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AN ERROR OF THE METHOD OF VANISHING VISCOSITY OF THE FIRST-ORDER HAMILTON-JACOBI EQUATIONS

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1. Introduction

In this paper, we study an error of vanishing viscosity method for the first-order Hamilton-Jacobi equations.

$$\text{(H-J)} \qquad u(x) + f(\nabla u(x)) = 0, \quad x \in \mathbb{R}^N,$$

It is well known that (H-J) does not have a classical solution even though the Hamiltonian f is smooth. However Crandall and Lions [1] introduced the class of viscosity solutions, which turns out to be the correct class of generalized solutions for such type of equations. They also showed the uniqueness of generalized solutions under a viscosity condition. The book by Lions [3] and a paper by Souganidis [4] provided a view of the scope of the references to much of the recent literature. Hong [2] showed some regularity result for Hamilton-Jacobi equations.

2. Viscosity solutions of (H-J)

The general reference for this section is [3].

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DEFINITION 2.1 (EXISTENCE AND UNIQUENESS). A *viscosity subsolution* (respectively, *supersolution*) of (H-J) with $f \in C(\mathbb{R}^N)$ is a bounded function $u \in C(\mathbb{R}^N)$ such that for every $\phi \in C^1(\mathbb{R}^N)$:

(2.1.1) If x_0 is a local maximum point of $u - \phi$ on $\mathbb{R}^N$ then $u(x_0) + f(\nabla\phi(x_0)) \leq 0$.

(respectively,

(2.1.2) If x_0 is a local minimum point of $u - \phi$ on $\mathbb{R}^N$, then $u(x_0) + f(\nabla\phi(x_0)) \geq 0$.)

DEFINITION 2.2. A *viscosity solution* of (H-J) is a bounded function $u \in C(\mathbb{R}^N)$ for which both (2.1.1) and (2.1.2) hold (i.e. u is both a viscosity subsolution and a viscosity supersolution).

Remark. If u is a bounded classical solution of (H-J), then it is a viscosity solution, and if u is a viscosity solution of (H-J), then $u(x_0) + f(\nabla u(x_0)) = 0$ at any point x_0 where u is differentiable.

3. The vanishing viscosity method and viscosity solutions of (H-J)

If we consider the problem

(V-V) $$u^\epsilon + f^\epsilon(\nabla u^\epsilon) - \epsilon\Delta u^\epsilon = 0 \quad \text{on} \quad \mathbb{R}^N,$$

then it is well known that this problem has the solution, $u^\epsilon \in C^2(\mathbb{R}^N)$ if $f^\epsilon \in W^{1,\infty}(\mathbb{R}^N)$ and this method is referrred to be the method of vanishing viscosity because of the viscosity term $\epsilon\Delta u^\epsilon$; see Lions [3]. Moreover Crandall and Lions [1] showed the convergence of the method of vanishing viscosity.

THEOREM 3.1 (CONVERGENCE). *Let $u^\epsilon \in C^2(\mathbb{R}^N)$ be a solution of* (V-V) *where $f^\epsilon \to f$ as $\epsilon \to 0$ in $C(\mathbb{R}^N)$. Assume u^ϵ converges uniformly to u as $\epsilon \to 0$. Then u is the viscosity solution of* (H-J). *Moreover if u is bounded, then so is u^ϵ.*

Proof. See Crandall and Lions [1], and Lions [3].

4. An error of the method of vanishing viscosity

In this section, we prove an error of the vanishing viscosity that is the main result of this paper. To prove this, we use the following Lemma.

LEMMA 4.1. *Suppose that f is bounded and Lipschitz continuous on $\mathbb{R}^N$. Let u and u^ϵ be solutions of* (H-J) *and* (V-V) *respectively. Assume that*

$$\text{(A)} \qquad \sup_{|x|\geq R} |u(x)| \quad \text{and} \quad \sup_{|x|\geq R} |u^\epsilon(x)| \to 0 \quad \text{as } R \to \infty.$$

