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LINEAR OPERATORS THAT STRONGLY PRESERVES THE SIGN-CENTRAL MATRICES

G.-Y. LEE, S.-G. LEE AND S.-Z. SONG

1.Introduction

Let $M_{m,n}$ be the set of all $m \times n$ real matrices. For a matrix $A = [a_{ij}] \in M_{m,n}$, the *sign* of a_{ij} is defined by

$$\operatorname{sgn} a_{ij} = \begin{cases} 0 & \text{if } a_{ij} = 0. \\ +1 & \text{if } a_{ij} > 0. \\ -1 & \text{if } a_{ij} < 0. \end{cases}$$

The *sign pattern* of $A, \mathbf{A}$ is the $m \times n$ $\{0, 1, -1\}$-matrix

$$\mathbf{A} = [\operatorname{sgn} a_{ij}] = \operatorname{sgn} A$$

obtained from A by replacing each entry with its sign. If $\mathbf{A}$ and $\mathbf{B}$ are sign pattern matrices with same size, then $\mathbf{A} + \mathbf{B}$ exists, that is, $\mathbf{A} + \mathbf{B}$ is qualitatively defined if $a_{ij}b_{ij} \neq -1$ for all i and j, $1 \leq i \leq m$, $1 \leq j \leq n$. If $a_{ij}b_{ij} = -1$, then $a_{ij} + b_{ij}$ is 0, -1 or $+1$. So, we cannot determine the sign of the entry $a_{ij} + b_{ij}$. That is, $\mathbf{A} + \mathbf{B}$ is undefined.

Let $Q(\mathbf{B})$ be the *qualitative class* of $\mathbf{B}$ such that the sign pattern of any matrix in $Q(\mathbf{B})$ equals to the sign pattern of $\mathbf{B} = [\mathbf{b}_{ij}]$, i.e.,

$$Q(\mathbf{B}) = \{A = [a_{ij}] \in M_{m,n} \mid \mathbf{b}_{ij} = \operatorname{sgn} a_{ij} \text{ for all } i,\ j\}.$$

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The column vectors $a^{(1)}, a^{(2)}, \dots, a^{(n)}$ of a matrix A in $Q(\mathbf{B})$ determine a convex polytope

$$\mathcal{CP}(A) = \{\sum_{i=1}^{n} c_i a^{(i)} | \sum_{i=1}^{n} c_i = 1,\ c_i \geq 0\ (1 \leq i \leq n)\}.$$

We define the matrix A to be *central* provided that the origin $(0, \dots, 0)^T$ is contained in the polytope $\mathcal{CP}(A)$. The matrix $A \in Q(\mathbf{B})$ is called *sign-central* provided that each matrix in $Q(\mathbf{B})$ is central. That is, a matrix $A \in Q(\mathbf{B})$ is a sign-central matrix if and only if each matrix in $Q(\mathbf{B})$ is sign-central. For example, the $m \times (m+1)$ matrix with exactly one 1 and exactly one -1 in each row defined by

$$F_m = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

is easily seen to be a sign-central matrix. The matrix

$$E_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

is also a sign-central matrix. More generally, for each positive integer m, the $m \times 2^m$ matrix E_m such that each m-tuple of 1's and -1's is a column of E_m is a sign-central matrix.

A diagonal matrix $D \neq \mathbf{0}$ each of whose diagonal entries equals 0, 1, or -1 is called a *signing*. A signing with no 0's on its main diagonal is called a *strict signing*. Let A be an $m \times n$ matrix, and let P and Q be permutation matrices of order m and n, respectively. Let D be a strict signing. Then it follows from the definition that A is a sign-central matrix if and only if $PDAQ$ is a sign-central matrix. That is, a sign-central matrix is permutation invariant.

In [1], the sign-central matrix was characterized as following;

THEOREM 1.1. [ANDO AND BRUALDI, 1, THEOREM 2.1]. *Let A be an $m \times n$ $\{0, 1, -1\}$-matrix. Then the following are equivalent:*

(i) A is a sign-central matrix.

(ii) For every strict signing D of order m, the matrix DA has a nonnegative column vector.

(iii) For every strict signing D of order m, the matrix DA has a nonpositive column vector.

