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THE DOUBLE GAMMA FUNCTION

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1. Introduction and definitions

The double Gamma function was defined and studied by Barnes [4, 5, 6] and others in about 1900, not appearing in the tables of the most well-known special functions, but cited in the exercise by Whittaker and Watson [32, p. 264]. Recently this function has been revived in the study of determinants of Laplacians [9, 11, 19, 20, 24, 25, 26, 29, 31]. Shintani [27] also used this function to prove the classical Kronecker limit formula. Its p-adic analytic extension appeared in a formula of Cassou-Noguès [8] for the p-adic L-functions at the point 0. More recently Choi *et al.* [12, 13] showed that the theory of the double Gamma function is turned out to be useful in evaluating some series involving the zeta function, the origin of which can be traced back to an over two century old theorem of C. Goldbach as noted in Srivastava [28]. Before Barnes, these functions had been introduced under a different form by Alexeiewsky [1], Glaisher [18], Hölder [21] and Kinkelin [22].

Barnes [4] defines the double Gamma function $\Gamma_2 = 1/G$ satisfying each of the following properties:

(a) $G(z+1) = \Gamma(z)G(z)$ for $z \in \mathbf{C}$,
(b) $G(1) = 1$,
(c) As $n \to \infty$,

$$\begin{aligned}\log G(z+n+2) = {} & \frac{n+1+z}{2}\log 2\pi \\ & + \left[\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right]\log n - \frac{3z^2}{4} - n \\ & - nz - \log A + \frac{1}{12} + O(\frac{1}{n}),\end{aligned}$$

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where Γ is the well-known Gamma function and A is called Glaisher's (or Kinkelin's) constant defined by

$$\log A = \lim_{n\to\infty} \log(1^1 2^2 \cdots n^n) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4}, \tag{1.1}$$

the numerical value of A being $1.282427130\cdots$.

From this definition, Barnes deduced

$$\begin{aligned}\{\Gamma_2(z+1)\}^{-1} &= G(z+1)\\ &= (2\pi)^{\frac{z}{2}} e^{-\frac{1}{2}\left[(1+\gamma)z^2+z\right]} \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}},\end{aligned} \tag{1.2}$$

where γ is the Euler-Mascheroni's constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577215664\ldots. \tag{1.3}$$

We also have two more equivalent forms of G:

$$\begin{aligned}G(z+1) &= (2\pi)^{\frac{z}{2}} e^{-\frac{z(z+1)}{2}-\gamma\frac{z^2}{2}} \prod_{k=1}^{\infty}\frac{\Gamma(k)}{\Gamma(z+k)} e^{z\psi(k)+\frac{z^2}{2}\psi'(k)}\\ &= (2\pi)^{\frac{z}{2}} e^{(\gamma-\frac{1}{2})z-(\frac{\pi^2}{6}+1+\gamma)\frac{z^2}{2}}\Gamma(z)z\\ &\qquad \prod_{m=0}^{\infty}\prod_{n=0}^{\infty}{}' \left(1+\frac{z}{m+n}\right) e^{-\frac{z}{m+n}+\frac{z^2}{2(m+n)^2}},\end{aligned} \tag{1.4}$$

where the accent $'$ denotes the exclusion of the case $n = m = 0$ and ψ is the logarithmic derivative of the Gamma function:

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{1.5}$$

Each form is a product convergent for all finite values of $|z|$, by the Weierstrass factorization theorem [14, p. 170].

We observe that $\Gamma_2(z)^{-1}$ is an entire function with zeros at $z = -k$ whose multiplicity is $k+1$, $k = 0, 1, 2, \cdots$.

For more known properties and formulas see [4].

The object of the present paper is to give another approach for the double Gamma function. Using this approach we can deduce and prove some properties and formulas for this function more easily because, in this approach, the study of the double Gamma function is reduced to that of the generalized zeta function whose many properties and formulas have already been developed and studied.

