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# TOTALLY UMBILIC LORENTZIAN SURFACES EMBEDDED IN $L^n$

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## 1. Introduction

Define $\overline{g}(v,w) = -v_1w_1 + \cdots + v_nw_n$ for $v$ , $w$ in $R^n$. $R^n$ together with this metric is called the *Lorentzian n-space*, denoted by $L^n$, and $R^n$ together with the Euclidean metric is called the *Euclidean n-space*, denoted by $E^n$. A *Lorentzian surface* in $L^n$ means an orientable connected 2-dimensional Lorentzian submanifold of $L^n$ equipped with the induced Lorentzian metric $g$ from $\overline{g}$.

Let $M$ be a Lorentzian surface in $L^n$, D the flat Levi-Civita connection on $L^n$, $\nabla$ the induced connection on $M$, and $h$ the second fundamental form on $M$. A point $p$ of $M$ is *umbilic* if there is a normal vector $z$ such that $h(v,w) = g(v,w)z$ for all $v$, $w$ in $T_pM$. The $z$ is called the *normal curvature vector* of $M$ at $p$ . A Lorentzian surface is *totally umblic* provided every point of $M$ is umblic. Note that if $M$ is totally umblic, then there is a smooth normal vector field $Z$ on $M$, called *the normal curvature vector field* of $M$ such that $h(V,W) = g(V,W)Z$ for all $C^\infty$ tangent vector fields $V, W$ on $M$.

Our purpose is to show that there is only one kind of nontrivial totally umbilic Lorentzian surfaces in $L^n$, say pseudospheres in $L^3 \subset L^n$.

## 2. Main Theorems

Let $M$ be a Lorentzian surface in $L^n$. Note that $T_p(M)$ is a 2 dimensional subspace of $T_pL^n \cong L^n$ for any $p \in M$. At first, consider the

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causal characters of $T_p(M)$ and its orthogonal complement with respect to $\overline{g}$ in $L^n$.

THEOREM 2.1. *Let $V$ be a $k$-dimensional subspace of $L^n$. Then exactly one of the following is true:*

(1) *$V = L^k$, and $\overline{g} \mid V$ is nondegenerate,*
(2) *$V = E^k$, and $\overline{g} \mid V$ is nondegenerate,*
(3) *$\overline{g} \mid V$ is degenerate, and in this case (and only in this case) we may write $V = E^{k-1} \oplus Span\xi$ , where $\overline{g}(\xi,\xi) = 0$ and $\xi$ is orthogonal to $E^{k-1}$.*

*Proof.* See [1].

According to Theorem 2.1, we know that $T_p(M)$ is isomorphic to $L^2$ and its orthogonal complement is isomorphic to $E^{n-2}$.

It is well known that local isothermal coordinates on $M$ compatible with the orientation of $M$ can be always given. By $(x,y)$, we always denote isothermal coordinates compatible with the orientation on $M$ with $\frac{\partial}{\partial x}$ timelike.

DEFINITION. Let $n \geq 2$. A pseudosphere of radius $r > 0$ in $L^{n+1}$ is the hyperquadric

$$S_1^n(p,r) = \{q \in L^{n+1} \mid \overline{g}(q-p, q-p) = r^2\}$$

with dimension $n$ and index 1.

A totally umbilic Lorentzian surface in $L^3$ can be easily classified using elementary techniques as in the proof of Theorem 2.2.

THEOREM 2.2. *If $M$ is a connected totally umbilic Lorentzian surface in $L^3$, then $M$ is a portion of a timelike plane or a pseudosphere.*

*Proof.* Choose an isothermal parameter $(x,y)$ so that $M$ is defined locally by a map $X(x,y) = (x_1, x_2, x_3) \in L^3$. Denote $X_x$ by $\partial_1$ and $X_y$ by $\partial_2$. Then

$$D_{\partial_i}\partial_j = \nabla_{\partial_i}\partial_j + h(\partial_i, \partial_j).$$

Let $N$ be the unit normal vector field on $M$. By Theorem 2.1, it must be spacelike. Note that it may not be defined globally if $M$ is not orientable (and time-orientable.) Consider the smooth function $f = \overline{g}(N, Z)$, where $Z$ is a normal curvature vector field on $M$ such that $h(V, W) = g(V, W)Z$ for any smooth tangent vector fields $V$, $W$ on $M$. Note that $Z = fN$. Since $0 = \partial_j \overline{g}(N, N) = 2\overline{g}(N, D_{\partial_j} N)$, $D_{\partial_j} N$ is a local smooth tangent vector field on $M$. Since

$$\overline{g}(D_{\partial_i}\partial_j, N) = f\overline{g}(\partial_i, \partial_j)$$

and

$$0 = \overline{g}(D_{\partial_i}\partial_j, N) + \overline{g}(\partial_j, D_{\partial_i} N) \ ,$$

we have

$$\overline{g}(D_{\partial_i} N, \partial_j) = -f\overline{g}(\partial_i, \partial_j)$$

for $i$, $j = 1, 2$. Therefore, $D_{\partial_i} N = -f\partial_i$.

