q})\tilde{\beta}(x) + \delta[(\frac{x-w-cq}{q})\pi_c(\frac{x-w}{q})]\tilde{\alpha}(x) \\
&\quad - \delta(\frac{x-w-cq}{q})\pi_c(x)\tilde{\alpha}(x) \\
&= (\frac{x-w-cq}{q})\pi_c(\frac{x-w}{q})\tilde{\beta}(x) - \delta(\frac{x-w-cq}{q})\pi_c(x)\tilde{\alpha}(x) \\
&\quad + \{\pi_c(x)\delta(\frac{x-w-cq}{q}) + (\frac{x-w-cq}{q})\delta(\pi_c(\frac{x-w}{q}))\}\tilde{\alpha}(x) \\
&= (\frac{x-w-cq}{q})[\pi_c(\frac{x-w}{q})\tilde{\beta}(x) + \delta(\pi_c(\frac{x-w}{q}))\tilde{\alpha}(x)],
\end{aligned}
$$

which implies $r_c = 0$ and $\beta_c(x) = \pi_c(\frac{x-w}{q})\tilde{\beta}(x) + \delta(\pi_c(\frac{x-w}{q}))\tilde{\alpha}(x)$. On the other hand, we have

$$
\begin{aligned}
\langle\sigma, \beta_c\rangle &= \langle\sigma, \pi_c(\frac{x-w}{q})\tilde{\beta}(x) + \delta(\pi_c(\frac{x-w}{q}))\tilde{\alpha}(x)\rangle \\
&= \langle\tilde{\beta}(x)\sigma, \pi_c(\frac{x-w}{q})\rangle - \langle{}^{+}\delta(\tilde{\alpha}\sigma), \pi_c(\frac{x-w}{q})\rangle \\
&= 0,
\end{aligned}
$$

which contradicts to $r_c \neq 0$ and $\langle\sigma, \beta_c\rangle \neq 0$ for any root c of $\alpha(x)$. □

Theorem 3.13 for $\delta = d/dx$ was proved in [13].

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