2. Another form of double gamma function

We can define the Gamma function Γ by using the Bohr-Mollerup theorem (see [3, p. 14); [14, p. 179]). By analogy, we can also give definitions for the double Gamma function and more generally for the n-ple Gamma functions Γ_n for any positive integer n by the following theorem provided by Vignéras [30]:

THEOREM 2.1. *For all positive integers n, there exists a unique meromorphic function $G_n(z)$ such that*

(a) $G_n(1) = 1$,

(b) $G_n(z+1) = G_{n-1}(z)G_n(z)$ *for all* $z \in \mathbf{C}$,

(c) *For* $x \geq 1$, $G_n(x)$ *is infinitely differentiable,* $\frac{d^{n+1}}{dx^{n+1}} \log G_n(x) \geq 0$,

(d) $G_0(x) = x$.

In particular, when $n = 1$, the uniqueness of $\Gamma(z) = G_1(z)$ satisfying the above conditions was shown by the Bohr-Mollerup. Starting with the following function

$$f_1(x) := -\gamma x + \sum_{n=1}^{\infty} \left[\frac{x}{n} - \log\left(1 + \frac{x}{n}\right) \right],$$

and using a result of Dufresnoy and Pisot [15], Vignéras [30] gives the Weierstrass canonical product forms of n-ple Gamma functions Γ_n by a recurrence formula, which is summarized as follows:

THEOREM 2.2. *For all positive integers n, the n-ple Gamma functions Γ_n are given by*

$$\Gamma_n(z) = G_n(z)^{(-1)^{n-1}},$$

where $G_n(z+1) = \exp(f_n(x))$ *and the functions* $f_n(x)$ *satisfy each of the following properties:*

(a) $f_n(x+1) - f_n(x) = f_{n-1}(x)$;

(b) $f_n(0) = 0$;

(c) $\frac{d^n}{dx^n} f_n(x) \geq 0$ *for all* $x \geq 0$;

(d) $f_n(x) = -xA_n(1) + \sum_{h=1}^{n-1} \frac{p_h(x)}{h!} \left[f_{n-1}^{(h)}(0) - A_n^{(h)}(1) \right] + A_n(x)$;

where

$$A_n(x) = \sum_{m \in \mathbf{N}^{n-1} \times \mathbf{N}^*} \left[\frac{1}{n} \left(\frac{x}{L(m)} \right)^n - \frac{1}{n-1} \left(\frac{x}{L(m)} \right)^{n-1} + \cdots + (-1)^{n-1} \frac{x}{L(m)} + (-1)^n \log\left(1 + \frac{x}{L(m)} \right) \right];$$

and $L(m) = m_1 + m_2 + \ldots + m_n$ *if* $m = (m_1, m_2, \cdots, m_n) \in \mathbf{N}^{n-1} \times \mathbf{N}^*$, $\mathbf{N}$ *denotes the set of nonnegative integers and* $\mathbf{N}^* = \mathbf{N} - \{0\}$, $p_h'(x) = B_h(x)$ *is the* h*th Bernoulli polynomial;*

(e)

$$\frac{d^{n+1}}{dx^{n+1}} f_n(x) = n! \sum_{m \in \mathbf{N}^{n-1} \times \mathbf{N}^*} (x + L(m))^{-n-1}$$

is decreasing for $x \geq 0$ *and tends to* 0 *as* $x \uparrow \infty$.

In terms of the Hermite formula for $\zeta(s,a)$ [32, p. 270], the Gamma function Γ is seen to be related to the generalized zeta function as follows:

$$\left\{ \frac{d}{ds} \zeta(s,a) \right\}_{s=0} = \log \Gamma(a) - \frac{1}{2} \log(2\pi) \text{ or } \Gamma(a) = \sqrt{2\pi} e^{\zeta'(0,a)}, \tag{2.1}$$

where $\zeta(s,a) = \sum_{k=0}^{\infty} (a+k)^{-s}$, $a > 0$ is the generalized (or Hurwitz) zeta function which is analytic for $\mathrm{Re}(s) > 1$. Furthermore by the contour integral representation [32, p. 266], $\zeta(s,a)$ can be continued analytically to the whole s-plane except a simple pole at $s = 1$ with its residue 1. $\zeta(s,1) = \sum_{k=1}^{\infty} k^{-s} = \zeta(s)$ is the Riemann zeta function. Note [10] that the formula (2.1) can also be obtained by using the Bohr-Mollerup theorem.

