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ATTAINABLE SETS FOR SEMILINEAR EVOLUTION EQUATIONS

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1. Introduction

Let H and V be complex Hilbert spaces such that the imbedding $V \subset H$ is compact. The inner product and norm in H are denoted by $(\cdot,\cdot)$ and $|\cdot|$, and those in V are by $((\cdot,\cdot))$ and $||\cdot||$, respectively. Let $-A_0$ be the operator associated with a bounded sesquilinear form $a(u,v)$ defined in $V \times V$ and satisfying Gårding inequality

$$\mathrm{Re}\, a(u,v) \geq c_0 ||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0$$

for any $u \in V$. It is known that A_0 generates an analytic semigroup in both of H and V^*, where V^* stands for the dual space of V. The object of this paper is to investigate the quality of reachable set of the following semilinear retarded parabolic type equation

$$\frac{d}{dt}x(t) = A_0 x(t) + f(t), \quad t \in (0,T], \tag{1.1}$$

where

$$f(t) = A_1 x(t-h) + \int_{-h}^{0} a(s) A_2 x(t+s)ds + f(t, x(t)) + B_0 u(t). \tag{1.2}$$

Then the initial condition of system (1.1) is given as follows:

$$x(0) = g^0, \quad x(s) = g^1(s), \quad \text{for} \quad s \in [-h, 0]. \tag{1.3}$$

The existence and uniqueness of solution of the above system are proved in [6]. The condition for equivalence between the reachable

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set of the semilinear system and that of its corresponding linear system(i.e., the case where $f(\cdot,\cdot)=0$ in (1.2)) was established in [6, 10]. This paper is dealt with another applicable condition for controller of approximate control problem. Thus the main result in this paper will show that the system(1.1) with some conditions for the operator A_0 satisfies a sufficient condition for approximate controllability obtained in [6].

2. Main results

Let A_0 be the self adjoint operator associated with a sesquilinear form defined on $V\times V$ such that

$$(A_0u,v)=-a(u,v),\quad u,\ v\in V,$$

where $a(\cdot,\cdot)$ is bounded sesquilinear form satisfying Gårding inequality. It is known that A_0 generates an analytic semigroup in both H and V^*. Let us assume that A_i, $i=1,\ 2$, are bounded linear operators from V to V^* and $A_iA_0^{-1}$ are also bounded in H. The real valued function $a(s)$ is assumed to be Hölder continuous in $[-h,0]$ where h is a fixed positive number. The controller B_0 is a bounded linear operator from a subspace U of H to H. Let f be a nonlinear mapping from $\mathcal{R}\times V$ into H. Hence, we assume more general Lipschitz condition: for any $x_1,\ x_2\in V$ there exists a constant $L>0$ such that

$$|f(t,x_1)-f(t,x_2)|\le L||x_1-x_2||, \tag{2.1}$$

$$f(t,0)=0. \tag{2.2}$$

Then as is seen in [6] we can obtain the following result.

PROPOSITION 2.1. *Under the assumptions (2.1) and (2.2), there exists a unique solution of (1.1) and (1.3) such that*

$$x\in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H)$$

for any $g=(g^0,g^1)\in Z\equiv H\times L^2(-h,0;V)$. Moreover, there exists a constant C such that

$$||x||_{L^2(0,T,V)\cap W^{1,2}(0,T,V^*)}\le C(|g^0|+||g^1||_{L^2(-h,0,V)}+||u||_{L^2(0,T,U)}),$$

where

$$\|\cdot\|_{L^2(0,T,V)\cap W^{1,2}(0,T;V^*)} = \max\{\|\cdot\|_{L^2(0,T,V)}, \|\cdot\|_{W^{1,2}(0,T,V^*)}\}.$$

Let $g \in Z$ and $x(T; g, f, u)$ be a solution of the system (1.1) and (1.3) associated with nonlinear term f and control u at time T. We define reachable sets for the system (1.1) and (1.3) as follows:

$$L_T(g) = \{x(T; g, 0, u) : u \in L^2(0, T; U)\},$$
$$R_T(g) = \{x(T; g, f, u) : u \in L^2(0, T; U)\}.$$

