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ZARISKY TOPOLOGY IN GROUPOIDS, SEMIGROUPS AND LATTICES

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1. Introduction

Descriptions of commutative rings in terms of hull-kernel topology can be found in [2, 3, 6]. Kist has studied semigroups by using hull-kernel topology[5]. In this paper, we proved our main result that the space $D(a)$ is compact when a is contained in groupoids or semigroups. The properties of groupoids in terms of the hull-kernel topology are described in [8]. Up to Theorem 1, without proofs, we repeat those of [8]. We can find the fact that the space $D(a)$ is compact when a is contained in commutative rings [2, 7]. And Also, we can find the fact that the space $D(a)$ is compact when a is contained in regular commutative semigroups[5]. In Theorem 5, by the direct calculation, we will prove the fact that the space $D(a)$ is compact when a is contained in commutative semigroups. Using the method of Stone's Boolean Representation Theorem in [7], which is used by Simmon [8], we prove the fact that the space $D(a)$ is compact when a is contained in groupoids or semigroups. We consider the reason of the failure by the direct calculation that the space $D(a)$ is compact when a is contained in non-commutative rings, non-commutative semigroups or non-commutative groupoids. There are the Krull's Separation Lemma [4, 8] in commutative rings and the similar Lemma 1.2 of [5] in commutative semigroups. Hence there are Proposition 1.7 of [2] and Theorem 1.5 of [5]. But there is no similar theorem in the non-commutative cases. The Lemma 4 is the modification of the Krull's Separation Lemma in commutative rings and the coresponding lemma in commutative semigroups.

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2. Main results

Throughout this paper, the symbol G will always denote a groupoid. And elements $0, 1$ mean such that $g0 = 0$ and $g1 = g$, for all g in G under a binary composition. A non empty subset I of a groupoid G is called an ideal of G if $GI \subset I$ and $IG \subset I$. For a set A, we mean A^c as the complement of A. $\tilde{A}$ is meant to be the subgroupoid generated by A, and $< A >$ is the ideal generated by A. The union and the intersection of ideals are ideals. And the intersection of subgroupoids is a subgroupoid. But IJ need not be an ideal when I and J are ideals.

EXAMPLE 1. Let R be the set of real numbers with the operation $*$ such that for any $a, b \in R$, $a * b = a^2b$. Then $*$ is neither associative nor commutative.

EXAMPLE 2. Let R be the set of real numbers with the operation $\circledast$ such that for any $a, b \in R$, $a \circledast b = ab + 1$. Then $\circledast$ is not associative but commutative.

The following result is an easy consequence of Zorn's Lemma and the fact that the union of totally ordered subgroupoids is a subgroupoid.

LEMMA 1. *Let G denote a groupoid. If F is a subgroupoid of G which does not meet the ideal I, then F is contained in a subgroupoid which is maximal with respect to the property of not meeting I.*

A proper ideal P is called prime ; if ab is in P then a or b is in P. A prime ideal P is said to be a minimal prime ideal if G has a zero and there is no prime ideal of G which is properly contained in P. Let $M(G)$ denote the set of all minimal prime ideals in G and $P(G)$ denote the set of all prime ideals in G.

EXAMPLE 3. Let X be the set of $\{0, 1, 2\}$ with the operation $+$ such that for any a, b in $X, a + 0 = 0 + a = 0 = 2 + 2$, otherwise $a + b = 1$. Then $(1 + 2) + 2 = 1, 1 + (2 + 2) = 0$. Hence $+$ is not associative but commutative. And $\{1\}$ is a subgroupoid. Let P be a prime ideal. Since $1 + 2 = 1, 2 + 2 = 0$ and $0 \in P, P$ equals X. Hence X has no prime ideals.

LEMMA 2. *$M(G)$ is not empty if $P(G)$ is not empty.*

LEMMA 3. *Let P be a prime ideal in a groupoid G. If P contains the IJ, then P contains I or J.*

REMARK. I and J of Lemma 3 need not be ideals.

COROLLARY 1. *Let P be a prime ideal in a groupoid G. If P contains the intersection of I and J, then P contains I or J.*

We call the subgroupoid S of G to be saturated ; if gh is in S then g and h are in S. And the saturation of a subgroupoid S which we denote S^-, is the smallest saturated subgroupoid of G containing S. Since the intersection of saturated subgroupoids is saturated, clearly the saturation of a subgroupoid S exists. If 0 is in a saturated subgroupoid S, then $S = G$. Hence if a saturated subgroupoid S is proper, 0 is not in S. Moreover, although $I \cap S$ is empty, $I \cap S^-$ may not be empty by Example 3. But if I is prime, $I \cap S = \emptyset$ implies that $I \cap S^-$ is empty from the following Lemma 4.

