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EXTENSION OF HOLOMORPHIC MAPPINGS IN BANACH MANIFOLDS

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1. Introduction

J. Kajiwara, M. Nishihara, L. Li and M. Yoshida[4] discussed extension of holomorphic mappings from a separable Fréchet space, with the bounded approximation property, into a Banach complex Lie group. M. Yoshida[9] investigated the extension of holomorphic mappings from domains over locally convex spaces where the Levi problem was affirmatively solved.

On the other hand, P. K. Ban[1] proved that a Banach manifold $\mathcal{X}$ with the weak disc property has the holomorphic extension property and P. K. Ban[1,2] proved that any holomorphic mapping $f : \Omega \to \mathcal{X}$ can be extended to a holomorphic mapping $\tilde{f}$ of the envelope of holomorphy $\hat{\Omega}$ of the domain Ω into $\mathcal{X}$.

Let Ω be a Riemann domain over a complex manifold manifold $\mathcal{M}$ modelled with a Banach space B with Schauder basis. $\tilde{\Omega}$ be its pseudoconvex hull and $\mathcal{X}$ be a Banach manifold with weak disc property. In the present paper we generalize the above theorem of Ban[1] to Riemann domains over a complex manifold $\mathcal{M}$, where the Levi problem is not necessarily solved and prove that any holomorphic mapping $f : \Omega \to \mathcal{X}$ is extended to a holomorphic mapping $\tilde{f} : \tilde{\Omega} \to \mathcal{X}$.

2. Ban's theory

In this section, we discuss an extension of holomorphic mappings with values in a Banach manifold $\mathcal{X}$ with the weak disc property.

We use the notations

$$(1) \qquad \mathbf{D} = \{z \in C; |z| < 1\},$$

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and

(2) $$\mathbf{D}^* = \{z \in D; z \neq 0\}.$$

Let $\mathcal{X}$ be a Banach manifold and let $H(\mathbf{D}, \mathcal{X})$ be the space of holomorphic maps from $\mathbf{D}$ into $\mathcal{X}$ equipped with the compact open topology. The manifold $\mathcal{X}$ is said to satisfy the *weak disc condition,* if every sequence $\{f_n\}$ of $H(\mathbf{D}, \mathcal{X})$ which converges in $H(\mathbf{D}^*, \mathcal{X})$, converges in $H(\mathbf{D}, \mathcal{X})$ too.

The following theorem was proved by P. K. Ban[1]:

THEOREM 1. *Let (Ω, φ) be a Riemann domain over a Banach space with a Schauder basis and let $(\hat{\Omega}, \hat{\varphi})$ be the envelope of holomorphy of the domain (Ω, φ) in the sense of Malgrange[5]. Let $\mathcal{X}$ be a Banach manifold satisfying the weak disc condition. Then every holomorphic mapping from Ω to $\mathcal{X}$ can be extended holomorphically to $\hat{\Omega}$.*

3. Envelopes of holomorphy

Let E and F be Hausdorff C-linear spaces. A continuous function f on an open subset Ω of E is said to be *holomorphic* if, for any finite dimensional C-linear subspace S of E, the restriction $f|\Omega \cap S$ is holomorphic in the open subset $\Omega \cap S$ of the finite dimensional complex space S. A continuous mapping $h : \Omega \to F$ is said to be *holomorphic* if, for any C-linear continuous mapping $\chi : F \to C$, which is naturally holomorphic by definition, the composite mapping $\chi \circ h : E \to C$ is holomorphic.