Consequently,

$$D_{\partial_j} D_{\partial_i} N = -\partial_j(f)\partial_i - fD_{\partial_j}\partial_i$$

and

$$D_{\partial_i} D_{\partial_j} N = -\partial_i(f)\partial_j - fD_{\partial_i}\partial_j \ .$$

From the above we have

$$(\partial_j f)\partial_i = (\partial_i f)\partial_j \ .$$

Then the linear independency of $\partial_1$ and $\partial_2$ tells us that $\nabla f = 0$. Hence $f$ is constant.

$f \equiv 0$ implies $Z \equiv 0$ and in turn $M$ is locally a plane.

When $f \equiv c$ ($c \neq 0$), consider $N$ and $\partial_i$ as a smooth function from an open set $U \subset R^2$ to $L^3$. Since $D_{\partial_i} N = -c\,\partial_i$, $N = -cX + v_0$ for some $v_0 \in L^3$.

From this we obtain

$$X(x, y) = \frac{N}{c} - \frac{v_0}{c}$$

and

$$\overline{g}(X(x,y) - \frac{v_0}{c}, X(x,y) - \frac{v_0}{c}) = \overline{g}(-\frac{N}{c}, -\frac{N}{c}) = \frac{1}{c^2}$$

Therefore $X$ lies in a pseudosphere $S_1^2(\frac{v_0}{c}, \frac{1}{|c|})$. This local argument can be extended to the global argument using continuation along a path in $M$. □

Now we know there is only one kind of nontrivial Lorentzian surfaces which are totally umbilic in $L^3$. But what about a totally umbilic Lorentzian surface in $L^n (n > 3)$? The next theorem tells us that only pseudosperes in 3-dimensional affine subspaces of $L^n$ are the nontrivial totally umbilic Lorentzian surfaces in $L^n$.

THEOREM 2.3. *Let $M$ be a totally umbilic Lorentzian surface in $L^n$ $(n > 3)$. Then it is in fact either a timelike affine plane isomorphic to $L^2$ or a pseudosphere in a 3-dimensional affine subspace isomorphic to $L^3$ in $L^n$.*

To prove the theorem, we need to know about the Lorentzian version of the structural equations and parallel distribution along $M$.

PROPOSITION 2.4. *Let $e^1, \cdots, e^n$ be an orthonormal moving frame on $L^n$ and let $\phi^i$'s be the dual 1-forms, where $e^1$ is timelike. Then there exist unique 1-forms $\omega_{ij}$ (called the connection forms) such that*

(1) $\omega_{ij} = -\omega_{ji}$,
(2) $d\phi^i = -\sum_k \varepsilon_i \omega_{ik} \wedge \phi^k$,
(3) $d\omega_{ij} = -\sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj}$,

*where $\varepsilon_1 = -1$ and $\varepsilon_j = 1$ if $j \neq 1$.*

*Proof.* Define $\omega_{ij}$ by $\omega_{ij}(X) = \overline{g}(e^i, D_X e^j)$. Since

$$0 = \overline{g}(e^i, D_X e^j) + \overline{g}(D_X e^i, e^j) \quad ,$$

we have $\omega_{ij}(X) = -\omega_{ji}(X)$ for any $C^\infty$ vector field $X$ in $L^n$.

Consider the moving frames $e^1, \cdots, e^n$, with a little abuse of notation, as an $R^n$-valued function $e^i : R^n \longrightarrow R^n$. Then we can consider

$dI$ and $de^i$'s as $R^n$-valued 1-forms. Since $d^2 = 0$, we have

$$\begin{aligned}
0 &= d^2 I \\
&= d\Big(\sum_i \phi^i \wedge e^i\Big) \\
&= \sum_i (d\phi^i) e^i - \sum_k \phi^k \wedge de^k \\
&= \sum_i (d\phi^i) e^i - \sum_k \phi^k \wedge \sum_i \varepsilon_i \omega_{ik} e^i \\
&= \sum_i \Big(d\phi^i - \sum_k \varepsilon_i \phi^k \wedge \omega_{ik}\Big) e^i
\end{aligned}$$

Setting the coefficient of each $e^i$ equal to 0, we obtain

$$d\phi^i = -\sum_k \varepsilon_i \omega_{ik} \wedge \phi^k$$

We also have

$$\begin{aligned}
0 &= d^2 e^j \\
&= \sum_i \varepsilon_i d\omega_{ij} e^i - \sum_k \varepsilon_k \omega_{kj} \wedge de^k \\
&= \sum_i \Big(\varepsilon_i d\omega_{ij} - \sum_k \varepsilon_i \varepsilon_k \omega_{kj} \wedge \omega_{ik}\Big) e^i \qquad ,
\end{aligned}$$

from which we immediately deduce

$$d\omega_{ij} = -\sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} \qquad \square$$

LEMMA 2.5. *Let $Z$ be a normal curvature vector field on a totally umbilic Lorentzian surface $M$ in $L^n$. Then $\bar{g}(Z, Z)$ is constant everywhere on $M$.*

*Proof.* Since $Z$ is a $C^\infty$ spacelike vector field in $L^n$ on $M$ by Theorem 2.1, $\overline{g}(Z,Z)$ is a nonnegative $C^\infty$ function.