LEMMA 4. *A proper subgroupoid S is saturated if and only if S^c is a prime ideal.*

LEMMA 5. *If S is a subgroupoid of G which does not meet the ideal I, then I is contained in an ideal J which is maximal with respect to the property of not meeting S; moreover, if S^- does not meet J then J is prime.*

For any subset L of $P(G)$, $k(L)$ is defined to be the set of all elements in G which are common to all of the ideals in L. And for any subset A of $G, h(A)$ is the set of all P in $P(G)$ such that $A \subset P$. If L is in $P(G)$, then we define a closure operator on $P(G)$ such that the closure of L is $h(k(L))$. The topology so defined on $P(G)$ is called the hull kernel topology. If g is in G, then let, for any subset A of G, $P(A)$=$D(A)$=$X(A)$=$\{ P \in P(G)$: A is not contained in P $\}$. Similarly, $M(g)$ is defined. For the following Lemma 6, it is easy to see that the collection $\{X(g) : g \in G\}$ is a basis for the open sets when $P(G)$ is equipped with the hull kernel topology. Moreover, it is obvious that the hull kernel topology satisfies the T_0 separation axiom. It is clear that $M(G)$ is a T_1 space when equipped with the hull kernel topology. We define $s^n = s^{n-1}s$ inductively. If I is an ideal in G, then the radical of I, denoted by $r(I)$, is defined to be the set of all g in G such that the intersection of $\tilde{g}$ and I is not empty.

LEMMA 6. a). $X(A) \cap X(B) = X(AB) \neq X(A \cap B)$ and $M(A) \cap M(B) = M(AB) = M(A \cap B)$. In particular, $X(a) \cap X(b) = X(ab)$ and $M(a) \cap M(b) = M(ab)$. b). $X(A) \cup X(B) = X(A \cup B)$ and $M(A) \cup M(B) = M(A \cup B)$. c). $X(A) = X(< A >)$ and $M(A) = M(< A >)$.

LEMMA 7. *If I is an ideal in the commutative groupoid G, then $r(I)$ is in $\cap$ $\{P \in P(G)$: P contains $I\}$; moreover if the saturation of $\tilde{s}$ for any s in I^c does not meet I, then* r(I) *equals* $\cap$ $\{P \in P(G)$: *P contains $I\}$.*

LEMMA 8. *For any elements f, g, h in a groupoid $G, M((fg)h) = M(f(gh))$ and $P(fg)h) = P(f(gh))$.*

THEOREM 1. *$\{M(g)$: g is in $G\}$ and $\{P(g)$: g is in $G\}$ are semilattices under the intersection.*

$D(G), E(G)$ denote the semilattices of sets $\{M(g)$: g is in $G\}$ and $\{P(g)$: g is in $G\}$, respectively .

THEOREM 2. *$P(G)$ and $P(E(G))$ are homeomorphic.*

Proof. For $P \in P(G)$, let $f(P)$=$\{$ $U \in E(G)$: U does not contain $P\}$ =$\{P(g) : g \in P\}$. If $P(g)$ is contained in $f(P)$, for any h in G, $P(g) \cap P(h) = P(gh)$ is in $f(P)$. Hence $f(P)$ is an ideal. If $P(g) \cap P(h) = P(gh)$ is in $f(P)$, then $gh \in P$. Hence $g \in P$ or $h \in P$. This means that $P(g) \in f(P)$ or $P(h) \in f(P)$. Thus $f(P)$ is a prime ideal. Conversely if Q' is a prime ideal of $P(E(G))$, we define $l(Q') = \{ g : P(g) \in Q'\}$, which is a prime ideal of G from the following argument. Let $g \in l(Q')$. Then for every $h \in G$, $P(gh) = P(g) \cap P(h) \in Q'$, since Q' is an ideal. Thus, $gh \in l(Q')$. Hence $l(Q')$ is an ideal. If $gh \in l(Q')$, then $P(gh) = P(g) \cap P(h) \in Q'$. Since Q' is prime, $P(g) \in Q'$ or $P(h) \in Q'$. Thus $g \in l(Q')$ or $h \in l(Q')$. Hence $l(Q')$ is a prime ideal. And we easily find $l(f(P)) = P$ and $f(l(Q')) = Q'$. Hence f is a bijection. If we show that $f(P(g)) = P(P(g))$ then $f^{-1}(P(P(g))) = P(g)$. Thus f is a homeomorphism of $P(G)$ into $P(E(G))$. To show that $f(P(g)) \subset P(P(g))$, let $P \in P(g)$. Then g is not in P hence $P(g)$ is not in $f(P)$. Thus $f(P)$ is in $P(P(g))$. Hence we proved that $f(P(g)) \subset P(P(g))$. To show that $f(P(g)) \supset P(P(g))$, let $Q' \in P(P(g))$. Hence $P(g)$ is not in Q', g is not in $f^{-1}(Q')$. Thus $f^{-1}(Q') \in P(g)$. Therefore $f(f^{-1}(Q')) = Q' \in f(P(g))$.