Let $\mathcal{M}$ be a Hausdorff space. Then, $\mathcal{M}$ is called a *complex manifold modelled with E*, if there exists a collection $\{(U_\imath, \chi_\imath); \imath \in I\}$ of holomorphic mappings $\chi_\imath$ of an open subsets $U_\imath$ of M onto an open subset $V_\imath$ of E such that there holds

(3) $$\cup_{\imath \in I} U_\imath = \mathcal{M}$$

and that, for any $\imath, \jmath \in I$, the composite map $\chi_\imath \circ (\chi_\jmath | V_\imath \cap V_\jmath)^{-1}$: $U_\imath \cap U_\jmath \to U_\imath \cap U_\jmath$ is holomorphic. In this case, each $(U_\imath, \chi_\imath)$ is called a *chart* and the collection $\{(U_\imath, \chi_\imath); \imath \in I\}$ of charts is called an *atlas*.

A pair (Ω, φ) of a Hausdorff space Ω and a local homeomorphism φ of Ω into the complex manifold $\mathcal{M}$ modelled with the Hausdorff complex linear space E is called an *open set over* $\mathcal{M}$ and is called a *domain over* $\mathcal{M}$ when Ω is connected. We can canonically induce a complex structure on the Hausdorff space Ω so that the mapping φ is a locally biholomorphic mapping of the domain Ω into the base manifold $\mathcal{M}$. A continuous mapping f of Ω into a Hausdorff complex linear space F is said to be *holomorphic* if, for any subset O of Ω such that the restriction $\varphi|O$ is a homeomorphism of O onto an open subset $\varphi(O)$ of $\mathcal{M}$, the composite mapping $f \circ (\varphi|\varphi(O))^{-1} : O \to F$ is holomorphic.

Let (Ω, φ) and (Ω', φ') be open sets over the complex manifold $\mathcal{M}$. The open set (Ω', φ') is said to be *larger than* the open set (Ω, φ) and written in the notation $(\Omega, \varphi) \prec (\Omega', \varphi')$ if there exists a locally biholomorphic mapping $\tau : \Omega \to \Omega'$ with $\varphi' \circ \tau = \varphi$.

Let (Ω, φ) be a domain over the complex manifold $\mathcal{M}$, $\mathcal{X}$ be a complex manifold and $\mathcal{F} := \{f_i; i \in I\}$ be a family of holomorphic mappings of Ω into the complex manifold $\mathcal{X}$. Let (Ω', φ') be a domain over the complex manifold $\mathcal{M}$, $\mathcal{F}' := \{f'_i; i \in I\}$ be a family of holomorphic mappings of Ω' into the complex manifold $\mathcal{X}$ and λ' be a locally biholomorphic mapping of $\Omega \to \Omega'$. If there holds $\varphi' \circ \lambda = \varphi$ and $f'_i \circ \lambda = f_i (i \in I)$, the family $\mathcal{F}'$ of mappings of Ω' into $\mathcal{X}$ is called a *holomorphic extension* of $\mathcal{F}$ to the triple $(\lambda, \Omega', \varphi')$ or briefly to the domain (Ω', φ'). If the holomorphic extension $\hat{\mathcal{F}}$ of $\mathcal{F}$ to the triple $(\hat{\lambda}, \hat{\Omega}, \hat{\varphi})$ is also a homolomorphic extension of any family $\mathcal{F}'$, which is a holomorphic extension of $\mathcal{F}$ to a triple $(\lambda', \Omega', \varphi')$ as above, the triple $(\hat{\lambda}, \hat{\Omega}, \hat{\varphi})$ is called the *envelope of holomorphy* of the domain (Ω, φ) *with respect to the family* $\mathcal{F}$ of holomorphic mappings. When $F = C$ and $\mathcal{F}$ is the family of holomorphic functions on Ω, the envelope of holomorphy with respect to the family $\mathcal{F}$ is called briefly the *envelope of holomorphy.* An envelope of holomorphy with respect to a family $\{f\}$, consiting of a single mapping f, the envelope of holomorphy with respect to the family $\{f\}$ is called a *domain of holomorphy* of f.