(iv) Each set of the blocker $b(\mathcal{A})$ contains as a subset at least one of the sets $\{1, \bar{1}\}, \ldots, \{m, \bar{m}\}$.

(v) There do not exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & X_1 \\ X_2 & A_2 \end{bmatrix}$$

where A_1 is a possibly vacuous matrix with at least one 1 in each column and A_2 is a possibly vacuous matrix with at least one -1 in each column.

In the above theorem, (ii) and (iii) are clearly equivalent. By the above theorem, if a matrix A has a zero column, then A is a sign-central matrix. And, if a matrix A is a sign-central matrix with no zero column vector, then the matrix DA have both a nonnegative column vector and a nonpositive column vector for every strict signing D of order m.

Let $T : M_{m,n} \rightarrow M_{m,n}$ be a linear operator. We say T *preserves* the subset $\mathcal{K}$ of $M_{m,n}$ if T maps each matrix in the set $\mathcal{K}$ to a matrix in $\mathcal{K}$. We say T *strongly preserves* the subset $\mathcal{K}$ of $M_{m,n}$ if T preserves both $\mathcal{K}$ and $M_{m,n} \setminus \mathcal{K}$, the complement of $\mathcal{K}$ in $M_{m,n}$.

Let E_{ij} denote the (0,1)-matrix whose only nonzero entry is in the (i, j) position. A *cell* is a scalar multiple of E_{ij} for some (i, j), so that the set of cells is the set

$$\{\alpha_{ij} E_{ij} | \alpha_{ij} \in \mathrm{R}, \text{ the reals, } 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}.$$

Let $\mathfrak{R}_i = \sum_{j=1}^{n} E_{ij}$ and $\mathfrak{C}_j = \sum_{i=1}^{m} E_{ij}$. That is, $\mathfrak{R}_i$ is the matrix whose ith row is all ones and all other entries are zero. Let J be an

$m \times n$ matrix whose entries are all ones and let I_m be the identity matrix of order m. Clearly, $\mathfrak{R}_i$, J and I_m are not sign-central matrices.

We denote the *Hadamard product* of $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{m,n}$ by $A \circ B$, i.e., $A \circ B = [a_{ij}b_{ij}]$.

The *term rank* is the minimum number, $t(A)$, of lines(columns or rows) which contain all non-zero entries of A.

In [3], Beasley and Pullman characterized the linear opertors that preserve term rank 1 as following;

THEOREM 1.2. [BEASLEY AND PULLMAN, 3, COROLLARY 3.1.2]. *Suppose that T is a nonsingular linear operator on $M_{m,n}$. The linear operator T preserves the set of matrices of term rank 1 if and only if T is one of or a composition of some of the following operators:*

(i) $X \to X^T$ if $m = n$.

(ii) $X \to PXQ$ for some fixed but arbitrary permutation matrices P and Q of order m and n, respectively.

(iii) $X \to X \circ M$ for some fixed but arbitrary matrix M in $M_{m,n}$ with no zero entries.

In this paper, we characterize linear operators T preserve the set of sign-central matrices using the above theorem.

2. STRONG PRESERVERS OF SIGN-CENTRAL MATRICES

In this section we will investigate the linear operators that strongly preserve sign-central matrices. We will prove that if T is a linear operator that strongly preserves the sign-central matrices then

$$T(X) = PDXQ \quad \text{for all} \quad X \in M_{m,n},$$

or

$$T(X) = PDX^TQ \quad \text{for} \quad m = n \quad \text{and} \quad X = X^T,$$

where P and Q are permutation matrices of order m and n, respectively, and D is a strict signing of order m.

Throughout this section, let T be a linear operator that strongly preserves sign-central matrices.