In virtue of the Riesz-Schauder theorem, if the imbedding $V \subset H$ is compact then the operator A_0 has discrete spectrum

$$\sigma(A_0) = \{\mu_n : n = 1,\ 2,\ \ldots\}$$

which has no point of accumulation except possibly $\mu = \infty$. Let μ_n be a pole of the resolvent of A_0 of order k_n and P_n the spectral projection associated with μ_n

$$P_n = \frac{1}{2\pi i}\int_{\Gamma_n} (\mu - A_0)^{-1} d\mu,$$

where Γ_n is a small circle centered at μ_n such that it surrounds no point of $\sigma(A_0)$ except μ_n. Then the generalized eigenspace corresponding to μ_n is given by

$$H_n = P_n H = \{P_n u : u \in H\},$$

and we have that from $P_n^2 = P_n$ and $H_n \subset V$ it follows that

$$P_n V = \{P_n u : u \in V\} = H_n.$$

Let us set

$$Q_n = \frac{1}{2\pi i}\int_{\Gamma_n} (\mu - \mu_n)(\mu - A_0)^{-1} d\mu.$$

Then we remark that $\dim H_n < \infty$ and

$$Q_n^i = \frac{1}{2\pi i}\int_{\Gamma_n} (\mu - \mu_n)^i(\mu - A_0)^{-1} d\mu.$$

It is also well known that $Q_n^{k_n} = 0$ (nilpotent) and $(A_0 - \mu_n)P_n = Q_n$.

DEFINITION 2.1. The system of the generalized eigenspaces of A_0 is complete in H if $\mathrm{Cl}\{\mathrm{span}\{H_n : n = 1,\ 2,\ \dots\}\} = H$, where Cl denotes the closure in H.

Let $G(t)$ be an analytic semigroup generated by A_0. We now define the fundamental solution $W(t)$ of (1.1) and (1.3) by

$$W(t) = \begin{cases} x(t;(g^0,0),0,0), & t \geq 0 \\ 0 & t < 0. \end{cases}$$

According to the above definition $W(t)$ is a unique solution of

$$W(t) = G(t) + \int_0^t G(t-s)\{A_1 W(s-h) + \int_{-h}^0 a(\tau) A_2 W(s+\tau) d\tau\} ds$$

for $t \geq 0$ (cf. Nakagiri [5]). We denote the bounded linear operator $\hat{W}$ from $L^2(0,T;H)$ to H by

$$\hat{W} p = \int_0^T W(T-s) p(s) ds$$

for $p \in L^2(0,T;H)$.

DEFINITION 2.2. The system (1.1) and (1.3) is approximately controllable on $[0,T]$ if $\overline{R_T(g)} = H$, that is, for any $\epsilon > 0$ and $x \in H$ there exists a control $u \in L^2(0,T;U)$ such that $|x - W(T)g^0 - \int_{-h}^0 U_T(s) g^1(s) ds - \hat{W} f(\cdot, x_u(\cdot)) - \hat{W} B_0 u| < \epsilon$, where $U_T(s) = W(T - s - h)A_1 + \int_{-h}^s W(T-s-\sigma) a(\sigma) A_2 d\sigma$ and $x_u(\cdot) = x(\cdot; g, f, u)$.

We need the following hypotheses:
(A) The system of the generalized eigenspaces of A_0 is complete.
(B1) For any $\epsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\left| \int_0^t G(t-s) p(s) ds - \int_0^t G(t-s) B_0 u(s) ds \right| < \epsilon, \quad 0 \leq t \leq T.$$

(B2) $B_0 P_n H \subset P_n H$ for $n = 1,\ 2,\ \dots$.

REMARK 1. We know that the condition (B2) is equivalent to the fact that $P_nB_0P_n = B_0P_n$, thus by the definition of Q_n it is also held that if $f \in P_nH$ then $Q_nB_0f = B_0Q_nf$.

PROPOSITION 2.2. *Under the assumption (B1), we have* $\overline{L_T(0)} = H$.

THEOREM 2.1. *Let us assume the hypotheses (A), (B1) and (B2). Then we have* $\overline{R_T(g)} = \overline{L_T(g)}$ *for any* $g \in H \times L^2(-h, 0; V)$.

In virtue of Proposition 2.2 and Theorem 2.1 we have known that the system (1.1) and (1.3) is approximately controllable in conclusion.

REMARK 2 For the semilinear equation without delay terms in case where $A_1 = A_2 = 0$ we may assume the condition (B1) at only time T, that is, we can rewite the condition (B1) as follows.

For any $\epsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$|\int_0^T G(t-s)p(s)ds - \int_0^T G(t-s)B_0u(s)ds| < \epsilon.$$

REMARK 3. In [4] Naito proved Theorem 2.2 under assumptions (B1) and compact operator $G(t)$ and also Zhou in [10] showed it under assumtion (B1) and another condition of range of controller.