Now, we construct domains of holomorphy of a family of holomorphic mappings in the manifold $\mathcal{X}$ by the method of Malgrange[5]:

THEOREM 2. *Let E be a Hausdorff $\mathbf{C}$-linear space, $\mathcal{M}$ be a complex manifold modelled with E, (Ω, φ) be a domain over the complex manifold $\mathcal{M}$, $\mathcal{X}$ be a complex manifold and $\mathcal{F} = \{f_\imath; \imath \in I\}$ be a family of holomorphic mappings of Ω into $\mathcal{X}$. Then, there exists uniquely an envelope of holomorphy $(\hat{\Omega}, \hat{\varphi})$ of the domain (Ω, φ) over the complex manifold $\mathcal{M}$ with respect to the family $\mathcal{F}$.*

Proof. Let a be a point of the manifold $\mathcal{M}$, U be a neighborhood of the point a and $\mathcal{G} = \{g_\imath; i \in I\}$ be a family of holomorphic mappings of U into the complex manifold $\mathcal{X}$ with the same set I of indices as that of the family $\mathcal{F}$. Let $(\mathcal{G}, U)$ and $(\mathcal{G}', U')$ be two such pairs of families of holomorphic mappings of U and U' into $\mathcal{X}$ indexed by the set I of indices. We say that they are equivalent, if there exists a neighborhood W of the point a so that $W \subset U \cap V$ and that $g_\imath | W = g'_\imath | W$ on W for any $\imath \in I$. Let $\mathcal{S}_a$ be the set of equivalence classes by the above equivalence relation between the set of pairs of such $(\mathcal{G}, U)$. An element $\mathcal{G}_a$ of $\mathcal{S}_a$ is called a *germ* of a family indexed by I at a of holomorphic mappings into $\mathcal{X}$.

We put

$$\mathcal{S} := \bigcup_{a \in \mathcal{M}} \mathcal{S}_a \tag{4}$$

and induce a topology on $\mathcal{S}$ so that, for point $a \in \mathcal{M}$, for an element $\mathcal{G}_a$ of $\mathcal{S}_a$ and a representative $(\mathcal{G}, U)$ of the equivalence class $\mathcal{G}_a$, the set $\{\mathcal{G}_x; x \in U\}$ forms a neighborhood of $\mathcal{G}_a$ and call $\mathcal{S}$ the *sheaf of germs of families indexed by I of holomorphic mappings in $\mathcal{X}$ over $\mathcal{M}$.*

We define a canonical mapping $\pi : \mathcal{S} \to \mathcal{M}$ by putting

$$\pi(g_a) = a \qquad (\mathit{for\ any}\ a \in \mathcal{M}). \tag{5}$$

Then, we can introduce canonically a complex structure on $\mathcal{S}$ so that $(\mathcal{S}, \pi)$ is an open set over the complex manifold $\mathcal{M}$.

Now, we define a holomorphic mapping $\psi : \Omega \to \mathcal{S}$ as follows:

Let x be a point of Ω. There exists a neighborhood U of x in Ω and a neighborhood V of $\varphi(x)$ in $\mathcal{M}$ such that $\varphi | U : U \to V$ is a biholomorphic mapping of U onto V. We put

$$\psi(x) := \{f_\imath \circ (\varphi | U)^{-1}; \imath \in I\}_{\varphi(x)} \in \mathcal{S}_{\varphi(x)} \subset \mathcal{S}. \tag{6}$$

By construction, the mapping $\psi : \Omega \to \mathcal{M}$ is locally biholomorphic. Since Ω is connected, its image $\psi(\Omega)$ by the continuous mapping ψ is also connected.

