Pusan Kyŏngnam Math. J. 13(1997), No. 1, pp. 99–108

APPLICATIONS OF RUSCHEWEYH DERIVATIVES

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1. Introduction

Let $\mathcal{A}(n)$ denote the class of the functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n = 1, 2, 3, ...) \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$[8,9].

An univalent function $f(z)$ belonging to $\mathcal{A}(n)$ is said to be starlike of order γ, $0 \leq \gamma < 1$, if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \tag{1.2}$$

for all $z \in U$. We denote this class by $\mathcal{S}^*(n, \gamma)$.

An univalent function $f(z)$ belonging to $\mathcal{A}(n)$ is called convex of order γ, $0 \leq \gamma < 1$, if it satisfies

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \tag{1.3}$$

for all $z \in U$. We denote this class by $\mathcal{C}(n, \gamma)$.

Let $f \in A(n)$ and $g \in S^*(n, \gamma)$, $0 \leq \gamma < 1$. Then we define $f \in \mathcal{K}(n, \beta, \gamma)$ if and only if

$$Re\left(\frac{zf'(z)}{g(z)}\right) > \beta, \tag{1.4}$$

Received Apr. 11, 1997.

The first author was supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1996, Project No. BSRI-96-1411

where $0 \le \beta < 1$ and $0 \le \gamma < 1$. Such functions are called close-to-convex functions of order β type γ.

Let $f \in A(n)$ and $g \in \mathcal{C}(n,\gamma)$, $0 \le \gamma < 1$. Then we define $f \in C^*(n,\beta,\gamma)$ if and only if

$$(1.5) \qquad Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \beta,$$

where $0 \le \beta < 1$.

Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ in $A(n)$. Then the Hadamard product(or convolution) $f * g(z)$ of $f(z)$ and $g(z)$ is defined by

$$(1.6) \qquad f * g(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k \ \ (n = 1, 2, 3, \cdots).$$

By using the Hadamard product, we define, for $\alpha \ge -1$,

$$(1.7) \qquad D^{\alpha} f(z) = \frac{z}{(1-z)^{1+\alpha}} * f(z)$$

for $f \in \mathcal{A}(n)$, $D^{\alpha} f(z)$ is called the Ruscheweyh derivative and was introduced by Ruscheweyh[11].

We easily note that, for $\alpha \ge -1$,

$$(1.8) \qquad D^{\alpha}(zf'(z)) = z(D^{\alpha} f(z))'.$$

2. Main results

In proving our results, we shall need the following lemmas due to Miller and Mocanu [5,6], and Fukui and Sakaguchi [2].

LEMMA 2.1 [5, 6]. *Let $\psi(u,v)$ be a complex function,*

$$\psi : D \longrightarrow C, \ \ D \subset C \times C \ \text{, where } C \text{ is a complex plane,}$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\psi(u,v)$ satisfies the following conditions :

i) *$\psi(u,v)$ is continuous in D,*

ii) *$(1,0) \in D$ and $Re\{\psi(1,0)\} > 0$,*

iii) *$Re\{\psi(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in D$ with $v_1 \le -n(1+u_2^2)/2$.*
Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots\cdots$ be analytic in the unit disk U such that $(p(z), z'p(z)) \in D$ for all $z \in U$. If $Re\{\psi(p(z), zp'(z))\} > 0$ $(z \in U)$, then $Re\{p(z)\} > 0$ $(z \in U)$.

LEMMA 2.2 [2]. *For a real number $\alpha (\alpha > -1)$, we have*

$$z(D^{\alpha} f(z))' = (\alpha+1) D^{\alpha+1} f(z) - \alpha D^{\alpha} f(z).$$

Now we consider the new classes:

$$\mathcal{S}^{*}_{\alpha}(n,\gamma) = \{f \in \mathcal{A}(n) : D^{\alpha} f \in \mathcal{S}^{*}(n,\gamma),\ \ \alpha \geq -1\}.$$

$$\mathcal{C}_{\alpha}(n,\gamma) = \{f \in \mathcal{A}(n) : D^{\alpha} f \in \mathcal{C}(n,\gamma),\ \ \alpha \geq -1\}.$$

$$\mathcal{K}_{\alpha}(n,\beta,\gamma) = \{f \in \mathcal{A}(n) : D^{\alpha} f \in \mathcal{K}(n,\beta,\gamma),\ \ \alpha \geq -1\}.$$

$$\mathcal{C}^{*}_{\alpha}(n,\beta,\gamma) = \{f \in \mathcal{A}(n) : D^{\alpha} f \in \mathcal{C}^{*}(n,\beta,\gamma),\ \ \alpha \geq -1\}.$$

Above all classes is equal to the classes of Noor[7] when $n = 1$, respectively.

We study some properties of these classes and an integral operator for these classes.

Applying the above Lemma 2.1 and Lemma 2.2, we have the following theorem.

