East Asian Math. J 13 (1997), No 2, pp 253–264

SOME ASSOCIATED FUNCTIONS WITH GEVREY–TYPE SEQUENCES AND TEMPERED ULTRADISTRIBUTIONS

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1. Introduction

In this article we investigate associated functions with Gevrey -type sequences and the space of tempered ultradistributions of which test function space is defined by a certain sequence of positive numbers.

The Beurling type spaces have been defined and studied by Björck[3] in terms of a weight function $\omega : R^n \to [0,\infty)$ under some assumptions. Roumieu[16,17] has also given ultradistribution theory in which growths of derivatives of test functions are restricted by means of certain sequences.

The associated function M on $[0,\infty)$, defined by a Gevrey-type sequence $\{M_k\}$ satisfying some conditions, corresponds to a weight function ω. We define some locally convex spaces by inductive limit or projective limit of Banach spaces which are defined by means of Gevrey-type sequences. We also consider the space of ultradistributions of class M_k and of Beurling type(resp. Roumieu type). We define the space $S(M_k, N_k)$ with the inductive limit topology and we study a number of properties of the space $S(M_k, N_k)$. We can identify the space $S(M_k, M_k)$ with the space $S^{M_q}_{M_k}$ [5,11,17] in a natural way.

The elements of the dual space $S'(M_k, M_k)$ of the space $S(M_k, M_k)$ are called tempered ultradistributions. Consequently we have $D_M \subset S(M_k, M_k) \subset S_M$, where D_M and S_M are the Beurling spaces[3]. Since D_M is dense in S_M, $S'_M \subset S'(M_k, M_k)$.

Received Sep 22, 1997

2. Associated functions

Let $\{M_k\}$, $k \in N_0$, be a sequence of positive numbers which satisfies some of the following conditions:

(M.1)$M_k^2 \leq M_{k-1}M_{k+1}, k \in N$;

(M.2)There are constants $K > 0$ and $H > 1$ such that

$$M_k \leq KH^k \min_{0 \leq l \leq k} M_l M_{k-l}, k \in N_0;$$

(M.3) There is a constant $K > 0$ such that

$$\sum_{l=k+1}^{\infty} \frac{M_{l-1}}{M_l} \leq Kk \frac{M_k}{M_{k+1}}, \quad k \in N;$$

(M.2)′ There are constants $K > 0$ and $H > 1$ such that

$$M_{k+1} \leq KH^k M_k, k \in N_0;$$

(M.3)′ $\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty$.

Clearly (M.3) ⇒ (M.3)′ and (M.2) ⇒ (M.2)′. The inverse statements are not true in general. We note that if $s > 1$ the Gevrey sequence $M_k = (k!)^s$ or k^{ks} satisfies the above conditions.

DEFINITION 2.1. For each sequence $\{M_k\}$, $k \in N_0$, of positive numbers, we define functions $M(t), Q(t)$ and $q(t), (t > 0)$ associated with the sequence $\{M_k\}$ as follows:

$$M(t) = \sup_k \ln \frac{t^k M_0}{M_k}, \quad M(0) = 0, \tag{2.1}$$

$$Q(t) = \sum_{k=0}^{\infty} \frac{t^k M_0}{M_k}, \tag{2.2}$$

$$q(t) = \sup_k \frac{t^k M_0}{M_k}, \tag{2.3}$$

Under the same condition we define a sequence $\{M_k^c\}$:

$$M_k^c = M_0 \sup_{t>0} \frac{t^k}{\exp M(t)}, \quad k \in N_0. \tag{2.4}$$

If there is a constant $C > 0$ such that $M_0 C^k \le M_k$ then $M(t)$ is an increasing concave function in $\ln t$ which vanishes for sufficiently small $t > 0$ and increases more rapidly than $\ln t^k$ for any k as t tends to infinity. It is well known that a sequence $\{M_k\}$ of positive numbers satisfies (M.1) if and only if $M_k = M_k^c, \quad k \in N_0$ The sequence $\{M_k^c\}, \quad k \in N_0$, is the maximum of all sequences which satisfy $M_0^c = M_0, M_k^c \le M_k$ and (M.1). We have the following theorem by the Denjoy-Carleman Theorem:

THEOREM 2.1. *If $\{M_k > 0\}$ satisfies (M.1), then the following conditions are equivalent:*

(1) $\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty (= (M.3)\prime)$.
(2) $C\{M_k\}$ *is not quasi-analytic.*
(3) $\int_0^{\infty} \ln Q(t) \frac{dt}{1+t^2} < \infty$.
(4) $\int_0^{\infty} \frac{M(t)}{1+t^2} dt < \infty$.
(5) $\sum_{k=1}^{\infty} \left(\frac{1}{M_k}\right)^{\frac{1}{k}} < \infty$.
(6) $\int_0^{\infty} \ln q(t) \frac{dt}{1+t^2} < \infty$.

THEOREM 2.2. *If $\{M_k > 0\}$ satisfies (M.1), then the following inequalities hold:*

$$\begin{aligned} &M_{k-l} M_l \le M_0 M_k, \ (l = 0, 1, ..., k) \ \text{and} \\ &M_{k-l} M_l \le \left(\frac{M_0}{M_1}\right)^k (M_k)^2, \ (l = 0, 1, \cdots, k). \end{aligned} \tag{2.5}$$

Proof. We shall show first inequalities by induction. If $k = 0, k = 1$ or $k = 2$, then they are clear. Suppose that $M_{n-l} M_l \le M_0 M_n$ hold for $k = n, (l = 0, 1, \cdots, n)$. We shall show that $M_{(n+1)} - l M_l M_0 \le M_{n+1}$ hold for $k = n + 1, (l = 0, 1, \cdots, n + 1)$. If $l = 0$, then it is clear. By hypothesis,$M_{n+1-l} M_{l-1} \le M_0 M_n \ (l = 1, \cdots, n + 1)$. Multiplying $M_l M_{n+1}$ both sides we obtain $M_{n+1-l} M_{l-1} M_l M_{n+1} \le M_0 M_n M_l M_{n+1}$.

Since $\frac{M_{l-1} M_{n+1}}{M_l M_n} \ge 1$, we have $M_{n+1-l} M_l \le M_0 M_{n+1}$ $(l = 0, 1, \cdots, n + 1)$. For the second inequalities, $M_l \le \left(\frac{M_0}{M_1}\right)^{k-l} M_k$ and $M_{k-l} \le \left(\frac{M_0}{M_1}\right)^l M_k$ imply the inequality.

PROPOSITION 2.3 [14, LEMMA 1.10]. *If $\{M_k > 0\}$ satisfies (M.1), then the following inequality holds.*

$$M(s+t) \leq M(2s) + M(2t), \ s,t > 0. \tag{2.6}$$

Proof. By using $2^n = \sum_{k=0}^{n} {}_nC_k$ and $\frac{1}{M_n} \leq \frac{M_0}{M_{n-k}M_n}$ we can prove the inequality

$$\exp(M(s+t)) \leq \exp(M(2s) + M(2t)).$$

Hereafter we assume that the sequence $\{M_k\}$ always satisfies (M.1). Condition (M.1) is equivalent to saying that the sequence $m_k = \frac{M_k}{M_{k-1}}$, $k \in N$, is increasing. We define $m(t) = \sup\{k \in N_0 | m_k \leq t\}$. Then $m(t) = k \Leftrightarrow m_k \leq t$ and $m_{k+1} > t$. We have

$$M(t) = \int_0^t \frac{m(s)}{s} ds \tag{2.7}$$

(Roumieu[16, p.65], Komatsu[8, p.50]).

PROPOSITION 2.4. *The sequence $\{M_k > 0\}$ satisfies (M.2)′ if and only if there are constants $K > 0$ and $H > 1$ such that*

$$m(t) > \frac{\ln(t/K)}{\ln H}, t > 0. \tag{2.8}$$

In this case, we have for any $t \geq K > 0$,

$$M(t) - M(K) \leq \frac{1}{2lnH}((ln(t/K))^2 \tag{2.9}$$

Proof. We can prove it similarly as proof of Proposition 3.4[8]. We have the following theorem by Petzsche([4, Lemma 1.4]).