COROLLARY 2. *$M(G)$ and $M(D(G))$ are homeomorphic.*

Let S is a semigroup and L is a collection of $P(I)$ which means the set of prime ideals of S not containing I, when I is an arbitrarily finitely generated ideal of S. We think L as a lattice under the union and the intersection [8]. $P(S)$ and $P(L)$ mean their prime ideals space with Zarisky topology. $M(S)$ and $M(L)$ mean their minimal prime ideals subspace of $P(S)$ and $P(L)$.

THEOREM 3. *$P(S)$ is bijective to $P(L)$.*

Proof. We define a map f from $P(S)$ to $P(L)$ as follows. For any prime ideal P of S, we define $f(P)$ as $\{U \in L$: U does not contain $P\}=\{P(I) \in L{:}I \subset P$ and I is finitely generated ideal of $S.\}$ For any prime ideal Q of L, we define $g(Q)$ as the set $\{a \in S$: $P(a)$ is contained in $Q\}$. If $a \in g(Q)$, then $P(a) \in Q$. For each $b \in S, P(a) \cap P(b) = P(ab) \in Q$. Hence $ab \in g(Q)$. Thus $g(Q)$ is an ideal of S. Suppose that ab is contained in $g(Q)$. Then $P(ab)$ is contained in Q. Since Q is prime, $P(a) \in Q$ or $P(b) \in Q$. Hence $a \in g(Q)$ or $b \in g(Q)$. Thus we have that $g(Q)$ is a prime ideal of L. Let $P(I) \in f(P)$, then $I \in P$. Since for each $P(J) \in L, P(I) \cap P(J) = P(IJ)$ and $IJ \subset P, P(IJ) = P(I) \cap P(J) \in f(P)$. Suppose that $P(I) \in f(P)$ and $P(J) \in f(P)$, then $I \subset P$ and $J \subset P$. Thus $I \cup J \subset P$. Hence $f(I \cup J) \in f(P)$ Thus $f(P)$ is an ideal of the lattice L. Suppose that $P(I) \cap P(J) \in f(P)$. Then $P(IJ) = P(I) \cap P(J)$ is contained in f(P). Hence $IJ \subset P$. Since P is prime, $I \subset P$ or $J \subset P$. Hence $P(I) \in f(P)$ or $P(J) \in f(P)$. Therfore we have that $f(P)$ is a prime ideal of L. Suppose that $P(I) \in f(g(Q))$ when Q is an arbitrary prime ideal of L. Then $I \subset g(Q)$. For $a \in I \subset g(Q)$, $P(a) \in Q$. And $P(I) = P(a_1) \cup ... \cup P(a_n)$ when $< \{a_1, ..., a_n\} >= I$. Hence $P(I) \in Q$. Thus we proved that $f(g(Q)) \subset Q$. Suppose that $P(I) \in Q$ and $< \{a_1, ..., a_n\} >= I$. Since $P(a_i) = P(a_i) \cap P(I), P(a_i) \in Q$. Hence $a_i \in g(Q)$. Then $P(ai) \in f(g(Q))$. Thus $P(I) = P(a_1) \cup ... \cup P(a_n) \in f(g(Q))$. Hence we proved that $f(g(Q)) \subset Q$ and $f(g(Q)) = Q$. Let $a \in g(f(P))$, then $P(a) \in f(P)$ and $a \in P$. Hence $g(f(P)) \subset P$. Let $a \in P$,then $P(a) \in f(P)$. Hence $a \in g(f(P))$ and $g(f(P)) = P$. Of course f has g as inverse map. Hence f is a bijection.