The double Hurwitz zeta function $\zeta_2(s,a)$ is defined by

$$\zeta_2(s,a) = \sum_{k_1,k_2=0}^{\infty} (a+k_1+k_2)^{-s} \tag{2.2}$$

which is analytic for $\text{Re}(s) > 2$ by the Eisenstein's theorem [16, p. 99]. Furthermore $\zeta_2(s,a)$ can be continued analytically to the whole s-plane except simple poles at $s = 1,\ 2$ by the contour integral representation [7]:

$$\zeta_2(s,a) = \frac{i\Gamma(1-s)}{2\pi} \int_C \frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^2} dz, \tag{2.3}$$

where the contour C is the same as in [32, p. 245].

It is easy to reduce $\zeta_2(s,a)$ to a linear combination of $\zeta(s,a)$:

$$\zeta_2(s,a) = \zeta(s-1,a) + (1-a)\zeta(s,a). \tag{2.4}$$

The Kinkelin's constant A was shown to be expressed as a derivative of the Riemann zeta function $\zeta(s)$ by Voros [31]:

$$\log A = \frac{1}{12} - \zeta'(-1). \tag{2.5}$$

Now we deduce a relationship between $\Gamma_2(a)$ and $\zeta_2'(0,a)$ similar to that of the formula (2.1).

THEOREM 2.3. *Let $\zeta_2(s,a)$ be the double Hurwitz zeta function where $a > 0$. Then we have*

$$\Gamma_2(a) = e^{-\frac{1}{12}} A (2\pi)^{\frac{1}{2}-\frac{1}{2}a} e^{\zeta_2'(0,a)} \tag{2.6}$$

where $\zeta_2'(s,a) = \frac{\partial}{\partial s}\zeta_2(s,a)$ and A is the Kinkelin's constant.

Proof. Let $g(a)$ be the right side of Eq. (2.6). We show that $g(a)^{-1}$ satisfies the criteria of Theorem 2.1 when $n = 2$:

It follows from Eq. (2.4) that $\zeta_2'(0,1) = \zeta'(-1)$. Considering Eq. (2.5) we have $e^{\zeta_2'(0,1)} = A^{-1}e^{\frac{1}{12}}$. Therefore we have $g(1) = 1$.

It can easily be verified that

$$\zeta(s,a) = \zeta(s, m+a) + \sum_{n=0}^{m-1} (a+n)^{-s}, \; m = 1,\, 2,\, 3,\, \ldots \tag{2.7}$$

Letting $m = 1$ in this formula, we have $\zeta(s,a) = \zeta(s, 1+a) + a^{-s}$. Then considering Eq. (2.4) we see that $g(a+1)^{-1} = \Gamma(a)g(a)^{-1}$.

We find that for $a > 0$

$$\frac{d^3}{da^3} \log g(a)^{-1} = -\frac{d^3}{da^3}\frac{d}{ds}\zeta_2(s,a)\bigg|_{s=0} = \sum_{k_1,k_2=0}^{\infty} \frac{2}{(a+k_1+k_2)^3} > 0.$$

Also by the analytic continuation of $\zeta_2(s,a)$ we see that $g(a)^{-1} \in C^{\infty}(0,\infty)$. Hence, in view of Theorem 2.1, the result follows from the uniqueness of the function which satisfies the properties of the case $n = 2$.

From Eq. (2.4) and Theorem 2.3, we have the following:

THEOREM 2.4. *Let $\zeta(s,a)$ be the Hurwitz zeta function where $a > 0$. Then we have*

$$\Gamma_2(a) = e^{-\frac{1}{12}} A (2\pi)^{\frac{1}{2} - \frac{1}{2}a} e^{\zeta'(-1,a) + (1-a)\zeta'(0,a)} \tag{2.8}$$

where $\zeta'(s,a) = \frac{\partial}{\partial s}\zeta(s,a)$.