THEOREM 2.5. *If $M_k > 0, k \in N_0$, satisfies (M.1), then the following inequalities are equivalent :*

(a)There are constants $K > 0$ and $H > 1$ such that

$$M_k \leq KH^k M_r M_s, \quad r, s, k \in N_0, \quad r + s = k.$$

(b) $M_{2k} \leq KH^k M_k^2, \quad k \in N_0.$

(b)′ $\Pi_{j=1}^{k} m_{k+j} \leq KH^k M_k, \quad k \in N_0.$

(c) $m(t) \geq \frac{1}{\ln H}(M(t) - \ln(KM_0)),\ t \geq 0.$

(c)′ $m(t) \geq \frac{1}{\ln H}(k \ln t - \ln(KM_k)),\ k \in N_0,\ t \geq 0$

(d) For every $L \geq 1$,

$$M(Lt) \geq (1 + \frac{\ln L}{\ln H})M(t) - \ln(KM_0)\frac{\ln L}{\ln H},\ t \geq 0.$$

(e) $2M(t) \leq M(Ht) + \ln(KM_0),\ t \geq 0.$

Proof. We can find a proof of (a)⇒ (b)⇒ (c) ⇒ (d) ⇒ (e) ⇒ (a) in [4, Lemma 1.4]. (b)⇔ (b)′ and (c) ⇔ (c)′ are clear.

PROPOSITION 2.6 [8, LEMMA 3.5]. *Suppose that $\{M_k > 0\}$ and $\{M'_k > 0\}$ are sequences satisfying (M.1). Denote by $M(t)$ and $M'(t)$ their associated functions and by $m(t)$ and $m'(t)$ the logarithmic derivatives of $M(t)$ and $M'(t)$. Then*

$$N_k = \min\{M_i M'_j : i + j = k, i, j \in N_0\}, k \in N_0$$

is also a sequence of positive numbers satisfying (M.1) and its associated function $N(t)$ and the logarithmic derivative $n(t)$ of $N(t)$ are given by

$$n(t) = m(t) + m'(t) \tag{2.10}$$

$$N(t) = M(t) + M'(t) \tag{2.11}$$

Proof. $N_k = N_O \min_{i+j=k}\{m_1, \cdots, m_i, m_1\prime, \cdots, m_j\prime\}$ and $n_k = \frac{N_k}{N_k - 1}$. We rearranging the set $S = \{m_1, m_2, \cdots, m_k, m_1\prime, m_2\prime, \cdots, m_k\prime$ }, in the order of growth. Say $r_1 \leq r_2 \leq \cdots \leq r_k \leq \cdots \leq r_{2k}$ where $r_j \in S, \quad j = 1, 2, \cdots, 2k$.

Then we have $n_k \leq r_k \leq n_{k+1} \leq r_{k+1}, \quad k \in N$. Hence $\{N_k\}$ satisfies (M.1) and (2.10) holds. Integrating (2.10) we obtain (2.11).

PROPOSITION 2.7. *Let $\{M_k > 0\}, k \in N_0$, be a sequence satisfying (M.1). There exist positive constants $K > 0, H > 1$ such that*

$$M_k \leq KH^k N_k, \quad k \in N_0 \tag{2.12}$$

where $N_k = \min_{i+j=k}\{M_i M_j\}$, if and only if

$$N(t) \leq \frac{1}{2}[N(Ht) + \ln(N_O K^2)] \tag{2.13}$$

Proof. If (2.12) holds, then we have

$$N(t) = \sup_k \ln \frac{t^k N_0}{N_k} \leq \sup_k \ln \frac{t^k H^k K M_0^2}{M_k} = \frac{1}{2}[\ln(N_0 K^2) + N(Ht)].$$

Conversely,

$$\begin{aligned} N_k = N_0 \sup_{t>0} \frac{t^k}{\exp N(t)} &\geq N_0 \sup_{t>0} \frac{t^k}{M_0 K \exp \frac{1}{2} N(Ht)} \\ &= \frac{1}{KH^k} M_0 \sup_{t>0} \frac{t^k}{\exp M(t)} = \frac{M_k}{KH^k} \end{aligned}$$

Hence $M_k \leq KH^k N_k$.