If $\overline{g}(Z,Z)$ vanishes everywhere, then there is nothing to prove.

Suppose there is $p \in M$ such that $\overline{g}(Z,Z)(p) > 0$. In a neighborhood of $p$, we choose an orthonomal moving frame $e^1, e^2$ on $M$, where $e^1$ is timelike, and complete to an adapted orthonormal moving frame $e^1, \cdots, e^n$ with the unit spacelike vector field $e^3$ in $Z$-direction. Then, for $j = 1, 2$ and $X$ tangent to $M$, we have

$$\begin{aligned} \omega_{ij}(X) &= \overline{g}(e^i, D_X e^j) \\ &= \begin{cases} \sqrt{\overline{g}(Z,Z)}\,\overline{g}(X, e_j) & i = 3 \\ 0 & i > 3 \end{cases} \end{aligned}$$

which means that on $TM$ we have

$$\begin{aligned} \omega_{3j} &= \varepsilon_j \sqrt{\overline{g}(Z,Z)}\,\phi^j \\ \omega_{ij} &= 0, \qquad \text{if } i > 3 \end{aligned} \tag{1}$$

Denote $\sqrt{\overline{g}(Z,Z)}$ by $\lambda$. From (1) and the second structural equation we find that on $TM$ we have

$$\begin{aligned} d\omega_{3j} &= \varepsilon_j d\lambda \wedge \phi^j + \varepsilon_j \lambda d\phi^j \\ &= -\sum_k \varepsilon_k \omega_{3k} \wedge \omega_{kj} \\ &= -\lambda \sum_{k=1}^{2} \phi^k \wedge \omega_{kj} \end{aligned} \tag{2}$$

while the first structual equation gives

$$\begin{aligned} d\phi^j &= -\sum_{k=1}^{2} \varepsilon_j \omega_{jk} \wedge \phi^k \\ &= \sum_{k=1}^{2} \varepsilon_j \phi^k \wedge \omega_{jk} \\ &= -\varepsilon_j \sum_{k=1}^{2} \phi^k \wedge \omega_{kj} \qquad , \end{aligned} \tag{3}$$

so we find that

$$\varepsilon_j d\lambda \wedge \phi^j = 0 \qquad \text{for } j = 1, 2.$$

Hence $d\lambda \equiv 0$ and so $\sqrt{\overline{g}(Z,Z)}$ is constant around $p$. This implies that $\{q \in M \mid \overline{g}(Z,Z)(q) = \overline{g}(Z,Z)(p)\}$ is a nonempty open and closed subset of $M$. Therefore $\overline{g}(Z,Z)$ is constant everywhere since $M$ is connected. □

LEMMA 2.6. *Let $\Delta$ be a k-dimensional distribution along the curve $c : [a,b] \longrightarrow L^n$ with $\frac{dc}{dt} \in \Delta(t)$. Suppose $\Delta$ is parallel along c. Then c is a curve in some k-dimensional plane $W \subset L^n$, and W is just* $\exp(\Delta(t))$ *for any t.*

*Proof.* Let $W = \Delta(a)$, considered as a k-dimensional plane in $L^n$. Then $W$ is isometric to $L^k$, $E^k$, or $E^{k-1} \oplus span\{\xi\}$, where $\xi$ is a nonzero lightlike vector in $L^n$. Without loss of generality we may assume $W$ is $L^k$, $E^k$ or $H^k = \{(x, x, y_1, \cdots, y_{k-1}, 0, \cdots, 0) \in L^n \mid x, y_i \in R\}$.

Case 1. $W = L^k$.

If $c$ does not lie entirely in $W$, then by the mean value theorem, some tangent vector $c'(t)$ has a nonzero $i$-th component for some $i > k$. But this is impossible, because $c'(t) \in \Delta(t)$ and $\Delta(t)$ is parallel to $W = \Delta(a)$.

Since each $\Delta(t)$ is parallel to $W = \Delta(a)$ and also contains the points c(t) in $W$, each $\Delta(t)$ must be equal to $W$, when $\Delta(t)$ is considered as a k-dimensional plane. In other words, $W = \exp(\Delta(t))$ for all $t$.