THEOREM 4. *$P(S)$ and $P(L)$ are homeomorphic.*

Proof. For $Q \in f(P(a))$, there exists $P \in P(a)$ such that $Q = f(P)$. Since a is not contained in P, $P(a)$ is not contained in $f(P) = Q$. Hence Q is contained in $P(P(a))$. Thus we proved that $f(P(a)) \subset P(P(a))$. Let $Q \in P(P(a))$. Then $P(a)$ is not contained in Q. Hence a is not contained in $g(Q)$. Thus $g(Q)$ is contained in $P(a)$. Hence $Q = f(g(Q))$ is contained in $f(P(a))$. Thus we proved that $P(P(a)) \subset f(P(a))$. Hence we proved that $f(P(a)) = P(P(a))$. Then it says that f is a homeomorphism.

COROLLARY 3. *([3], Theorem 4.4) $M(S)$ and $M(L)$ are homeomorphic.*

Proof. Since the map$f$ of Theorem 3 is inclusion preserving, the restriction of f to $M(S)$ is a function into $M(L)$.Since the map g of Theorem 3 is also inclusion preserving, the restriction of g to $M(L)$ is a function into $M(S)$. And they are continuous. Then it says that f is a homeomorphism.

REMARK. When I is an ideal of a commutative semigroup, it is well known that $r(I)$ equals to the intersection of prime ideals which contain I([5], Theorem 1.5). But the above fact is not proved for non commutative semigroups.

LEMMA 9. *Let I and J be ideals of a commutative semigroup S. If $P(I)$ contains $P(J)$ then $r(I)$ contains $r(J)$.*

Proof. If a prime ideal P contains I, then P contains J. So $r(I)$ contains $r(J)$.

THEOREM 5. *$P(a)$ is a compact space when a is an element of a commutative semigroup S.*

Proof. Assume $P(a)$ is the union of $P(a_i)$, when a_i is an element of A which is a subset of S. Then $P(a)$ is contained in $P(A)$. Hence a^k is contained in $< A > = \cup\{< a >: a \in A\}$. So a^k is contained in a $< \{a_n\} >$ when a_n in A . Hence $P(a)$ is contained in $P(< a_n >) = P(\{a_n\}) = P(a_n)$. So we have proved that $P(a)$ is a compact space.

REMARK. We do not know that $M(a)$ has this property. Since $\cap$ $\{P \in P(G) : P$ contains $a\}$ is not equal to $\cap$ $\{P \in M(G) : P$ contains $a\}$, we does not have the proof that $M(a)$ is a compact space.

COROLLARY 5. *When a is an element of groupoid G, $P(a)$ is compact.*

Proof. Because of Theorem 5 and 2.

REMARK. When a is an element of groupoid G, we do not know that $M(a)$ has this property because it is not proved that $M(a)$ is a compact space when a is an element of a commutative semigroup S, although $M(G)$ and $M(D(G))$ are homeomorphic (Corollary 2).

COROLLARY 6. *When a is an element of groupoid G or noncommutative semigroup $S, P(a)$ is compact.*

Proof. Because of Theorem 5, 2 and the fact that S can be regarded as a groupoid.

REMARK. When a is an element of a noncommutative ring R, we can not find that $P(a)$ is compact. $\{D(I){:}I \subset P$ and I is finitely generated$\}$ is not an ideal under the union.

THEOREM 6. *The spectrum of a semigroup S has a compactification.*

Proof. By Corollary 5, the spectrum of a semigroup with 1 is compact. But every semigroup without 1 can be made to be a semigroup with 1. Clearly spectrum is not changed during the process only but adding maximal ideal S which is contained only in $P(1)$ among the set $\{P(a) : a \in S \quad \text{or} \quad a = 1\}$. Since $(P(1) - S) \cap P(S)$ is not empty, S is a limit point of $P(S)$.

REMARK. We does not know that $M(S)$ have this property.

COROLLARY 7. *The spectrum of a groupoid has a compactification.*

Proof. It follows from Theorem 2.

REMARK. It is known that every T_0-space has a connected compactification ([5], Theorem 6.2).

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