DEFINITION 2.8 [8]. Let $\{M_k > 0\}$ and $\{N_k > 0\}$ be sequences satisfying(M.1).

We write $M_k \subset N_k$ if there are constants K and H such that

$$M_k \leq KH^k N_k, \quad k \in N_0 \tag{2.14}$$

PROPOSITION 2.9. *Let $\{M_k > 0\}, k \in N_0$, be a sequence satisfying (M.1). Let $N_k = \min_{i+j=k}\{M_i M_j\}, k \in N_0$. If $N(t) \leq \frac{1}{2}[N(Ht) + \ln(N_0 K^2)]$ for some constants $K > 0$ and $H > 1$, then $M_k \subset N_k$.*

Proof. Clear by Proposition 2.7.

DEFINITION 2.10 [8]. Let $\{M_k > 0\}$ and $\{N_k > 0\}$ be sequences satisfying(M.1). We write $M_k \prec N_k$ if for any $H > 0$ there is a constant K such that $M_k \leq KH^k N_k, \quad k \in N_0$.

PROPOSITION 2.11. *Let* $\{M_k > 0\}$ *be a sequence satisfying (M.1).* $N_k = \min_{i+j=k}\{M_i M_j\}$, $k \in N_0$. *Then* $M_k \prec N_k$ *if and only if for any* $H > 0$ *there is a constant* K *such that* $M(t) \leq \frac{1}{2}M(Ht) + \ln K$, $0 < t < \infty$.

Proof. Clear.

3. Ultradifferentiable functions

Let $\{M_k\}, k \in N_0$,be a sequence of positive numbers satisfying (M.1) and (M.3)′. An infinitely differentiable function f on an open set Ω in R^n is called an ultradifferentiable function of class M_k and of Beurling type (resp. Roumieu type) if on each compact set K in Ω its derivatives are estimated by

$$\|D^\alpha f\|_{C(K)} \leq Ch^{|\alpha|}M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \cdots,$$

for every $h > 0$ and $C = C(h) > 0$ (resp. for some $h > 0$ and $C > 0$) depending on f. An ultradifferentiable function of class M_k and of Beurling type(resp. Roumieu type) is called an ultradifferentiable function of class (M_k) (resp. $[M_k]$) for short.

DEFINITION 3.1. Let K be a compact set in R^n, let $\{M_k\}$ be a sequence of positive numbers and let $h > 0$ We denote by $D(K; M_k, h)$ the space of all $f \in C^\infty(R^n)$ with support in K which satisfies (3.1).

$$\|f\|_{K,M_k,h} = \sup_{x \in K, \alpha \in N_0^n} \frac{|D^\alpha f(x)|}{h^{|\alpha|}M_{|\alpha|}} < \infty \qquad (3.1)$$

Clearly $D(K; M_k, h)$ is a Banach space under the norm (3.1.)

PROPOSITION 3 2. *If* $h < r$ *and* $K \subset\subset L$, *then the inclusion mappings*

$$D(K; M_k, h) \longrightarrow D(K; M_k, r),$$

$$D(K; M_k, h) \longrightarrow D(L; M_k, h)$$

are compact operators

Proof. We can find a proof in Komatsu[8, p41].

DEFINITION 3.3. Let K be a compact set in R^n and let Ω be an open set in R^n. As locally convex spaces we define :

$$D(K;(M_k)) = proj \lim_{n\to\infty} D(K;M_k,\frac{1}{n}), \tag{3.2}$$

$$D(\Omega;M_k,h) = ind \lim_{K\subset\subset\Omega} D(K;M_k,h) \tag{3.3}$$

$$D(\Omega;(M_k)) = proj \lim_{n\to\infty} D(\Omega;M_k,\frac{1}{n}) \tag{3.4}$$

$$D(K;[M_k]) = ind \lim_{n\to\infty} D(K;M_k,n) \tag{3.5}$$

$$D(\Omega;[M_k]) = ind \lim_{\substack{K\subset\subset\Omega \\ n\to\infty}} D(K;M_k,n), \tag{3.6}$$

THEOREM 3.4. *$D(K;(M_k))$ is an (FS)-space, $D(\Omega;M_k,h)$, $D(K;[M_k])$ and $D(\Omega;[M_k])$ are (DFS)-spaces and $D(\Omega;(M_k))$ is a (DLFS)-space.*

Proof. We call a locally convex space X an (LFS)-space ((DLFS)-space) if it is the strict inductive (projective) limit of a sequence of (FS)-spaces ((DFS)-spaces). By the Proposition 3.2 the theorem is clear.

