Bull. Korean Math. Soc. **34**(1997), No. 1, pp. 29-34

ON CERTAIN ANALYTIC FUNCTIONS WITH POSITIVE REAL PART

EUGEN DRĂGHICI

1. Introduction

Let $U = \{z \in C : |z| < 1\}$ be the open unit disc in the complex plane and let $\mathcal{N}$ be the class of all analytic functions p in U with $p(0) = 1$. A function $p \in \mathcal{N}$ is called a Carathéodory function if $\text{Re}p(z) > 0$ in U. Let now denote by A the class of all analytic functions f in U with $f(0) = f'(0) - 1 = 0$, A function $f \in A$ is said to be starlike in U if:

$$\text{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in U.$$

A function $f \in A$ is said to be with bounded argument rotation in U if:

$$\text{Re}f'(z) > 0,\ z \in U.$$

Let S^* be the class of starlike functions in U and $\mathcal{R}$ the class of functions with bounded argument rotation in U. It is well–known (cf. Duren [1]) that all the functions in S^* and in $\mathcal{R}$ are univalent and $\mathcal{R}$ is not a subclass of S^* (cf. Krzys [2]). Also, $\mathcal{R}$ is a subclass of the class of close–to–convex functions in U.

In [6] Nunokawa showed that $p \in \mathcal{N}$ is a Carathéodory–function if:

$$\left|\text{Im}\frac{zp'(z)}{p(z)}\right| < 1,\ \ z \in U \tag{1.1}$$

This result was improved in [7] by S.Owa and J.Kang. They showed that:

Received February 21, 1996.
1991 AMS Subject Classification: 30C45.
Key words and phrases: subordination, Carathéodory function.

THEOREM 1.1. *If* $p \in \mathcal{N}$ *satisfies:*

$$\frac{z'(z)}{p(z)} \neq i\alpha, \; z \in I$$

then $\mathrm{Re}p(z) > 0, \; z \in U$, *where* α *is real and* $|\alpha| \geq 1$

DEFINITION 1. 1 [1]. if f and g are analytic functions in U and g is univalent, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(U) \subset g(U)$.

DEFINITION 1.2 [5]. Let λ be a complex number such that $\mathrm{Re}\lambda > 0$ and let

$$N = N(\lambda) = \frac{|\lambda|\sqrt{1 + 2\mathrm{Re}\lambda} + \mathrm{Im}\lambda}{\mathrm{Re}\lambda}$$

If h is the univalent function defined by $h(z) = 2Nz/(1 - z^2)$ and $b = h^{-1}(\lambda)$, then we define the "open door" function Q_λ as:

$$Q_\lambda(z) = h[(z + b)/(1 + \bar{b}z)], \; z \in U$$

It is clear that Q_λ is univalent, $Q_\lambda(0) = \lambda$ and $Q_\lambda(U) = h(U)$ is the complex plane slit along the half–lines $\mathrm{Re}w = 0, \mathrm{Im}w \geq N$ and $\mathrm{Re}w = 0,, \mathrm{Im}w \leq -N$.

The reason for the title "open door" will become apparent in the following theorem:

THEOREM 1.2 [5]. *Let* Q_λ *be the function given by (2.3) and let* $B(z)$ *be an analytic function in* U *satisfying:*

$$B(z) \prec Q_\lambda(z), \; z \in U$$

If p *is analytic in* U, $p(0) = 1/\lambda$, *and* p *satisfies the differential equation:*

$$zp'(z) + B(z)p(z) = 1$$

then $\mathrm{Re}p(z) > 0$ *in* U.

This lemma was first proved by P.T.Mocanu in [5]. More details can be found in ([4], **Definition 7.1, Lemma 7.2** and **Theorem 7.3**). The purpose of this paper is to generalize **Theorem 1.1** and **Theorem 1.2** by using a differential–subordination technique.

2. Preliminaries

In order to prove our main results we will need the following definition and lemma.

DEFINITION 2.1 [3]. Let Ω be a set in $\mathbf{C}$. We define $\Psi[\Omega]$ to be the class of all functions $\psi: \mathbf{C}^2 \times U \to \mathbf{C}$ that satisfy the following condition:

$$\psi\,(\mathrm{i}x, y; z) \notin \Omega \quad \text{for all} \quad z \in U \quad \text{and} \quad x, y \in \mathbf{R}\,,\; y \leq -\frac{1+x^2}{2}.$$

$\Psi[\Omega]$ is called the class of admissible functions. More details can be found in ([4] **Definition 3.1**).

LEMMA 2.1 [3]. *Let $p(z) = 1 + p_1 z + \cdots$ be analytic in U. If $\psi \in \Psi[\Omega]$, then the following implication holds:*

$$\psi[p(z), zp'(z); z] \in \Omega \Longrightarrow \mathrm{Re}p(z) > 0, \quad z \in U$$

More general forms of this lemma can be found in [4].

3. Principal result

THEOREM 3.1. *Let b, c, d be real numbers, $b > -1/2$, $c > 0$, $d \geq 0$ and let*

$$K = \frac{\sqrt{d^2 + c^2(2b+1)}}{c^2}$$

If $p \in \mathcal{N}$ satisfies:

$$\frac{b - zp'(z)}{cp(z) + d\mathrm{i}} \neq \mathrm{i}\alpha, \; z \in U \tag{3.1}$$

for all real α with $\left|\alpha - \frac{d}{c^2}\right| \geq K$, then $\mathrm{Re}p(z) > 0$ in U.