LEMMA 2.1. *Let $X = [x^{(1)} \cdots x^{(n)}]$ be a nonzero sign-central matrix. Then there is a sign-central matrix $Y = [y^{(1)} \cdots y^{(n)}]$ such that $X + Y$ is not a sign-central matrix.*

Proof.. First, suppose that the matrix X has no zero column vector. Then, for any strict signing D, DX have both a nonnegative column vector and a nonpositive column vector. For some fixed D, without loss of generality, let $Dx^{(1)}, \dots, Dx^{(i)}$ be nonnegative vectors and let $Dx^{(i+1)}, \dots, Dx^{(i+j)}$ be nonpositive vectors, $i \geq 1$, $j \geq 1$, $i + j \leq n$. For some $p \geq 2$, let

$$\begin{aligned} y^{(1)} &= -px^{(1)}, \dots, y^{(i)} = -px^{(i)}, \\ y^{(i+1)} &= \cdots = y^{(i+j)} = 0, \\ y^{(i+j+1)} &= x^{(i+j+1)}, \dots, y^{(n)} = x^{(n)}. \end{aligned}$$

Then the matrix Y is a sign-central and the matrix $X + Y$ has no zero column. Since $D(X + Y)$ does not have a nonnegative column vector, $X + Y$ is not a sign-central matrix.

Now, suppose that the matrix X have zero columns. Without loss of generality, let $x^{(1)} = \cdots = x^{(i)} = 0$ and $x^{(i+1)}, \dots, x^{(n)}$ are nonzero vectors, $1 \leq i \leq n - 1$. First, assume that $Dx^{(i+1)}, \dots, Dx^{(n)}$ are not nonpositive (respectively, nonnegative) vectors for some strict signing D. Then, let

$$y^{(1)} = \cdots = y^{(i)} = x^{(i+1)}, y^{(i+1)} = \cdots = y^{(n)} = 0.$$

Then the matrix Y is a sign-central matrix and the matrix $X+Y$ has no zero column. Since $D(X+Y)$ does not have a nonpositive(respectively, nonnegative) vector, $X + Y$ is not a sign-central matrix. Next, assume that there are nonpositive(respectively, nonnegative) vectors and there is no nonnegative (respectively, nonpositive) vector among the vectors $Dx^{(i+1)}, \dots, Dx^{(n)}$. Without loss of generality, we may assume that $Dx^{(i+1)}, \dots, Dx^{(i+j)}$ are nonpositive (respectively, nonnegative) vectors, $1 \le j \le n-i$. Let

$$y^{(1)} = \cdots = y^{(i)} = x^{(i+1)}, y^{(i+1)} = \cdots = y^{(n)} = 0.$$

Then the matrix Y is a sign-central and $X+Y$ has no zero column vector. Since $D(X+Y)$ does not have a nonnegative (respectively, nonpositive) vector, the matrix $X+Y$ is not a sign-central matrix. Finally, assume that the $\{Dx^{(i+1)}, \dots, Dx^{(n)}\}$ have both a nonnegative vector and a nonpositive vector. Without loss of generality, we may assume that $Dx^{(i+1)}, \dots, Dx^{(i+j)}$ are nonnegative vectors and $Dx^{(i+j+1)}, \dots, Dx^{(i+k)}$ are nonpositive vectors. Then, for some $p \ge 2$, let

$$\begin{aligned} y^{(1)} &= \cdots = y^{(i)} = x^{(i+j+1)} \\ y^{(i+1)} &= -px^{(i+1)}, \dots, y^{(i+j)} = -px^{(i+j)} \\ y^{(i+j+1)} &= \cdots = y^{(i+k)} = y^{(i+k+1)} = \cdots = y^{(n)} = 0. \end{aligned}$$

Then, the matrix Y is a sign-central matrix and the matrix $X+Y$ has no zero column vector. Since $D(X+Y)$ does not have a nonnegative vector, the matrix $X+Y$ is not a sign-central matrix.

Therefore, if X is a sign-central matrix then there is a sign-central matrix Y such that $X+Y$ is not a sign-central matrix. ■

LEMMA 2.2. *T is a nonsingular linear operator.*

Proof. Suppose $T(X) = 0$ for some $X \ne 0$. Since T is a strongly preserver, X is a sign-central matrix. So, there is a sign-central matrix Y such that $X+Y$ is not a sign-central. Then

$$T(X+Y) = T(X) + T(Y) = T(Y).$$

This is a contradiction. Therefore, T is nonsingular ■

By above lemma, since T is a nonsingular and dimension of domain of T equals dimension of image of T, the linear operator T is bijective on $M_{m,n}$. And, an immediate consequence of the above lemmas is the following:

THEOREM 2.3. *The mapping T is bijective on the set of cells.*

For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order, write $A \leq B$ if $a_{ij} \leq b_{ij}$ for all i and j.