Remark.

$$\begin{aligned} D^{(M_k)}(\Omega) &= ind \lim_{K\subset\subset\Omega} D(K;(M_k)) \\ &= ind \lim_{K\subset\subset\Omega} [proj \lim_{h\to 0} D(K;M_k,h)] \end{aligned}$$

is an (LFS)-space (Komatsu[8], p44). We assume that $\{M_k\}$ satisfies (M.1) and (M.3)′. We denote by $D'(\Omega;(M_k))$ (resp.$D'(\Omega;[M_k])$) the strong dual of $D(\Omega;(M_k))$ (resp.$D(\Omega;[M_k])$). The elements of $D'(\Omega;(M_k))$ (resp.$D'(\Omega;[M_k])$) are called ultradistributions of clss M_k of Beurling type (resp. Roumieu type) or of class (M_k) (resp. $[M_k]$) for short.

THEOREM 3.5. *Let $\{M_k\}, k \in N_0$, be a sequence satisfying (M.1). If $f \in D(K; M_k, h)$, then its Fourier-Laplace transform*

$$\hat{f}(\zeta) = Ff(\zeta) = \int_{R^n} f(x) e^{-i<x,\zeta>} dx$$

is an entire function on C^n which satisfies the following estimate

$$|\hat{f}(\zeta)| \leq M_0 |K| \|f\| \exp H_K(\zeta) / q(\frac{|\zeta|}{h}), \tag{3.7}$$

where $H_K(\zeta) = \sup_{x \in K} Im < x, \zeta >$, *$|K|$ is the Lebesgue measure of K and $\|f\|$ is the norm of f in $D(K; M_k, h)$.*

Conversely, if

$$\hat{f} \in L^1(\exp M(\frac{|\zeta|}{h}) d\zeta) \tag{3.8}$$

for some $h > 0$, then its inverse Fourier transform

$$f(x) = F^{-1} f(x) = (2\pi)^{-n} \int_{R^n} \hat{f}(\zeta) e^{i<x,\zeta>} d\zeta$$

is an ultradifferentiable function on R^n which satisfies

$$\sup_{x \in R^n} |D^\alpha f(x)| \leq C h^{|\alpha|} M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \cdots \tag{3.9}$$

Proof. The proof is similar to [8, Lemma 3.3].

Let $\{M_k > 0\}$ and $\{N_k > 0\}, k \in N_0$, be two sequences satisfying (M.1), (M.2) and (M.3)′. We define for $A, B \in N$,

$$S(K; M_k, N_k, A, B) = \{f \in C^\infty(R^n); supp f \subset K \quad \text{and} \quad \|f\|_{\delta,\rho} < \infty\} \tag{3.10}$$

$\delta, \rho = 1, \frac{1}{2}, \frac{1}{3}, \cdots,$
where

$$\|f\|_{\delta,\rho} = \sup_{x,\alpha,\beta} \frac{|x^\alpha D^\beta f(x)|}{(A+\delta)^{|\alpha|}(B+\rho)^{|\beta|} M_{|\alpha|} N_{|\beta|}} \tag{3.11}$$

$$S(\Omega; M_k, N_k) = ind \lim_{\substack{K \subset\subset \Omega \\ A,B \to \infty}} S(K; M_k, N_k, A, B) \tag{3.12}$$

Then $S(K;M_k,N_k,A,B)$ is a Fréchet space.