Let $\eta(z)$ be a smooth nonnegative function on $\mathbb{R}$ such that $\eta(-z) = \eta(z)$, $0 \leq \eta(z) \leq 1$, $\eta(0) = 1$ and $\eta(z) = 0$ if $|z| > 1$, and let $M = \max\{\|u\|_{L^\infty(\mathbb{R}^N)}, \|u^\epsilon\|_{L^\infty(\mathbb{R}^N)}\}$. Suppose that

$$\sigma := \|u(x) - u^\epsilon(x)\|_{L^\infty(\mathbb{R}^N)}.$$

For any $\delta > 0$, if we define

$$\psi(x,y) = u(x) - u^\epsilon(y) + (3M + \frac{\sigma}{2})\beta_\delta(x-y),$$

where $\beta_\delta(x)$ is defined on $\mathbb{R}^N$ by $\beta_\delta(x) = \prod_{i=1}^N \eta(\frac{x_i}{\delta})$, then there exists a point $(x_0, y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that $\psi(x_0,y_0) \geq \psi(x,y)$ on $\mathbb{R}^N \times \mathbb{R}^N$.

Proof. Fix $\delta > 0$. Let a sequence $\{(x_i,y_i)\}_{i\geq 1}$ in $\mathbb{R}^N \times \mathbb{R}^N$ be such that

$$\text{(4.1.1)} \qquad \psi(x_i,y_i) \to \sup_{\mathbb{R}^N\times\mathbb{R}^N} \psi,$$

then (x_i,y_i) remains bounded by the following arguments.

First,

$$\begin{aligned}\sup_{\mathbb{R}^N\times\mathbb{R}^N} \psi &\geq u(x) - u^\epsilon(x) + (3M + \frac{\sigma}{2})\beta_\delta(x-x) \\ &= u(x) - u^\epsilon(x) + 3M + \frac{\sigma}{2} \quad \text{for all } x \in \mathbb{R}^N.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\mathbb{R}^N\times\mathbb{R}^N}\psi &\geq \sup_{\mathbb{R}^N}(u(x)-u^\epsilon(x))+3M+\frac{\sigma}{2}\\ &=\sigma+3M+\frac{\sigma}{2}\\ &=3M+\frac{3}{2}\sigma.\end{aligned}\tag{4.1.2}$$

If $\beta_\delta(x-y)=0$, then

$$\begin{aligned}\psi(x,y)&=u(x)-u^\epsilon(y)\\ &\leq 2M.\end{aligned}$$

Hence, (4.1.1) implies that $\beta_\delta(x_i-y_i)>0$ for large i, hence

$$|x_i-y_i|<\sqrt{(x_{i_1}-y_{i_1})^2+(x_{i_2}-y_{i_2})^2+\ldots+(x_{i_N}-y_{i_N})^2}\leq\sqrt{N}\delta.$$

If $|x_i|\to\infty$ and $|y_i|\to\infty$, then

$$\lim_{i\to\infty}\sup_{\mathbb{R}^N\times\mathbb{R}^N}\psi(x_i,y_i)\leq 3M+\frac{\sigma}{2}\quad\text{by A.}$$

This contradicts (4.1.1) and (4.1.2). Therefore, $\{(x_i,y_i)\}_{i\geq1}$ is a bounded sequence and there is a convergent subsequence of $\{(x_i,y_i)\}_{i\geq1}$. Let (x_0,y_0) be the limit of the above subsequence. This completes the proof.

THEOREM 4.2 (MAIN RESULT). *Let a bounded continuous function u be the viscosity solution of*

$$u(x)+f(\nabla u(x))=0,\quad x\in\mathbb{R}^N.$$

Suppose that, for any $\epsilon>0$, u^ϵ is the solution of

$$u^\epsilon(x)+f^\epsilon(\nabla u^\epsilon(x))-\epsilon\Delta u^\epsilon(x)=0,\quad x\in\mathbb{R}^N.$$

If u and u^ϵ satisfy (A), *then*

$$\|u-u^\epsilon\|_{L^\infty(\mathbb{R}^N)}\leq\|f-f^\epsilon\|_{L^\infty(\mathbb{R}^N)}+C\sqrt{\epsilon}$$

where C will be specified later.

