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VALIDITY OF OKA'S PRINCIPLE FOR RELATIVE COHOMOLOGY

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In the previous paper[4], the author extended Kajiwara-Shon[3]'s infinite dimensional vanishing cohomology theorem, which asserts that

$$\mathrm{H}^p(D - A, \mathcal{O}) = 0 \tag{1}$$

for the cohomology of degree $p \leq \mathrm{codim}\ A - 2$ of the complement $D - A$ of an analytic subset A of a pseudoconvex open set D of a vector space E equipped with the finite open topology.

Let F be a Fréchet space and $\mathcal{O}_F$ be the sheaf of germs of holomorphic mappings into F. By Fujimoto[1], the study of F-valued holomorphic functions is not only to generalize the results on ordinary holomorphic functions , but also to contribute to the study of ordinary holomorphic functions on a product space.

In the previous paper[5], the author generalized the above results[4] to the category of the F-valued holomorphic functions over infinite dimensional domains as follows:

Let E be a C-vector Hausdorff space and $\mathcal{O}$ be its structural sheaf. Let D be a domain of E. A real valued C^∞ function φ on D is said to be $q - convex$ if there exists a positive integer n_0 such that, for any integer n with $n \geq n_0$ and for any n-dimensional C-linear subspace H of E, the Levi form of the restriction $\varphi|D \cap H$ of the function φ to the n-dimensional domain $D \cap H$ has $n - q + 1$ positive eigen values at every point of $D \cap H$.

In the previous paper[5], the author generalized the above results and proved the vanishing cohomology

$$\mathrm{H}^p(\{x \in D; \varphi(x) > c\}, \mathcal{O}_{\mathcal{F}}) = 0 \tag{2}$$

for any pseudoconvex domain D, for any positive integers p and q, and for any q-convex function φ on D.

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On the other hands, Kajiwara-Kazama[2] established the validity of the Oka's principle for relative cohomology sets for pairs of Stein spaces X and analytic subset A for complex Lie group L modelled with Banach spaces.

The aim of the present paper is to extend the above results[2] to infinite dimensional domains as follows:

Let X be the infinite dimensional locally convex space with the finite open topology, D be the pseudoconvex domain of X and A be an analytic subset of D. Moreover, let L be a complex Lie group modelled with a complex Banach space with the unit element e and $\mathcal{A}_L(A)$ and $\mathcal{E}_L^0(A)$ be, respectively, the sheaves of germs of holomorphic and continuous mappings in L, which takes the value e on the analytic set A. We prove the quasi-injectivity of the canonical mapping

$$(3) \qquad \iota : H^1(\Omega, \mathcal{A}_L(A)) \to H^1(\Omega, \mathcal{E}_L^0(A))$$

and infinite dimensionalize the Kajiwara-Kazama's results[2]. The class of represented by coboundarys of type of coverings, of cohomology set $H^1(\Omega, \mathcal{E}_L^0(A))$, is called the neutral element of the cohomology set.

MAIN THEOREM. *Let E be the infinite dimensional locally convex space with the finite open topology, D be the pseudoconvex domain of X, A be an analytic subset of D, L be a complex Lie group modelled with a complex Banach space, $\mathcal{A}_L(A)$ and $\mathcal{E}_L^0(A)$ be, respectively, the sheaves of germs of holomorphic and continuous mappings in L, which take the value e on the analytic set A. Then the inverse image of the neutral element of the cohomology set $H^1(\Omega, \mathcal{E}_L^0(A))$ is also a neutral element of the cohomology set $H^1(\Omega, \mathcal{A}_L(A))$.*

Proof. Since E is locally convex, the family of convex open coverings of D is cofinal in the family of open coverings of D. We may exclusively discuss a convex open covering $\mathcal{U} := \{U_\imath; \imath \in I\}$ of D and a 1-cocycle $h := \{h_{\imath\jmath}; \imath, \jmath \in I\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{A}_L(A))$, for which there exists a 0-cochain $g := \{g_\imath; \imath \in I\} \in \mathrm{C}^0(\mathcal{U}, \mathcal{E}_L^0(A))$ of the covering $\mathcal{U}$ with values in $\mathcal{E}_L^0(A)$, the coboundary of which is the cochain $h := \{h_{\imath\jmath}; \imath, \jmath \in I\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{A}_L(A))$.