3. Properties and formulas

In this section we first give some properties and formulas for the double Gamma function by using Theorems 2.3 and 2.4. We can express $\log \Gamma_2(a)$ as improper integrals in many ways. In the following theorem we give only two representations for $\log \Gamma_2(a)$:

THEOREM 3.1. *We have, for $a > 0$,*

$$\log\Gamma_2(a) = -\frac{1}{12} + \log A - \frac{a^2}{4} + \left(\frac{1}{2}a^2 - \frac{1}{2}a\right)\log a + (1-a)\log\Gamma(a)$$
$$+ 2\int_0^\infty \left\{\frac{1}{2}\log(a^2+y^2)\cdot(a^2+y^2)^{\frac{1}{2}}\sin\left(\arctan\frac{y}{a}\right)\right.$$
$$\left. + (a^2+y^2)^{\frac{1}{2}}\arctan\frac{y}{a}\cos\left(\arctan\frac{y}{a}\right)\right\}\frac{dy}{e^{2\pi y}-1}.$$
$$\log\Gamma_2(a) = \log A - \frac{a^2}{4} + \left(\frac{a^2}{2} - \frac{a}{2} + \frac{1}{12}\right)\log a + (1-a)\log\Gamma(a)$$
$$- \int_0^\infty \left[\frac{1}{e^y-1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12}\right] y^{-2}e^{-ay}\,dy$$

Proof. From [32, p. 270] we have the Hermite's formula for $\zeta(s,a)$:

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1}$$
$$+ 2\int_0^\infty (a^2+y^2)^{-\frac{1}{2}s}\left\{\sin\left(s\arctan\frac{y}{a}\right)\right\}\frac{dy}{e^{2\pi y}-1}. \tag{3.1}$$

Differentiating the formula (3.1) with respect to s, and then making $s \to -1$, finally using the formula (2.1) and Theorem 2.4, we obtain the first formula. Setting $m = 2$ in the second formula which appears in [23, p. 24] (the condition $\mathrm{Re}(s) > -(2m+1)$ may be corrected as $\mathrm{Re}(s) > -(2m-1)$), we have for $\mathrm{Re}(s) > -3,\ a > 0$,

$$\zeta(s,a) = \frac{sa^{-s-1}}{12} + \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1}$$
$$+ \frac{1}{\Gamma(s)}\int_0^\infty \left[\frac{1}{e^y-1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12}\right] y^{s-1}e^{-ay}\,dy. \tag{3.2}$$

Then using this identity (3 2) we obtain the second formula in a similar way as in getting the first one by considering

$$\frac{d}{ds}\frac{1}{\Gamma(s)}\bigg|_{s=-1} = -1 \qquad \text{and} \qquad \frac{1}{\Gamma(s)}\bigg|_{s=-1} = 0.$$

Glaisher [17, p. 47] expressed the Kinkelin's constant A as an integral:

$$A = 2^{\frac{7}{36}} \pi^{-\frac{1}{6}} \exp\left\{ \frac{1}{3} + \frac{2}{3} \int_0^{\frac{1}{2}} \log \Gamma(1+x)\, dx \right\}. \tag{3.3}$$

Taking $a = 1$ in the formulas of Theorem 3.1, we can express $\log A$ as improper integrals:

COROLLARY 3.2. *We express*

$$\begin{aligned}\log A = \frac{1}{3} - 2 \int_0^\infty &\left\{ \frac{1}{2} \log(1+y^2) \cdot (1+y^2)^{\frac{1}{2}} \sin(\arctan y) \right. \\ &+ \left. (1+y^2)^{\frac{1}{2}} \arctan y \cos(\arctan y) \right\} \frac{dy}{e^{2\pi y} - 1}; \\ \log A = \frac{1}{4} + \int_0^\infty &\left[\frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12} \right] y^{-2} e^{-y}\, dy.\end{aligned}$$

We now obtain an expansion which represents the function $\log \Gamma_2(a)$ asymptotically for large values of a, and which can be used in the calculation of the double Gamma function.