We denote by $S(R^n;M_k,N_k)=S(M_k,N_k)$ shortly. We define, for multi-indices $\gamma=(\gamma_1,\cdots,\gamma_n)$ and $\delta=(\delta_1,\cdots,\delta_n)$,

$$S(\gamma,\delta)=\{f\in C^\infty(R^n); f\text{satisfies}(P)\}: \tag{3.13}$$

$$|x^\alpha D^\beta f(x)|\le CA^{|\alpha|}B^{|\beta|}\alpha^{|\gamma|\alpha}\beta^{|\delta|\beta}, \tag{P}$$

where A,B,C are certain positive constants which depend on the function f. For $k=0$, the expression $k^{|\gamma|k}$ is considered to equal 1. For given $A>0$ and $B>0$, we define

$$S(\gamma-A,\delta-B)=\{f\in S(\gamma,\delta): f\text{satisfies}(Q)\}: \tag{3.14}$$

For every $\epsilon>0,\rho>0$,there exists $C_{\epsilon\rho}$ constant such that

$$|x^\alpha D^\beta f(x)|\le C_{\epsilon\rho}(A+\epsilon)^{|\alpha|}(B+\rho)^{|\beta|}\alpha^{|\gamma|\alpha}\beta^{|\delta|\beta} \tag{Q}$$

We define topology in the space $S(x-A,\delta-B)$ by the system of norms

$$\|f\|_{\epsilon\rho}=\sup_{x,\alpha,\beta}\frac{|x^\alpha D^\beta f(x)|}{(A+\epsilon)^{|\alpha|}(B+\rho)^{|\beta|}\alpha^{|x|\alpha}\beta^{|\delta|\beta}}\quad (\epsilon,\rho=1,\frac{1}{2},\frac{1}{3},\cdots) \tag{3.15}$$

If $A_1<A_2,B_1<B_2$, then $S(\gamma-A_1,\delta-B_1)\subset S(\gamma-A_2,\delta-B_2)$ and the include mapping is continuous.

We define

$$S(\gamma,\delta)=ind\lim_{\substack{A,B\in N\\ A,B\to\infty}} S(\gamma-A,\delta-B) \tag{3.16}$$

If $M_k=k^{k|\gamma|},N_q=q^{q|\delta|},|\gamma|>1$ and $|\delta|>1$, then for S_γ^δ defined in [17] and for $n=1,S_\gamma^\delta=S(\gamma,\delta)$.

THEOREM 3.6. *The inductive limit space $S(M_k,N_k)$ can be identified with the space $S_{M_k}^{N_q}$ with the topology defined by means of the norms:*

$$\|f\|_{A,B}=\sup_{x,\alpha,\beta}\frac{|x^\alpha D^\beta f(x)|}{A^{|\alpha|}B^{|\beta|}M_{|\alpha|}N_{|\beta|}},\quad A,B\in N \tag{3.17}$$

Proof. Obvious.

The dual of $S(M_k, N_k)$ is the space of tempered ultradistributions denoted by

$S'(M_k, N_k)$. If the sequences $\{M_k\}$ and $\{N_k\}$ satisfy (M.1), (M.2) and (M.3)/ then by Gel/fand[5], p254 or Roumieu [17] or Pathak [11] we know that the Fourier transform from $S(M_k, N_k)$ to $S(N_k, M_k)$ is an isomorphism and hence the Fourier transform from $S(M_k, M_k)$ onto itself is an automorphism.

For $u \in S'(M_k, N_k)$, the Fourier transform of u is defined to be the element $\hat{u}$ such that the Parseval relation

$$< \hat{u}, \hat{f} >= (2\pi)^n < u, \check{f} > \tag{3.18}$$

holds, where $f \in S(M_k, N_k)$ and $\hat{f} = Ff \in S(N_k, M_k)$. We denote by $\hat{u} = F(u)$. By Roumieu[17] we have the following theorem

THEOREM 3.7. *The Fourier transform* $F : S'(M_k, N_k) \longrightarrow S'(N_k, M_k)$ *is an isomorphism and the Fourier transform* $F : S(M_k, M_k) \longrightarrow S'(M_k, M_k)$ *is an automorphism.*

We can easily see that $D_M \subset S(M_k, M_k) \subset S_M$, where D_M is the Beurling space [3]. Since D_M is dense in $S_M, S'_M \subset S'(M_k, M_k)$.

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