Proof. If $\sigma = 0$, then done. Otherwise, by Lemma 4.1, for any $\delta > 0$ we can find a point $(x_0, y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

$$u(x) - \Big(u^\epsilon(y_0) - (3M + \frac{\sigma}{2})\beta_\delta(x - y_0)\Big)$$

attains a local maximum at $x = x_0$, whence

$$u(x_0) + f\Big(-(3M + \frac{\sigma}{2})\nabla_x\beta_\delta(x_0 - y_0)\Big) \leq 0. \tag{4.2.1}$$

On the other hand, since $-\psi(x_0, y) = -u(x_0) + u^\epsilon(y) - (3M + \frac{\sigma}{2})\beta_\delta(x_0 - y)$ attains a local minimum at $y = y_0$ and $u^\epsilon \in C^2$,

$$\nabla u^\epsilon(y_0) - (3M + \frac{\sigma}{2})\nabla_y\beta_\delta(x_0 - y_0) = 0.$$

That is, since $\nabla_x\beta_\delta(x_0 - y_0) = -\nabla_y\beta_\delta(x_0 - y_0)$,

$$\nabla u^\epsilon(y_0) + (3M + \frac{\sigma}{2})\nabla_x\beta_\delta(x_0 - y_0) = 0. \tag{4.2.2}$$

Moreover one can easily see that

$$\Delta u^\epsilon(y_0) - (3M + \frac{\sigma}{2})(\Delta\beta_\delta)(x_0 - y_0) \geq 0. \tag{4.2.3}$$

Now we proceed by $u^\epsilon + f^\epsilon(\nabla u^\epsilon) - \epsilon\Delta u^\epsilon = 0$, (4.2.2) and (4.2.3) to deduce that

$$\begin{aligned} -u^\epsilon(y_0) =& f^\epsilon(\nabla u^\epsilon(y_0)) - \epsilon\Delta u^\epsilon(y_0) \\ \leq& f^\epsilon(-(3M + \frac{\sigma}{2})\nabla_x\beta_\delta(x_0 - y_0)) \\ &- \epsilon(3M + \frac{\sigma}{2})(\Delta\beta_\delta)(x_0 - y_0). \end{aligned} \tag{4.2.4}$$

Combining (4.2.1) and (4.2.4) gives

$$\begin{aligned} u(x_0) - u^\epsilon(y_0) \leq& f^\epsilon(-(3M + \frac{\sigma}{2})\nabla_x\beta_\delta(x_0 - y_0)) \\ &- f(-(3M + \frac{\sigma}{2})\nabla_x\beta_\delta(x_0 - y_0)) \\ &- \epsilon(3M + \frac{\sigma}{2})(\Delta\beta_\delta)(x_0 - y_0). \end{aligned}$$

Now, for all $x \in \mathbb{R}^N$,

$$\begin{aligned} u(x) - u^\epsilon(x) + 3M + \frac{\sigma}{2} =& \psi(x, x) \\ \le& \psi(x_0, y_0) \\ \le& u(x_0) - u^\epsilon(y_0) + 3M + \frac{\sigma}{2}. \end{aligned}$$

Hence

$$\begin{aligned} u(x) - u^\epsilon(x) \le& u(x_0) - u^\epsilon(y_0) \\ \le& f^\epsilon(-(3M + \frac{\sigma}{2})\nabla_x \beta_\delta(x_0 - y_0)) \\ &- f(-(3M + \frac{\sigma}{2})\nabla_x \beta_\delta(x_0 - y_0)) \\ &- \epsilon(3M + \frac{\sigma}{2})(\Delta\beta_\delta)(x_0 - y_0). \end{aligned}$$

Therefore

$$\sigma \le \|f - f^\epsilon\|_{L^\infty(\mathbb{R}^N)} + \epsilon(3M + \frac{\sigma}{2})|(\Delta\beta_\delta)(x_0 - y_0)|.$$

Since $\sigma \le \|u\|_{L^\infty(\mathbb{R}^N)} + \|u^\epsilon\|_{L^\infty(\mathbb{R}^N)} \le 2M$, we have

$$\|u - u^\epsilon\|_{L^\infty(\mathbb{R}^N)} \le \|f - f^\epsilon\|_{L^\infty(\mathbb{R}^N)} + \frac{C\epsilon}{\delta^2}$$

where C is depending on M and β_δ. Setting $\delta = \epsilon^{\frac{1}{4}}$ completes the proof.

5. Remark

For simplicity, we impose the assumption (A). We will drop (A) in a later paper. We expect that its proof will be quite different and more complicated of course.

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