Each g_i is a continuous mapping of U_i in L with values the unit element e on the analytic set A and there holds,

$$h_{ij} = g_j^{-1} g_i \qquad in \qquad U_{ij} := U_i \cap U_j \tag{4}$$

for each $i, j \in I$.

Now, we remember the construction of the locally convex C-linear vector space E equipped with the finite open topology. It is nothing but the space ΣC explained in the previous papers[4,5]:

For any integers m and n with $m < n$, we regard the complex m-space C^m as a subspace

(5)

$$C^m = \{z = (z_1, z_2, \cdots, z_n) \in C^n; z_{m+1} = z_{m+2} = \cdots = z_n = 0\}$$

of the superspace C^n. Let $\pi_{m,n} : C^m \to C^n$ be the canonical injection. We put

$$\Sigma C := \bigcup_{n \geq 1} C^n, \tag{6}$$

$$D^{(n)} := D \cap C^n \tag{7}$$

and, for any $n \geq 1$, denote by $\pi_n : C^n \to \Sigma C$ the canonical injection. We induce the strongest topology in ΣC so as each injection π_n is continuous.

The C-linear space ΣC is the locally convex space E with the finite open topology. Its structural sheaf is denoted by $\mathcal{O}$, which is the sheaf of germs of holomorphic functions over ΣC.

We prove by induction with respect to a positive integer n the proposition P_n which asserts that, for any positive integer $m \leq n$, the restriction of the 1-cocycle h to the covering

$$\mathcal{U}(m) := \{U_i \cap C^m; i \in I\} \tag{8}$$

is the coboundary $h^{(m)}$ of a 0-cochain $\{k_i^{(m)}; i \in I\}$ of the covering $\mathcal{U}(m)$ with values in the sheaf $\mathcal{A}_L(A)$ and that, for any $m < n$, each rear $k^{(m+1)}$ is an extension of the preceding $k^{(m)}$.

Since the convex covering is a Leray covering, we have the isomorphism

$$H^1(\mathcal{U}(n+1), \mathcal{A}_L(A)) \cong H^1(D^{(n+1)}, \mathcal{A}_L(A)). \tag{9}$$

By the assumption of the theorem, the cocycle

$$h^{(n+1)} \in \mathrm{B}^1(\mathcal{U}(n+1), \mathcal{E}_L^0(A)) \tag{10}$$

is a coboundary in the category of $\mathcal{E}_L^0$ and by the finite dimensional results of Kajiwara-Kazama[2] it is a coboundary in the category of $\mathcal{A}_L$ too. In other words, there exists a cochain $g^{(n+1)} := \{g_{ij}^{(n+1)}; i, j \in I\} \in \mathrm{C}^0(\mathcal{U}(n+1), \mathcal{A}_L(A))$ whose coboundary is the cocycle $h^{(n+1)} = \{h_i^{(n+1)}; i \in I\} \in \mathrm{Z}^1(\mathcal{U}(n+1), \mathcal{A}_L(A))$. Hence, we have

$$h^{(n+1)} = g_j^{(n+1)^{-1}} g_i^{(n+1)} \tag{11}$$

in each $U_i \cap U_j \cap C^{n+1}$. Then we have

$$g_i^{(n+1)^{-1}} g_i^n = g_j^{(n+1)^{-1}} g_j^n \tag{12}$$

in each $U_i \cap U_j \cap C^n$. Hence the mapping $g^{(n+1)}$ defined by

$$g^{(n+1)} := g_i^{(n+1)^{-1}} g_i^{(n)} \tag{13}$$

in each $U_i \cap C^n$ is a well defined holomorphic mapping of $D^{(n)}$ in L which takes the value e on the analytic set $A \cap C^n$. Since it is continuously extended to an element of $\mathrm{H}^0(D^{(n+1)}, \mathcal{E}_L^0(A))$ according to the above equation, it is also extended to an element $k^{(n+1)}$ of $\mathrm{H}^0(D^{(n+1)}, \mathcal{A}_L(A))$ by Kajiwara-Kazama[2].

We revise the 0-cochain $g^{(n+1)}$ by putting

$$k_i^{(n+1)} := g_i^{(n+1)} k^{(n+1)} \tag{14}$$

in each $U_i \cap C^{n+1}$ and

$$k^{(n+1)} := \{k_i^{(n+1)}; i \in I\}. \tag{15}$$

Then the coboundary of the revised 0-cochain $k^{(n+1)}$ is the said 1-cocycle $h^{(n+1)}$, what was to be proved.

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