LEMMA 2.4. *If $T(\mathfrak{R}_i) = X$ for each i, then $X \geq \mathbf{0}$ or $X \leq \mathbf{0}$.*

Proof. Suppose that $T(\mathfrak{R}_i) = X_1 - X_2$ for $X_1, X_2 \geq \mathbf{0}$. Let

$$A = E_{i1} + \cdots + E_{ik} \text{ and } B = E_{ik+1} + \cdots + E_{in}$$

for some k, $1 \leq k \leq n$. Since T is nonsingular and bijective on the cells, without loss of generality, let $T(A) = X_1$ and $T(B) = -X_2$. Since $\mathfrak{R}_i$ is not a sign-central matrix, $T(\mathfrak{R}_i) = X_1 - X_2$ is not a sign-central matrix. So, the matrix $X_1 - X_2$ does not have a zero column vector and hence $X_1 + X_2$ does not have a zero column vector. Now, we consider a sign-central matrix $A - B$. Then, $T(A - B) = X_1 + X_2$. So, $X_1 + X_2$ is a sign-central matrix. But, the matrix $X_1 + X_2$ is not a sign-central matrix, since $X_1 + X_2$ does not have a zero column vector and $X_1 + X_2 \geq 0$. ■

We now show that T preserves the term rank of any matrix. We say that a matrix A is a *row matrix* if $\mathfrak{R}_i \geq A$ for some i. Also, we say that a matrix A is a *column matrix* if $\mathfrak{C}_j \geq A$ for some j.

THEOREM 2.5. *If T strongly preserves the set of sign-central matrices, then T preserves the set of matrices of term rank 1.*

Proof. If $m = 1$ or $n = 1$, then, clearly, the mapping T preserves the set of matrices of term rank 1. If $n = 2$, then, for any real numbers $a, b, \ldots, c$, the matrices

$$\begin{bmatrix} a & 0 \\ b & 0 \\ \vdots & \vdots \\ c & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & -a \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

are only sign-central matrices. Since T strongly preserve the sign-central matrices and T is nonsigular, T is a term rank 1 preserving operator. Thus, we consider the case $m \geq 2$ and $n \geq 3$.

Suppose that $m \geq 2$ and $n \geq 3$. If T preserve row matrices and column matrices, respectively, then T is a term rank 1 perserver.

First, assume that $T(\mathfrak{R}_i)$ is not a row matrix for some i. Let $A_{ip} = \sum_{j=1}^{p} E_{ij}$ and $B_{ip} = \sum_{j=p+1}^{n} E_{ij}$ for some i, p. Since T is bijective on the set of cells, we may assume that $T(A_{ip}) = A_{kp}$ and $T(B_{ip}) = B_{rp}$ for $k \neq r$. Then, by lemma 2.4,

$$T(A_{ip} - B_{ip}) = A_{kp} - B_{rp}.$$

Since the matrix $A_{ip} - B_{ip}$ is a sign-central matrix, the matrix $A_{kp} - B_{rp}$ is a sign-central matrix. Let $D = \mathrm{diag}\{d_1, \ldots, d_m\}$ be a strict signing of order m with $d_k = 1$ and $d_r = -1$. Then, the matrix $D(A_{kp} - B_{rp})$ does not have a nonpositive column vector, i.e., the matrix $A_{kp} - B_{rp}$ is not a sign-central matrix. Thus, T preserves row matrices.

Now, suppose that $T(\mathfrak{C}_j)$ is not a column matrix for some j. Let $G_{kp} = \sum_{i=1}^{k} E_{ip}$ and $H_{kp} = \sum_{i=k+1}^{m} E_{ip}$ for some k, p. Without loss of generality, we may assume that $T(G_{kp}) = G_{kl}$ and $T(H_{kp}) = H_{ks}$ for $l \neq s$. Then, there exist cells $C_1, \ldots, C_k$ such that $T(C_1 + \cdots + C_k) = G_{ks}$. Since $C_i \neq E_{ip}$ for $i = 1, \ldots, k$,

$$J \setminus (C_1 + \cdots + C_k + H_{kp})$$

is not a sign-central matrix. But,

$$T(J \setminus (C_1 + \cdots + C_k + H_{kp}))$$

have a zero column vector. That is, $T(J \setminus (C_1 + \cdots + C_k + H_{kp}))$ is a sign-central matrix. Thus, T preserves column matrices.