3. Proof of main results

First of all, for the meaning of assumption (B1) we need to show the existence of controller satisfying $\mathrm{Cl}\{B_0u : u \in L^2(0,T;U)\} \neq L^2(0,T;H)$. In fact, consider about the controler B_0 defined by

$$B_0u(t) = \sum_{n=1}^{\infty} u_n(t),$$

where

$$u_n = \begin{cases} 0, & 0 \le t \le \frac{T}{n} \\ P_nu(t), & \frac{T}{n} < t \le T. \end{cases}$$

Hence we see that $u_1(t) \equiv 0$ and $u_n(t) \in \operatorname{Im} P_n$. By completion of generalized eigenspaces of A_0 we may write that $f(t) = \sum_{n=1}^{\infty} P_n f(t)$ for $f \in L^2(0,T;H)$. Let us choose $f \in L^2(0,T;H)$ satisfying

$$\int_0^T \|P_1 f(t)\|^2 dt > 0.$$

Then since

$$\int_0^T \|f(t) - B_0 u(t)\|^2 dt = \int_0^T \sum_{n=1}^{\infty} \|P_n(f(t) - B_0 u(t))\|^2 dt$$

$$\geq \int_0^T \|P_1(f(t) - B_0 u(t))\|^2 dt = \int_0^T \|P_1 f(t)\|^2 dt > 0,$$

the statement mentioned above is reasonable.

Proof of Proposition 2.2. Let $x_0 \in D(A_0)$, Then putting $f(s) = (x_0 + sA_0 x_0)/t$ it follows that

$$x_0 = \int_0^t G(t-s) f(s) ds.$$

Thus by the condition (B1) there exists $u \in L^2(0,T;U)$ such that

$$\|x_0 - \int_0^t G(t-s) B_0 u(s) ds\| < \epsilon.$$

Therefore, the density of the domain $D(A_0)$ in H implies approximate controllablity of (1.1) and (1.3), the proof of Proposition 2.2 is complete.

From now on we go to proof of Theorem 2.1. In what follows in this section, let us assume that the system of the generalized eigenspaces of A_0 is complete. Then we will prove that the assumptions (B1) and (B2) are a sufficient condition for the following statement (H) in Theorem 1, 2 as in [6]:

(H) For any $\epsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$|\int_0^t G(t-s) p(s) ds - \int_0^t G(t-s) B_0 u(s) ds| < \epsilon, \quad 0 \leq t \leq T,$$

$$\|B_0 u\|_{L^2(0,T,H)} \leq q \|p\|_{L^2(0,T;H)}$$

where $G(t)$ is an analytic semigroup with infinitesimal generator A_0 and q is a constant independent of p.

If $\mu_n \in \sigma(A_0)$ then we have the Laurent expansion for $R(\mu - A_0) \equiv (\mu - A_0)^{-1}$ at $\mu = \mu_n$ whose principal part (the part consisting of all the negative power of $(\mu - \mu_n)$) is a finite series:

$$R(\mu - A_0) = \frac{P_n}{\mu - \mu_n} + \sum_{i=1}^{k_n-1} \frac{Q_n^i}{(\mu - \mu_n)^{i+1}} + R_0(\mu),$$

where $R_0(\mu)$ is a holomorphic part of $R(\mu - A_0)$ at $\mu = \mu_n$.

Since the system of generalized eigenspaces of A_0 is complete, it holds that for any $\epsilon > 0$

$$|f(s) - \sum_{n=1}^{\infty} P_n f(s)| < \frac{\epsilon}{2M\sqrt{T}} \tag{3.1}$$

for $f \in L^2(0,T;H)$, where M is a constant such that $|G(t)| \le M$ for the sake of simplicity. Here, in what follows we put $u_n = P_n B_0 u$.

Since A_0^{-1} is compact we note that there exists an arc C_n which joints μ_n and some z_0 with $\operatorname{Re} z_0 < \inf\{\operatorname{Re}\mu_n : \mu_n \in \sigma(A_0)\}$ and $C_n - \{\mu_n\} \subset \rho(A_0)$ where $\rho(A_0)$ is the resolvent set of A_0.

LEMMA 3.1. *Let $G(t)$ be the semigroup generated by A_0. Then we give an expression of the semigroup that*

$$G(t)f = e^{\mu_n t} \sum_{i=1}^{k_n-1} \frac{t^i}{i!} Q_n^i f, \quad t \ge 0$$

for any $f \in P_n H$.