Proof. Let $\psi : \mathbf{C}^2 \times U \to \mathbf{C}$, $\psi(u, v; z) = (b - v)/(cu + d\mathrm{i})$. Then:

$$\psi(\, ix, y; z) = \; i\frac{y-b}{cx+d}.$$

If we let $x \in \mathbf{R}$ and $y \leq -(1+x^2)/2$ we immediately obtain:

$$\mathrm{Re}\psi(\mathrm{i}x, y; z) = 0 \quad \text{and} \quad \mathrm{Im}\psi(\mathrm{i}x, y; z) = \frac{y-b}{cx+d}.$$

If $cx + d > 0$ then:

$$\mathrm{Im}\psi(\, \mathrm{i}x, y; z) \leq \frac{-\frac{1+x^2}{2} - b}{cx+d}.$$

The maximum of the right–hand side of this inequality is $d/c^2 - K$, and thus, $\mathrm{Im}\psi(\mathrm{i}x, y; z)$ has the upper bound $d/c^2 - K$. Using a similar argument, we finally obtain that:

$$\left|\mathrm{Im}\psi(\mathrm{i}x, y; z) - \frac{d}{c^2}\right| \geq K$$

for all real x and $y \leq -(1+x^2)/2$.

Let Ω be the complex plane slit along the half–lines $\mathrm{Re}w = a, \mathrm{Im}w \geq \frac{d}{c^2} + K > 0$ and $\mathrm{Re}w = 0, \mathrm{Im}w \leq \frac{d}{c^2} - K < 0$. From (3.1) we have:

$$\psi(\mathrm{i}x, y; z) \notin \Omega \quad \text{for all} \quad z \in U \quad \text{and} \quad y \leq -\frac{1+x^2}{2}$$

From **Definition 2.2** we have that $\psi \in \Psi[\Omega]$.

Suppose now that $cp(z) + d\mathrm{i}$ has a zero of order $n \in \mathbf{N}$ at $z = \beta$. Then:

$$cp(z) + d\mathrm{i} = (z - \beta)^n q(z)$$

where $q(z) \neq 0$ in a neighbourhood V of β. A simple computation shows that:

$$\text{(3.2)} \qquad \frac{b - zp'(z)}{cp(z) + d\mathrm{i}} = \frac{b}{(z-\beta)^n q(z)} - \frac{nz}{c(z-\beta)} - \frac{zq'(z)}{cq(z)}, \; z \in V$$

Because the right–hand side of (3.3) can take any values when z tends to β, we contradict condition (3.1) and obtain that $cp(z) + di \neq 0$ in U. Thus, we can write:

$$\psi[p(z), zp'(z); z] = \frac{b - zp'(z)}{cp(z) + di}$$

From (3.1) we deduce that $\psi[p(z), zp'(z); z] \in \Omega$. Because $\psi \in \Psi[\Omega]$, by applying **Lemma 2.1** we conclude that $\mathrm{Re}p(z) > 0$ in U and the theorem is completely proved.

4. Some particular cases

1. If we let in **Theorem 3.1** $b = d = 0$ and $c = 1$, we obtain **Theorem 1.1** by S.Owa and J.Kang [7] which improves the result of M.Nunokawa in[6] cited in (1.1).

2. If we let in **Theorem 3.1** $b = c = 1$ and $d = 0$ we obtain **Theorem 1.2** of P.T.Mocanu [5], for $\lambda = 1$.

3. If we let in **Theorem 3.1** $p(z) = zf'(z)/f(z)$ for a function $f \in A$ and $c = 1, d = 0$ we obtain the following result:

COROLLARY 4.1. *If $f \in A$, $b > -1/2$ and:*

$$b\frac{f(z)}{zf'(z)} + \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \neq 1 + i\alpha, \; z \in U$$

for all real α with $|\alpha| \geq \sqrt{2b+1}$, then $f \in S^*$.

4. If we let in **Theorem 3.1** $p(z) = f'(z)$ for $f \in A$ and $c = 1, d = 0$ we obtain the following result:

COROLLARY 4.1. *If $f \in A$, $b > -1/2$ and:*

$$\frac{b - zf''(z)}{f'(z)} \neq i\alpha, \; z \in U$$

for all real α with $\alpha \geq \sqrt{2b+1}$, then $f \in \mathcal{R}$.

References

1. P.L.Duren, *Univalent functions*, Springer Verlag, 1983.
2. J. Krzys, *A counterexemple concerning univalent functions*, Folia. Soc. Scient. Lubliniensis **2** (1962), 57-58.
3. S.S.Miller, P.T.Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157-171.
4. ———, *The theory and applications of second-order differential subordinations*, Studia Univ.Babeş–Bolyai k(Cluj–Romania), Mathematica, **XXXIV** (1989), 3-33.
5. P.T.Mocanu, *Some integral operators and starlike functions*, Rev. Roumaine Math. Purres Appl. **31** (1986), 231-235.
6. M.Nunokawa, *On properties of Non-Carathéodory functions*, Proc. Japan Acad. **68** (1992), 152-153.
7. S.Owa, J.Kang, *A property of certain analytic functions*, Bull. Korean Math. Soc. **32** (1995), 201-204.

Department of Mathematics, University "Lucian Blaga" of Sibiu, 2400 Sibiu, Romania