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ON THE AUTOMORPHISM GROUPS OF CAYLEY MAPS

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1. Introduction

A Cayley map is a connected Cayley graph embedded in some orientable surface with the same local orientation at every point. Let $M = \mathrm{Cay}(G, S, r)$ be a Cayley map, where G is a finite group, S a subset generating G such that $1 \notin S$ and the inverse s^{-1} is contained in S for every s in S, and r a cyclic permutation on S. The underlying graph $X = \mathrm{Cay}(G, S)$ of the map is a connected Cayley graph and the permutation r induces the rotation ρ by $\rho_g(gs) = gr(s)$ for $g \in G$, $s \in S$.

A fundamental problem in the study of Cayley maps is to understand the automorphism groups. However, it seems very difficult to determine the automorphism group of a given map in general. In this paper, we will investigate an accessible case, namely the case when the left regular representation $L(G)$ of G is a normal subgroup of the automorphism group.

2. Background theory

We first recall some basic notions of maps. The presentation is largely based on the treatments of Biggs and White [1].

A *rotation* on a graph $X = (V, E)$ is an assignment of each vertex v in V to a cyclic permutation ρ_v of the vertices adjacent to v.

A *map* is a pair (X, ρ), where X is a connected graph and ρ is a rotation on X.

An *automorphism* of a map $M = (X, \rho)$ is an automorphism α of the graph X such that $\rho_{\alpha(v)} = \alpha\rho_v\alpha^{-1}$ for each vertex v in X.

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Note that Aut$X$, the automorphism group of X, acts on the set of all rotations on X in a natural way; each automorphism of the map M is an automorphism of X which fixes ρ in the action of Aut$X$ on the set of all rotations on X.

We now state the following basic property.

LEMMA 2.1. [1] *Let $M = (X, \rho)$, $\alpha \in$ Aut$M$, and suppose that $\{v, w\}$ is an edge of X. If α fixes both v and w, then α is the identity automorphism.*

Let A_v the set of all elements of Aut$M$ that fix the vertex v. Since each α in A_v is a graph automorphism, it permutes the vertices adjacent to v. Each α in A_v is completely determined by its action on the set $N(v)$ of all vertices adjacent to v. In fact, the restricted permutation $\bar{\alpha}$ on $N(v)$ is equal to ρ_v^k for some integer k and the restriction $\alpha \mapsto \bar{\alpha}$ defines a group-isomorphism from A_1 to the cyclic group $\langle \rho_v \rangle$ generated by ρ_v. We can summarize this observation as follows:

LEMMA 2.2. [1] *Let $A =$ Aut$M$, where M is the map (X, ρ), and let v be a vertex of X. The stabilizer A_v is isomorphic to a subgroup of the cyclic group $\langle \rho_v \rangle$ generated by ρ_v. Thus A_v is a cyclic group of order dividing the valency of v.*

3. The automorphism groups of Cayley maps

Let $M = \mathrm{Cay}(G, S, r)$ be a Cayley map, $X = \mathrm{Cay}(G, S)$ the underlying connected Cayley graph. The automorphism group of M is denoted by Aut$M$, and the subgroup consisting of all automorphisms in Aut$M$ that fix the identity 1 of G is denoted by A_1.

The following lemma is immediate from the definition of automorphisms of Cayley maps.

LEMMA 3.1. *A_1 consists of all permutations on G such that for all $g \in G$ and $s \in S$,*

(1) $\alpha(1) = 1$,

(2) $\alpha(g)^{-1}\alpha(gs) \in S$,

(3) $\alpha(gr(s)) = \alpha(g)r(\alpha(g)^{-1}\alpha(gs))$.

Let $L(G)$ be the left regular representation of G. Note that $L(G)$ is a regular subgroup of the automorphism group Aut$M$ of the Cayley map $M = \mathrm{Cay}(G, S, r)$. Let

$$\mathrm{Aut}(G, S, r) := \{\alpha \in \mathrm{Aut}G | S^\alpha = S,\ \alpha r = r\alpha \text{ on } S\}.$$

Then we have:

THEOREM 3.2. (1) *$L(G)$ is normal in* Aut$M$ *if and only if A_1 is contained in* Aut(G).

(2) Aut$M = L(G)$Aut(G, S, r) *if $L(G)$ is normal in* Aut$M$.