Therefore, T is a term rank 1 preserver. ■

We note that the sign-central matrices can be varied by the transpose, in general. That is, there is a sign-central matrix A such that A^T is not a sign-central matrix. For example, if

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

then the matrix A is a sign-central matrix. But the matrix A^T is not a sign-central matrix, since the identity matrix I_3 is a strict signing and the matrix I_3A^T does not have a nonpositive column vector. If

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then B is not a sign-central matrix. But B^T is a sign-central matrix.

Note that if $m = n$ and A is a symmetric sign-central matrix, then A^T is a sign-central matrix. We have thus established the following lemma;

LEMMA 2.6. *Let A does not have a zero vector in rows and columns. The transpose operator preserves a sign-central matrix A if and only if A is a symmetric sign-central matrix.*

LEMMA 2.7. *Let $X \in M_{m,n}$. If T strongly preserves sign-central matrices and if $T(X) = X \circ M$, then there exists a strict signing D of order m such that $M = DJ$; thus $T(X) = DX$.*

Proof. Let $T(X) = X \circ M$. A real matrix X is a sign-central matrix if and only if each matrix in $Q(X)$ is sign-central. So, without loss of generality, let X be a $(0, 1, -1)$-matrix and $M = [m_{ij}]$ be a $(1, -1)$-matrix. Since T preserves term rank 1, $T(E_{ij} + E_{ik}) = \alpha E_{pq} + \beta E_{ps}$, for $j \neq k$ and $q \neq s$. Suppose that $\operatorname{sgn} \alpha \neq \operatorname{sgn} \beta$. Then, without loss of generality, let $T(E_{ij} + E_{ik}) = E_{pq} - E_{ps}$. So, $\mathfrak{R}_i$ and $T(\mathfrak{R}_i)$

are not sign-central matrices and $T(\mathfrak{R}_i)$ has exactly one nonnero entry in each column. Since

$$T(\mathfrak{R}_i) = T(E_{ij}+E_{ik}+\mathfrak{R}_i\backslash(E_{ij}+E_{ik})) = E_{pq}-E_{ps}+T(\mathfrak{R}_i\backslash(E_{ij}+E_{ik})),$$

for every strict signing D of order m,

$$DT(\mathfrak{R}_i) = D(E_{pq} - E_{ps} + T(\mathfrak{R}_i \setminus (E_{ij} + E_{ik})))$$

have both a nonnegative column vector and a nonpositive column vector. That is, $T(\mathfrak{R}_i)$ is a sign-central matrix. This is a contradiction. Thus $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$. Since, for a sign-central matrix X, $T(X) = X \circ M$,

$$\operatorname{sgn} m_{i1} = \cdots = \operatorname{sgn} m_{in} \quad \text{for} \quad i = 1, \cdots, m.$$

Let $D = \operatorname{diag}\{\operatorname{sgn} m_{11}, \operatorname{sgn} m_{21}, \ldots, \operatorname{sgn} m_{m1}\}$. Then, $X \circ M = DX$ and hence $T(X) = DX$ for strict signing D. ■

An immediate consequence of the above lemmas and theorems is the following;

THEOREM 2.8. *Let a linear operator T strongly preserves sign-central matrices. Then,*

$$T(X) = PDXQ \quad \textit{for any} \quad X \in M_{m,n},$$

or

$$T(X) = PDX^TQ \quad \textit{for} \quad m = n \textit{ and } X = X^T,$$

where P and Q are permutatiom matrices of order m and n, respectively, and D is a strict signing of order m.

Proof. Since a real matrix X is a sign-central matrix if and only if each matrix in $Q(X)$ is sign-central matrix, without loss of generality, let X be a sign-pattern matrix. Then, we have an immediate consequence by the above lemmas and theorems. ■

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G.-Y. LEE
DEPARTMENT OF MATHEMATICS, HANSEO UNIVERSITY CHUNG-NAM 356-820, KOREA

S.-G. LEE
DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON, 440-746, KOREA

S.-Z. SONG
DEPARTMENT OF MATHEMATICS, CHEJU NATIONAL UNIVERSITY, CHEJU 690-756, KOREA