THEOREM 3.3. *We have, for large values of $a > 0$,*

$$\begin{aligned}\log \Gamma_2(a) = \log A &+ \frac{3a^2}{4} - a - \frac{1}{12} - \left[\frac{a^2}{2} - a + \frac{5}{12} \right] \log a \\ &+ \frac{1}{2}(1-a) \log(2\pi) + O\left(\frac{1}{a}\right).\end{aligned}$$

Proof. It follows from Stirling's formula for $\Gamma(a)$ [23, p. 12] that, for large values of $a > 0$,

$$\log \Gamma(a) = \left(a - \frac{1}{2}\right) \log a - a + \frac{1}{2} \log(2\pi) + \frac{1}{12a} + O\left(a^{-3}\right). \tag{3.4}$$

We can find a number $M > 0$ such that

$$\left| \left(\frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12} \right) \frac{1}{y^2} \right| < M$$

for all real y in $(0,\ \infty)$. Thus we see that

$$\left|\int_0^\infty \left[\frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12}\right] y^{-2} e^{-ay} dy\right| < M\frac{1}{a}.$$

It follows from the second equation of Theorem 3.1 that

$$\begin{aligned}\log\Gamma_2(a) = \log A - \frac{a^2}{4} + \left(\frac{a^2}{2} - \frac{a}{2} + \frac{1}{12}\right)\log a \\ + (1-a)\log\Gamma(a) + O\left(\frac{1}{a}\right).\end{aligned} \tag{3.5}$$

Hence the desired result is obtained from combining of Eqs. (3.4) and (3.5).

For any complex x we define the functions $B_l(x)$ by the equation

$$\frac{ze^{xz}}{e^z - 1} = \sum_{l=0}^\infty \frac{B_l(x)}{l!} z^l, \qquad where \quad |z| < 2\pi.$$

The functions $B_l(x)$ are called l-th Bernoulli polynomials and the numbers $B_l(0)$ are called Bernoulli numbers and are denoted by B_l. Thus,

$$\frac{z}{e^z - 1} = \sum_{l=0}^\infty \frac{B_l}{l!} z^l, \qquad where \quad |z| < 2\pi.$$

The Bernoulli polynomials and numbers of order n are defined respectively by, for any complex number x,

$$\frac{z^n e^{xz}}{(e^z - 1)^n} = \sum_{l=0}^\infty B_l^{(n)}(x)\frac{z^l}{l!}, \qquad where \quad |z| < 2\pi;$$

$$\frac{z^n}{(e^z - 1)^n} = \sum_{l=0}^\infty B_l^{(n)}\frac{z^l}{l!}, \qquad where \quad |z| < 2\pi.$$

Note that $B_l^{(1)}(x) = B_l(x),\ B_l^{(1)} = B_l \qquad and\ B_l^{(n)}(0) = B_l^{(n)}$.

Recall the formula (see [2, p. 264, Eq.(17)]): For every integer $m \geq 0$, we have

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}, \tag{3.6}$$

where $B_{m+1}(a)$ are Bernoulli polynomials.

Now we can obtain the similar formula for $\zeta_2(-l, a)$ and $B_l^{(2)}(a)$ as the formula (3.6).

THEOREM 3.4. *For every integer $l \geq 0$, we have*

$$\zeta_2(-l, a) = \frac{(-1)^l}{(l+2)(l+1)} B_{l+2}^{(2)}(2-a).$$

Proof. In Eq. (2.3), we can justify that the function defined by the contour integral

$$I(s, a) = -\frac{1}{2\pi i}\int_C \frac{(-z)^{s-1} e^{-az}}{(1-e^{-z})^2} dz$$

is an entire function of s, where $a > 0$. Then we find that $\zeta_2(s, a) = \Gamma(1-s) I(s, a)$. Taking $s = -l$ in the formula just obtained, where l is a nonnegative integer, we have $\zeta_2(-l, a) = \Gamma(1+l) I(-l, a) = l! I(-l, a)$. We also have