Let $\hat{\Omega}$ be the connected component of $\mathcal{S}$ containing the domain $\psi(\Omega)$, $\hat{\varphi}$ be the restriction of π to the connected set $\hat{\Omega}$ and $\tau : \Omega \to \hat{\Omega}$ be the mapping induced canonically by $\psi : \Omega \to \mathcal{S} \supset \hat{\Omega}$. Then, the pair $(\hat{\Omega}, \hat{\varphi})$ is a domain over $\mathcal{M}$. Let $\hat{x}$ be a point of $\hat{\Omega}$. It is nothing but a germ $\{g_i; i \in I\}_y$ of a family $\{g_i; i \in I\}$ of holomorphic mappings indexed by I of a neighborhood V of a point y of the complex manifold $\mathcal{M}$ into the complex manifold $\mathcal{X}$. By definition, there holds $y = \hat{\varphi}(\hat{x})$. We put

$$(7) \qquad \hat{\mathcal{F}}(\hat{x}) := \{\hat{f}_i(\hat{x}); i \in I\} := \{g_i(y); i \in I\}$$

Then, by construction, $\hat{\mathcal{F}}$ is the maximal holomorphic extension of $\mathcal{F}$ and $(\tau, \hat{\Omega}, \hat{\varphi})$ is the desired envelope of holomorphy of the domain (Ω, φ) over the complex manifold $\mathcal{M}$ of the family of holomorphic mappings indexed the set I as we will show it below:

At first, we show that the family $\hat{\mathcal{F}}$ is a holomorphic continuation of the family $\mathcal{F}$ to $(\psi, \hat{\Omega}, \hat{\varphi})$

For any point $x \in \Omega$, $\psi(x) = \{(f_i \circ (\varphi|U)^{-1}); i \in I\}_x \in \mathcal{S}_x \subset \mathcal{S}$ for a neighborhood U of x such that $\varphi|U$ is biholomorphic. Then we have

$$(8) \qquad (\hat{\mathcal{F}} \circ \psi)(x) = \{f_i(x); i \in I\} = \mathcal{F}(x)$$

and the family $\hat{\mathcal{F}}$ is a holomorphic extension of $\mathcal{F}$ to $\hat{\Omega}$.

Next, we prove the maximality of the extension.

Let (Ω', φ') be a domain over the complex manifold $\mathcal{M}$, $\mathcal{G} = \{g_i; i \in I\}$ be a family of holomorphic mappings, with the same set I of indices, on Ω' and ξ be a locally biholomorphic mapping of Ω in Ω' such that $\varphi' \circ \xi = \varphi$ and $g_i \circ \xi = f_i$ for any $i \in I$. We define a mapping $\sigma : \Omega' \to \tilde{\Omega}$ as follows: Let x be a point of Ω'. Since (Ω', φ') is a domain over the complex manifold $\mathcal{M}$, there exists a neighborhood U of x and a neighborhood V of $\varphi'(x)$ such that $\varphi'|U$ is a biholomorphic mapping of U onto $V \subset \mathcal{M}$. We define the value of σ at x as the germ $\{g_i \circ (\varphi'|U)^{-1}; i \in I\}_{\varphi'(x)}$ of families of holomorphic mappings. Since Ω' is connected, $\sigma(\Omega')$ is a connected set containing $\psi(\Omega)$. Hence, we have $\sigma(\Omega')$ $subset\tilde{\Omega}$. So $\sigma : \Omega' \to \hat{\Omega}$ is a locally biholomorphic mapping with $\hat{\varphi} \circ \sigma = \varphi'$ and $\hat{\mathcal{F}} \circ \sigma = \mathcal{G}$. In this way, we can prove that $\hat{\mathcal{F}}$ is a holomorphic extension of the holomorphic extension $\mathcal{G}$ of $\mathcal{F}$ to $\hat{\Omega}$.