Proof. Since $L(G)$ acts regularly on G, we see that the normalizer of $L(G)$ in the symmetric group on G is the holomorph of G, that is, the semidirect product $L(G)\mathrm{Aut}G$ (see, for example, Corollary 4.2B in [2]). The claim (1) now follows from $\mathrm{Aut}M = L(G)A_1$. For (2), we first show that $A_1 \cap \mathrm{Aut}G = \mathrm{Aut}(G, S, r)$. In fact, from Lemma 3.1, we see that $\alpha(r(s)) = r(\alpha(s))$ for each $\alpha \in A_1$ and $s \in S$; so $A_1 \cap \mathrm{Aut}G \leq \mathrm{Aut}(G, S, r)$. It is also obvious that if $\alpha \in \mathrm{Aut}(G, S, r)$ then α satisfies the conditions (1), (2) and (3) in Lemma 3.1; so $\alpha \in A_1$, that is, $\mathrm{Aut}(G, S, r) \leq A_1$. Now we suppose that $L(G)$ is normal in Aut$M$. Then by (1), we have $A_1 \leq \mathrm{Aut}G$; so $A_1 = \mathrm{Aut}(G, S, r)$. Since $A = L(G)A_1$, we have $A = L(G)\mathrm{Aut}(G, S, r)$, which proves (2). □

We also prove the following theorem.

THEOREM 3.3. *$L(G)$ is normal in* Aut$M$ *if and only if $\alpha(s^{-1}) = \alpha(s)^{-1}$ for all $\alpha \in A_1$ and $s \in S$.*

Proof. Let α be an element in A_1. By Theorem 3.2(1), we only need to show that α is a homomorphism of G if $\alpha(s^{-1}) = \alpha(s)^{-1}$ for all $s \in S$. We want to prove that $\alpha(s_1 s_2 \cdots s_n) = \alpha(s_1)\alpha(s_2)\cdots\alpha(s_n)$ where $s_1, s_2, \cdots, s_n \in S$, by induction on n. To start the induction we note that it is vacuously true for $n = 1$. So we assume that $\alpha(s_1 s_2 \cdots s_m) = \alpha(s_1)\alpha(s_2)\cdots\alpha(s_m)$ for $m < n$. Write t for the inverse of s_{n-1}. Since r is a cyclic permutation on S, we have $s_n = r^k(t)$ for some nonnegative integer k. By Lemma 3.1 (3), $\alpha(gr^k(t)) = \alpha(g)r^k(\alpha(g)^{-1}\alpha(gt))$ for every $g \in G$. Thus we have

$$\begin{aligned}\alpha(s_1 s_2 \cdots s_n) &= \alpha(s_1 \cdots s_{n-1} r^k(t)) \\ &= \alpha(s_1) \cdots \alpha(s_{n-1}) r^k(\alpha(s_1 \cdots s_{n-1})^{-1} \alpha(s_1 \cdots s_{n-2})) \\ &= \alpha(s_1) \cdots \alpha(s_{n-1}) r^k(\alpha(t)) \\ &= \alpha(s_1) \cdots \alpha(s_{n-1}) \alpha(r^k(t)) \\ &= \alpha(s_1) \cdots \alpha(s_{n-1}) \alpha(s_n).\end{aligned}$$

The proof is now complete. □

We now consider the case when the greatest symmetry occurs.

Let $M = (X, \rho)$ be a map. Each ordered pair (v, w) such that $\{v, w\}$ is an edge of X is called an arc of X. A rotation ρ on X induces a permutation of the set of all arcs of X by $\rho(v, w) = (v, \rho_v(w))$ for each arc (v, w). If Aut$M$ acts regularly on the arc set of X, then such a map $M = (X, \rho)$ is called *regular*. A Cayley map $M = (X, \rho)$ is called a *balanced* Cayley map if $r(s^{-1}) = r(s)^{-1}$ for all $s \in S$. Then we have the following consequence.

COROLLARY 3.4. *Let $M = \mathrm{Cay}(G, S, r)$ be a regular Cayley map. Then $L(G)$ is normal if and only if M is balanced.*

Proof. By virtue of Lemma 2.2, a Cayley map $M = \mathrm{Cay}(G, S, r)$ is regular precisely when $|A_1| = |S|$. Since M is regular, there is a generator α of the cyclic group A_1 such that $\alpha(s) = r(s)$ for every s in S. The corollary now follows from Theorem 3.3. □

Note that for each balanced Cayley map, $L(G)$ is normal by Lemma 2.2 and Theorem 3.3.

We also state the following immediate consequence, which was also observed in [3].

COROLLARY 3.5. *Let $M = \mathrm{Cay}(G, S, r)$ be a Cayley map. Then, there exists an automorphism of the group G that agrees with r on S if and only if M is regular and balanced.*

References

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