$$\begin{aligned}
I(-l, a) &= -\frac{1}{2\pi i}\int_{C_2} \frac{(-z)^{-l-1} e^{-az}}{(1-e^{-z})^2} dz \\
&= -\mathrm{Res}_{z=0} \frac{(-z)^{-l-1} e^{-az}}{(1-e^{-z})^2} \\
&= (-1)^l \mathrm{Res}_{z=0}\, z^{-l-3} \frac{z^2 e^{(2-a)z}}{(e^z - 1)^2} \\
&= (-1)^l \mathrm{Res}_{z=0}\, z^{-l-3} \sum_{k=0}^{\infty} B_k^{(2)}(2-a) \frac{z^k}{k!} \\
&= (-1)^l \frac{B_{l+2}^{(2)}(2-a)}{(l+2)!},
\end{aligned}$$

from which we have the desired formula.

It can also be easily seen that, for every integer $l \geq 0$,

$$B_l^{(2)}(2-x) = (-1)^l B_l^{(2)}(x). \tag{3.7}$$

From Theorem 3.4 and Eq. (3.7) we have the following:

COROLLARY 3.5. *For every integer $l \geq 0$, we have*

$$\zeta_2(-l,a) = \frac{1}{(l+2)(l+1)} B_{l+2}^{(2)}(a).$$

From the formulas (2.4), (3.6) and Corollary 3.5, we obtain a relationship between $B_k^{(2)}(a)$ and $B_k(a)$ for some special values of k:

COROLLARY 3.6. *For every integer $l \geq 0$, we have*

$$B_{l+2}^{(2)}(a) = (l+2)(a-1)B_{l+1}(a) - (l+1)B_{l+2}(a).$$

Now we will give a formula for $\zeta_2(s,a)$ similar to the Hurwitz formula for $\zeta(s,a)$:

THEOREM 3.7. *Let $s = \sigma + \imath t$ and $0 < a \leq 1$. Then we have, for $\sigma < 0$,*

$$\begin{aligned}
&\zeta_2(s,a) \\
&= \frac{2(1-a)\Gamma(1-s)}{(2\pi)^{1-s}} \\
&\quad \left\{ \sin\left(\frac{1}{2}s\pi\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi an)}{n^{1-s}} + \cos\left(\frac{1}{2}s\pi\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi an)}{n^{1-s}} \right\} \\
&\quad + \frac{2(s-1)\Gamma(1-s)}{(2\pi)^{2-s}} \\
&\quad \left\{ \cos\left(\frac{1}{2}s\pi\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi an)}{n^{2-s}} - \sin\left(\frac{1}{2}s\pi\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi an)}{n^{2-s}} \right\},
\end{aligned}$$

each of these series being convergent.

Proof. Consider

$$I(s,a) = -\frac{1}{2\pi i}\int_{\Delta} \frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^2}dz$$

taken round a contour Δ consisting of a circle of radius $(2N+1)\pi$, (N a positive integer), starting at the point $(2N+1)\pi$ and encirling the origin in the positive direction, $\arg(-z)$ being zero at $z=-(2N+1)\pi$.

In the region between Δ and the contour $(2N\pi+\pi;0+)$, of which the contour C of Eq. (2.3) is the limiting form, $(-z)^{s-1}e^{-az}(1-e^{-z})^{-2}$ is analytic and one-valued except at the poles of order 2; $\pm 2\pi i$, $\pm 4\pi i, \cdots$, $\pm 2N\pi i$. Hence

$$\frac{1}{2\pi i}\int_{\Delta} \frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^2}dz - \frac{1}{2\pi i}\int_{(2N+1)\pi}^{(0+)} \frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^2}dz$$
$$= \sum_{n=1}^{N}(R_n+R_n'),$$

where R_n, R_n' are the residues of the integrand at $2n\pi i$, $-2n\pi i$ respectively, $\int_{(2N+1)\pi}^{(0+)}$ usually denotes that the path of integration starts at '$(2N+1)\pi$' on the real axis, encirles the origin in the positive direction and returns to the starting point, and so the contour in Eq. (2.3) is written as $\int_C = \int_{\infty}^{(0+)}$.