Lastly, we prove uniqueness of the maximal holomorphic extension. Let $(\tau', \hat{\Omega}', \hat{\varphi}')$ be another domain of maximal extension. Since $(\hat{\Omega}, \hat{\varphi})$ is a maximum, there exists a local biholomorphic mapping $\chi : \hat{\Omega}' \to \hat{\Omega}$ with $\chi \circ \tau' = \tau$ and $\hat{\varphi} \circ \chi = \hat{\varphi}'$ and since $\hat{\Omega}', \hat{\varphi}'$ is also a maximum, there exists a local biholomorphic mapping $\chi' : \hat{\Omega} \to \hat{\Omega}'$ with $\chi' \circ \tau = \tau'$ and $\hat{\varphi}' \circ \chi' = \hat{\varphi}$. Then, the mapping $\chi' \circ \chi$ is the identity mapping on the non-empty open set $\chi'(\Omega)$ and $\chi \circ \chi'$ is also the identity mapping on the non-empty open set $\chi(\Omega)$. By the theorem of identity, that is, by the principle of the uniqueness of continuation of holomorphic mappings, $\chi' \circ \chi$ and $\chi \circ \chi'$ are identity mappings, respectivel y, on Ω' and Ω, what was to be proved.

4. Pseudoconvex hull

Let E be a sequentially complete locally convex space, (Ω, φ) be a domain over the space E and $\mathcal{P}$ be the family $\{(\lambda_j, \Omega_j, \varphi_j); j \in P\}$ of triples such that each (Ω_j, φ_j) is a pseudoconvex domain over E and that each λ_j is a locally biholomorphic map of Ω in Ω_j with $\varphi = \varphi_j \circ \lambda_j$. In finite dimensional case, Kajiwara[3] defined a pseudoconvex hull of a domain over a holomorphically convex manifold. The colleague Ohgai[8], to whom the author would like to express his hearty gratitude for her fruitful discussions, constructed the Durchshnitt $(\tilde{\Omega}, \tilde{\varphi})$ as the minimum of the family $\mathcal{P}$:

THEOREM 3. *Let E be a sequentially complete locally convex space and (Ω, φ) be a domain over E. Then there exists uniquely a minimum pseudoconvex domain $(\tilde{\Omega}, \tilde{\varphi})$ over E between domains larger than (Ω, φ).*

Proof. By definition, for any $j \in \mathcal{P}$, each pseudoconvex domain (Ω_j, φ_j) over E is larger than the original domain (Ω, φ), that is, there exits a local biholomorphic map $\lambda_j : \Omega \to \Omega_j$ with $\varphi = \varphi_j \circ \lambda_j$

Now, we introduce a semi-order in $\mathcal{P}$. For $j, k \in P$, we write $(\Omega_j, \varphi_j) \prec (\Omega_j, \varphi_j)$ if there exits a local biholomorphic map $\lambda_j^k : \Omega_j \to \Omega_k$ with $\varphi_j = \varphi_k \circ \lambda_j^k$.

In the previous paper[7], the author proved that the semi-ordered set $\mathcal{P}$ is inductive, and hence, by the theorem of Zorn, there exists a mimimal element of $\mathcal{P}$ and we can prove that it is also the minimum element of $\mathcal{P}$.

We call the above domain $(\tilde{\Omega}, \tilde{\varphi})$ or the triple $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\varphi}) \in \mathcal{P}$ the *pseudoconvex hull* of the domain (Ω, φ) over E.

5. Continuation to pseudoconvex hulls

THEOREM 4. *Let B be a Banach space with Schauder base, $\mathcal{M}$ be a complex manifold modelled with B, (Ω, φ) be a domain over $\mathcal{M}$, $\mathcal{X}$ be a Banach manifold with the disc propery, and $\mathcal{F} := \{f_i; i \in I\}$ be a family of holomorphic mappings of the domain Ω into the complex manifold $\mathcal{X}$. The envelope of holomorphy $(\hat{\Omega}_{\mathcal{F}}, \hat{\varphi}_{\mathcal{F}})$ of the domain (Ω, φ) with respect to the family $\mathcal{F}$ is a pseudoconvex domain over the complex manifold $\mathcal{M}$.*

Proof. We prove the pseudoconvexity of the domain $(\hat{\Omega}_{\mathcal{F}}, \hat{\varphi}_{\mathcal{F}})$ over the manifold $\mathcal{M}$. For this purpose, it suffices to prove its local pseudoconvexity:

Let (U, χ, V) be a holomorphic chart of $\mathcal{M}$ such that χ is a biholomorphic mapping of a domain $U \subset \mathcal{M}$ onto an open convex set V of B, which is of course a domain of holomorphy in the complex Banach space B.