Case 2. $W = E^k$.

The exact same proof as in case 1 may be applied here with $W = 0 \oplus E^k$.

Case 3. $W = H^k \subset L^{k+1}$.

Since $c'(t) \in \Delta(t)$ and $\Delta(t)$ is parallel to $W = \Delta(a)$, $c_1'(t) = c_2'(t)$ for any $t$ and $c_i'(t) = 0$ for $i > k+1$, and result is proved in this case. □

We also need the converse of this assertion.

LEMMA 2.7. *Let $\Delta$ be a smooth k-dimensional distribution along $c : [a,b] \longrightarrow L^n$. Suppose the induced covariant derivative $\frac{DV}{dt}$ belongs* *to $\Delta$ whenever $V$ is a smooth vector field along $c$ belonging to $\Delta$. Then $\Delta$ is parallel along $c$.*

*Proof.* The proof given in [5, pp.41–42] works here. □

LEMMA 2.8. *Let $M$ be a connected Lorentzian surface in $L^n$ and let $\Delta$ be a smooth k-dimensional distribution along $M$ such that $T_pM \subset \Delta(p)$ for all $p \in M$. Suppose that $\Delta$ is parallel along every curve $c$ in $M$. Then $M$ lies in some k-dimensional plane $W \subset L^n$.*

*Proof.* Choose a point $p \in M$ and let $W$ be the k-dimensional plane of $L^n$ with $\exp(\Delta(p)) = W$. For any $q \in M$, choose a curve $c : [0,1] \to M \subset L^n$ with $c(0) = p$, and $c(1) = q$. Since $T_pM \subset \Delta(p)$ for all $p \in M$, $c'(t) \in \Delta(c(t))$ for all $t \in [0,1]$. Hence, Lemma 2.6 applied to the distribution $t \to \Delta(c(t))$ along $c$, implies that $c$ lies in the k-dimensional plane $W = \exp(\Delta(0)) \subset L^n$, because $\exp \Delta(c(t)) = W$ for all $t$. (Of course $W$ may be degenerate.) □

Now we are ready to prove the Theorem 2.3.

*Proof of Theorem 2.3.* Let $Z$ be the normal curvature vector field on $M$.

Case 1. $\overline{g}(Z,Z) \equiv 0$ on $M$.

Let $p \in M \subset L^n$. Choose an adapted moving frame $e^1, e^2, \cdots, e^n$ of $L^n$ so that $e^1, e^2$ becomes a moving frame of $M$ around $p$. Then, for $q$ near $p$, $Z(q) = \sum_{k=3}^n Z_k(q)e^k(q)$ and from $0 = \overline{g}(Z,Z)(q) = \sum_{k=3}^n Z_k^2(q)$ we have $Z \equiv 0$ near $p$, and on the whole $M$. Therefore, $M$ is a part of a timelike plane in $L^n$.

Case 2. $\overline{g}(Z,Z)$ is a positive constant function on $M$.

Denote the constant function $\sqrt{\overline{g}(Z,Z)}$ by $\lambda$. By (1),

$$D_X e^3 = \sum_{k=1}^{n} \varepsilon_k \omega_{k3}(X) e^k = \sum_{k=1}^{2} \omega_{k3}(X) e^k = -\lambda X. \tag{4}$$

We also have

$$D_X e^j = -\omega_{1j}(X) e^1 + \sum_{k=2}^{3} \omega_{kj} e^k, \qquad j = 1, 2. \tag{5}$$

Let $\Delta$ be the 3-dimensional $C^\infty$ distribution on $M$ with $\Delta(P) = M_p + \mathbb{R} \cdot e^3(p)$. Equation(4),(5) and Lemma 2.7 shows that $\Delta$ is parallel along every curve lying in $M$. So Lemma 2.8 implies that $M$ lies in a 3-dimensional plane $W$ of $L^n$. Since $\Delta(p) \cong L^3$ and $\exp(\Delta(p)) = W$, we know that $W$ has an index 1 and therfore $W \cong L^3 \subset L^n$.

Next,we have to show that $M$ lies in a pseudosphere of radius $\frac{1}{\lambda}$. Let $P$ be the position vector field on $L^n$. Then $D_X P = X$ for all tangent vector field $X$ to $M$ in $L^n$, and so we can write (4) as

$$D_X(e^3 + \lambda P) = 0$$

Thus the vector field $e^3 + \lambda P$ is parallel along $M$. Identifying tangent vectors of $M$ with elements of $L^n$, this means that $e^3 + \lambda P$ is a constant vector $v_0$ on $M$, so we have $p = \frac{v_o - e^3(p)}{\lambda}$ for all $p \in M$, which means that $M$ lies in pseudosphere with radius $\frac{1}{\lambda}$, center $\frac{v_0}{\lambda}$. This completes the proof. □

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