Proof. From the well known fact that

$$\begin{aligned} A_0 P_n &= A_0 \frac{1}{2\pi i} \int_{\Gamma_n} (\mu - A_0)^{-1} d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_n} \mu(\mu - A_0)^{-1} d\mu \end{aligned}$$

we have

$$G(t)P_n = \frac{1}{2\pi i}\int_{\Gamma_n} e^{\mu t}(\mu - A_0)^{-1} d\mu.$$

If $f \in P_n H$ then $f = P_n f$ and hence

$$\begin{aligned} G(t)f = G(t)P_n f &= \frac{1}{2\pi i}\int_{\Gamma_n} e^{\mu t}(\mu - A_0)^{-1} f d\mu \\ &= e^{\mu_n t}\frac{1}{2\pi i}\int_{\Gamma_n} e^{(\mu-\mu_n)t}(\mu - A_0)^{-1} f d\mu \\ &= e^{\mu_n t}\{\sum_{i=0}^{\infty}\frac{t^i}{i!}(\frac{1}{2\pi i}\int_{\Gamma_n}(\mu-\mu_n)^i(\mu - A_0)^{-1} f d\mu)\} \\ &= e^{\mu_n t}\sum_{i=0}^{k_n-1}\frac{t^i}{i!}Q_n^i f. \end{aligned}$$

Here, we used the nilpotent property of the operator Q_n in the last equality. The proof of lemma is complete.

REMARK 4. Let $f \in P_n H$. Then in virtue of Lemma 3.1 it holds that $B_0 G(t) f = G(t) B_0 f$ for every $t \geq 0$.

Let $f \in L^2(0, t; H)$. Then by the assumption (B1) for any $\epsilon > 0$ there exists a control $v \in L^2(0, t; U)$ such that

$$(3.2)\quad |\int_0^t G(t-s)f(s)ds - \int_0^t G(t-s)B_0 v(s)ds| < \frac{\epsilon}{2}, \quad 0 \leq t \leq T,$$

and

$$(3.3)\qquad |v(s) - \sum_{n=1}^{\infty} P_n v(s)| < \frac{\epsilon}{2M\|B_0\|\sqrt{T}}.$$

Let us define $h \in H$ by

$$\begin{aligned} h &= \sum_{n=1}^{\infty}\int_0^t G(t-s)P_n v(s)ds \\ &= \sum_{n=1}^{\infty} h_n. \end{aligned}$$

Here, we put $h_n = \int_0^t G(t-s)P_n v(s)ds$. Since $P_n v(s) \in P_n H$, in terms of Lemma 3.1 we have that

$$(3.4) \qquad h_n = \int_0^t G(t-s)P_n v(s)ds$$
$$= \sum_{i=1}^{k_n-1} \int_0^t e^{\mu_n(t-s)} \frac{(t-s)^i}{i!} Q_n^i P_n v(s)ds.$$

Define

$$u(s) = \sum_{n=1}^{\infty} u_n(s), \quad u_n(s) = (\sum_{i=1}^{k_n-1} \frac{t^{i+1}}{(i+1)!})^{-1} e^{-\mu_n(t-s)} Q_n^{k_n-1} h_n.$$

Then $u_n(s) \in P_n H$ and from Remark 1 it follows

$$\int_0^t G(t-s)B_0 u(s)ds = \sum_{n=1}^{\infty} \int_0^t G(t-s)B_0 u_n(s)ds$$
$$= \sum_{n=1}^{\infty} \int_0^t e^{\mu_n(t-s)} \sum_{i=1}^{k_n-1} \frac{(t-s)^i}{i!} Q_n^i B_0 u_n(s)ds$$
$$= B_0 \sum_{n=1}^{\infty} h_n = B_0 h.$$

Thus from (3.2), (3.3) it follows that

$$|\int_0^t G(t-s)B_0 u(s)ds - \int_0^t G(t-s)f(s)ds|$$
$$\le |\int_0^t G(t-s)B_0 u(s)ds - B_0 h| +$$
$$|B_0 h - \int_0^t G(t-s)B_0 v(s)ds| +$$
$$|\int_0^t G(t-s)B_0 v(s)ds - \int_0^t G(t-s)f(s)ds|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