We put

$$W := (\chi \circ \hat{\varphi}_{\mathcal{F}})^{-1}(V) \tag{9}$$

and

$$\xi := (\chi \circ \hat{\varphi}_{\mathcal{F}})|(\chi \circ \hat{\varphi}_{\mathcal{F}})^{-1}(V). \tag{10}$$

Let $(\hat{\nu}, \hat{W}, \hat{\xi})$ be the envelope of holomorphy of the open set (W, ξ) over the Banach space B with a Schauder base. By the previous paper [6,7], $(\hat{\nu}, \hat{W}, \hat{\xi})$ coincides with the pseudoconvex hull $(\tilde{\nu}, \tilde{W}, \tilde{\xi})$ of the open set (W, ξ) over B.

For any $i \in I$, each $h_i := f_i|W \to \mathcal{X}$ is continued to a holomorphic mapping $\hat{h}_i : \tilde{W} \to \mathcal{X}$ by Theorem 1. Since the domain $(\hat{\Omega}_{\mathcal{F}}, \hat{\varphi}_{\mathcal{F}})$ is the simultaneous existence domain of holomorphic mappings of $\mathcal{F}$, there holds $(\tilde{W}, \tilde{\xi}) \prec (\hat{\Omega}_{\mathcal{F}}, \hat{\varphi}_{\mathcal{F}})$ and hence $(W, \xi) = (\tilde{W}, \tilde{\xi})$. Thus, (W, ξ) is pseudoconvex and, by definition, the domain $(\hat{\Omega}_{\mathcal{F}}, \hat{\varphi}_{\mathcal{F}})$ is pseudoconvex.

As a corollary we have the following main theorem:

MAIN THEOREM. *Let B be a Banach space with Schauder base, $\mathcal{M}$ be a complex manifold modelled with B, (Ω, φ) be a domain over $\mathcal{M}$, $\mathcal{X}$ be a Banach manifold with the disc propery, and $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\varphi})$ be the pseudoconvex hull of the domain (Ω, φ). Then a holomorphic mapping f of Ω into $\mathcal{X}$ is continued to a holomorphic mapping of $\tilde{\Omega}$ into $\mathcal{X}$.*

Proof. By Theorem 4, the envelope of holomorphy $(\hat{\lambda}, \hat{\Omega}_f, \hat{\varphi}_f)$ of the domain (Ω, φ) with respect to the family $\{f\}$, consisting of a single mapping f, is pseudonconvex. Let $\hat{f}$ be the continuation of the mapping f to the domain $\hat{\Omega}_f$. Since $(\tilde{\Omega}, \tilde{\varphi})$ is the smallest pseudoconvex domain larger than $(\Omega, \varphi) \prec (\hat{\Omega}_f, \hat{\varphi}_f)$, we have $(\tilde{\Omega}, \tilde{\varphi}) \prec (\hat{\Omega}_f, \hat{\varphi}_f)$. By definition, there exists a locally biholomorphic mapping $\tau : \tilde{\Omega} \to \hat{\Omega}_f$ such that $\tilde{\varphi} = \tau \circ \hat{\varphi}_f$ and that $\tau \circ \tilde{\lambda} = \hat{\lambda}$. Then the mapping $\tilde{f} := \hat{f} \circ \tau$ is a holomorphic mapping of the pseudoconvex hull $\tilde{\Omega}$ into the manifold $\mathcal{X}$ satisfying $\tilde{f} \circ \tilde{\lambda} = \hat{f} \circ \tau \circ \tilde{\lambda} = \hat{f} \circ \hat{\lambda} = f$. Hence the mapping $\tilde{f}$ is a holomorphic continuation of the mapping f to $\tilde{\Omega}$.

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