We can readily compute that

$$R_n = (-2n\pi i)^{s-1}e^{-2\pi a n i}\left\{(1-a)+\frac{s-1}{2n\pi i}\right\};$$
$$R_n' = (2n\pi i)^{s-1}e^{2\pi a n i}\left\{(1-a)-\frac{s-1}{2n\pi i}\right\}.$$

Therefore we have

$$R_n+R_n' = (2n\pi)^{s-1}2(1-a)\cos\left(\frac{\pi}{2}(s-1)+2\pi a n\right)$$
$$- (2n\pi)^{s-2}2(s-1)\sin\left(\frac{\pi}{2}(s-1)+2\pi a n\right).$$

Hence

$$-\frac{1}{2\pi i}\int_{(2N+1)\pi}^{(0+)}\frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^2}dz=\frac{2(1-a)\sin\frac{\pi}{2}s}{(2\pi)^{1-s}}\sum_{n=1}^{N}\frac{\cos(2\pi an)}{n^{1-s}}$$
$$+\frac{2(1-a)\cos\frac{\pi}{2}s}{(2\pi)^{1-s}}\sum_{n=1}^{N}\frac{\sin(2\pi an)}{n^{1-s}}+\frac{2(s-1)\cos\frac{\pi}{2}s}{(2\pi)^{2-s}}\sum_{n=1}^{N}\frac{\cos(2\pi an)}{n^{2-s}}$$
$$-\frac{2(s-1)\sin\frac{\pi}{2}s}{(2\pi)^{2-s}}\sum_{n=1}^{N}\frac{\sin(2\pi an)}{n^{2-s}}-\frac{1}{2\pi i}\int_{\Delta}\frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^2}dz.$$

Now, since $0<a\leq 1$, it is easy to see that we can find a number K independent of N such that $\left|e^{-az}(1-e^{-z})^{-2}\right|<K$ when z is on Δ. Hence

$$\left|\frac{1}{2\pi i}\int_{\Delta}\frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^2}dz\right|<\frac{1}{2\pi}K\int_{-\pi}^{\pi}\left|\{(2N+1)\pi\}^{s}e^{si\theta}\right|d\theta$$
$$<K\{(2N+1)\pi\}^{\sigma}e^{\pi|s|}\to 0$$

as $N\to\infty$ if $\sigma<0$.
Making $N\to\infty$, we obtain the desired formula.

Let $\zeta_2(s,1)=\zeta_2(s)$. Taking $a=1$ in Theorem 3.7, considering the formula (2.4) and $\Gamma(z+1)=z\Gamma(z)$, we obtain functional relations for $\zeta_2(s)$ and $\zeta(s)$:

COROLLARY 3.6. *We have*

$$\zeta_2(s)=-2^{s-1}\pi^{s-2}\cos\left(\frac{\pi s}{2}\right)\Gamma(2-s)\zeta_2(3-s);$$
$$\zeta(s)=2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

where the second identity is referred to as the Riemann's functional equation for $\zeta(s)$.

It follows from Corollary 3.5 and Theorem 3.7 that the Bernoulli polynomials of order 2 are expressible in trigonometric series.

COROLLARY 3.7. *For $0 < a \leq 1$ and k any positive integer, we have*

$$B_{2k+1}^{(2)}(a) = \frac{2(-1)^k(2k+1)!}{(2\pi)^{2k}} \left\{ (1-a) \sum_{n=1}^{\infty} \frac{\cos(2\pi a n)}{n^{2k}} + \frac{k}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi a n)}{n^{1+2k}} \right\};$$

$$B_{2k+2}^{(2)}(a) = \frac{2(-1)^k(2k+2)!}{(2\pi)^{1+2k}} \left\{ (1-a) \sum_{n=1}^{\infty} \frac{\sin(2\pi a n)}{n^{1+2k}} + \frac{1+2k}{2\pi} \sum_{n=1}^{\infty} \frac{\cos(2\pi a n)}{n^{2+2k}} \right\}.$$

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