Moreover, by Hölder inequality we also have

$$\begin{aligned}
||B_0u||_{L^2(0,t,H)} &\le \int_0^t |\sum_{n=1}^{\infty} B_0 u_n(s)|^2 ds \\
&\le \int_0^t |\sum_{n=1}^{\infty} B_0 (\sum_{i=1}^{k_n-1} \frac{t^{i+1}}{(i+1)!})^{-1} e^{-\mu_n(t-s)} Q_n^{k_n-1} h_n|^2 ds \\
&\le c \int_0^t |\sum_{n=1}^{\infty} B_0 h_n|^2 ds,
\end{aligned}$$

where c is a constant. From Remark 4 we also note that

$$\begin{aligned}
B_0 h_n &= B_0 \int_0^t G(t-s) P_n v(s) ds \\
&= \int_0^t G(t-s) B_0 P_n v(s) ds,
\end{aligned}$$

and, hence from (3.2) and (3.3) it holds

$$\begin{aligned}
|\sum_{n=1}^{\infty} B_0 h_n| &= |\sum_{n=1}^{\infty} \int_0^t G(t-s) B_0 P_n v(s) ds| \\
&\le |\int_0^t G(t-s) B_0 \sum_{n=1}^{\infty} P_n v(s) ds - \int_0^t G(t-s) B_0 v(s) ds| + \\
&\quad |\int_0^t G(t-s) B_0 v(s) ds - \int_0^t G(t-s) B_0 f(s) ds| + \\
&\quad |\int_0^t G(t-s) f(s) ds| \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} + |\int_0^t G(t-s) f(s) ds| \\
&\le q||f||_{L^2(0,t,H)} + \epsilon,
\end{aligned}$$

where q is a constant. Thus, from the above equality we can conclude that

$$||B_0 u||^2_{L^2(0,t,H)} \le q||f||_{L^2(0,t,H)} + \epsilon.$$

Here, we note the constant q is independent of f. Since ϵ is arbitrary we have proof that the assumption of Theorem 2.1 implies the condition (H). In virtue of Theorem 4.2 of [6] the proof of Theorem 2.1 is complete.

4. Examples of controller

EXAMPLE 1. Define the controller B_0 by

$$B_0 u(t) = \sum_{n=1}^{\infty} u_n(t),$$

where

$$u_n(t) = \begin{cases} 0, & 0 \le t \le \frac{T}{n}, \\ P_n u(t), & \frac{T}{n} \le t \le T. \end{cases}$$

Then as is seen in section 3 we define h, u by

$$h = \sum_{n=1}^{\infty} h_n, \quad u(s) = \sum_{n=1}^{\infty} u_n(s),$$

where $h_n(s)$ is defined by as (3.4),

$$u(s) = \sum_{n=1}^{\infty} u_n(s), \quad u_n(s) = \sum_{i=1}^{k_n-1} (T - \frac{T}{n})^{i+1} i! e^{-\mu_n(T-s)} Q_n^{k_n-i} h_n,$$

respectively. Then $u_n(s) \in P_n H$ and $\int_0^T G(T-s) B_0 u(s) ds = \sum_{n=1}^{\infty} h_n$, and this controller is satisfied the conditions in Thereom 2.1.

EXAMPLE 2. We consider the heat controll system studied by Zhou [10, Example 1] and Naito [5 Example 1]. Let $H = L^2(0,\pi)$ and $A_0 = -d^2/dx^2$ $H = L^2(0,\pi)$ and $A_0 = -d^2/dx^2$ with

$$D(A_0) = \{y \in H : d^2y/dx^2 \in H \text{ and } y(0) = y(\pi) = 0\}.$$

Then $\{e_n = (2/\pi)^{1/2} \sin nx : 0 \le x \le \pi, \ n = 1, \dots\}$ is orthonomal base for H. Define an infinite dimensional space U by

$$U = \{\sum_{n=2}^{\infty} u_n e_n : \sum_{n=2}^{\infty} u_n^2 < \infty\}$$

with norm defined by $||u||_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$. Define a continuous linear operator B_0 from U to H as follows:

$$B_0 u = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n \quad \text{for } u = \sum_{n=2}^{\infty} u_n e_n \in U.$$

It is directly seen that the above controller B_0 satisfies the conditions (B1) and (B2). We can also check breifly by using the assumption (H). In fact, let $f \in L^2(0,T;H)$ and $f = \sum_{n=1}^{\infty} f_n(s) e_n$. Then we choose a function $u \in L^2(0,t;U)$ for $0 \le t \le T$ such that $u_2 = \frac{1}{2} f_1 + f_2$ and $u_n = f_n$ for $n = 2, 3, \ldots$. Hence, choosing a constant in condition (H) such that $q > \frac{7}{2}$, not only the system (1.1) and (1.3) with the operator A_0 mentioned above but also the general semilinear case is approximate controllable.

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