THEOREM 2.3. *For $\alpha \geq 0$, we get $\mathcal{S}^{*}_{\alpha+1}(n,\gamma) \subset \mathcal{S}^{*}_{\alpha}(n,\gamma)$.*

Proof. Let us define the function $h(z)$ by

$$\frac{z(D^{\alpha} f(z))'}{D^{\alpha} f(z)} = \gamma + (1-\gamma) h(z), \tag{2.1}$$

where $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots\cdots$ is analytic in U. Hence, from Lemma 2.2, we get

$$\begin{aligned}\frac{D^{\alpha+1} f(z)}{D^{\alpha} f(z)} &= \frac{1}{\alpha+1}\left(\frac{z(D^{\alpha} f(z))'}{D^{\alpha} f(z)} + \alpha\right)\\ &= \frac{1}{\alpha+1}\left((1-\gamma)h(z) + \gamma + \alpha\right)\end{aligned}$$

or

$$D^{\alpha+1} f(z) = \frac{1}{\alpha+1}\left((1-\gamma)h(z) + \gamma + \alpha\right) D^{\alpha} f(z). \tag{2.2}$$

Differentiating both sides of (2.2) logarithmically and multiplying z to both sides of that equation, we have

$$\begin{aligned}\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} &= \frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} + \frac{(1-\gamma)zh'(z)}{(1-\gamma)h(z)+\gamma+\alpha} \\ &= (1-\gamma)h(z)+\gamma+\frac{(1-\gamma)zh'(z)}{(1-\gamma)h(z)+\gamma+\alpha}.\end{aligned}$$

If $f \in S^*_{\alpha+1}(n,\gamma)$, then we have

$$\begin{aligned}&Re\left((1-\gamma)h(z)+\frac{(1-\gamma)zh'(z)}{(1-\gamma)h(z)+\gamma+\alpha}\right) \\ &= Re\left(\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)}-\gamma\right) > 0. \qquad (2.3)\end{aligned}$$

Defining the function $\psi(u,v)$ by

$$\psi(u,v) = (1-\gamma)u + \frac{(1-\gamma)v}{(1-\gamma)u+\gamma+\alpha} \qquad (2.4)$$

where $u = h(z)$ and $v = zh'(z)$, we have

i) $\psi(u,v)$ is continuous in $D = \left(C - \{\frac{\gamma+\alpha}{\gamma-1}\}\right) \times C$,

ii) $(1,0) \in D$ and $Re\psi(1,0) = 1-\gamma > 0$,

iii) for all (iu_2, v_1) such that $v_1 \le -n(1+u_2^2)/2$,

$$\begin{aligned}Re\psi(iu_2,v_1) &= \frac{(1-\gamma)v_1(\gamma+\alpha)}{(\gamma+\alpha)^2+(1-\gamma)^2u_2^2} \\ &\le -\frac{1}{2}\frac{(1-\gamma)n(1+u_2^2)(\gamma+\alpha)}{(\gamma+\alpha)^2+(1-\gamma)^2u_2^2} < 0.\end{aligned}$$

Therefore, the function $\psi(u,v)$ satisfies the conditions in Lemma 2.1. This implies that $Re(h(z)) > 0$ $(z \in U)$, which is equivalent to

$$Re\left(\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}\right) > \gamma \quad (z \in U). \qquad (2.5)$$

Hence $f \in S^*_{\alpha}(n,\gamma)$.

COROLLARY 2.4. *For $\alpha \geq 0$, we get $\mathcal{C}_{\alpha+1}(n,\gamma) \subset \mathcal{C}_{\alpha}(n,\gamma)$.*

Proof. We easily note that

$$f \in \mathcal{C}(n,\gamma) \text{ if and only if } zf' \in \mathcal{S}^*(n,\gamma). \tag{2.6}$$

By Theorem 2.3 and (2.6), we have that

$$D^{\alpha+1}(zf'(z)) = z(D^{\alpha+1}f(z))' \in \mathcal{S}^*(n,\gamma)$$

if $f \in \mathcal{C}_{\alpha+1}(n,\gamma)$. Thus

$$zf' \in \mathcal{S}^*_{\alpha+1}(n,\gamma) \subset \mathcal{S}^*_{\alpha}(n,\gamma).$$

Hence

$$z(D^{\alpha}f(z))' = D^{\alpha}(zf'(z)) \in \mathcal{S}^*(n,\gamma).$$

Therefore $f \in \mathcal{C}_{\alpha}(n,\gamma)$.

THEOREM 2.5. *For $\alpha \geq 0$, we have*

$$\mathcal{K}_{\alpha+1}(n,\beta,\gamma) \subset \mathcal{K}_{\alpha}(n,\beta,\gamma). \tag{2.7}$$

Proof. Assume that $f \in \mathcal{K}_{\alpha+1}(n,\beta,\gamma)$. Then there exists a function $k \in S^*(n,\gamma)$ such that

$$Re\left(\frac{z(D^{\alpha+1}f(z))'}{k(z)}\right) > \beta. \tag{2.8}$$

Letting $k(z) = D^{\alpha+1}g(z)$, we have $g \in \mathcal{S}^*_{\alpha+1}(n,\gamma) \subset \mathcal{S}^*_{\alpha}(n,\gamma)$, by Theorem 2.3. Hence we have

$$Re\left(\frac{z(D^{\alpha}g(z))'}{D^{\alpha}g(z)}\right) > \gamma$$

or

$$\frac{z(D^{\alpha+1}g(z))'}{D^{\alpha}g(z)} = (1-\gamma)h(z) + \gamma, \tag{2.9}$$

where $Re(h(z)) > 0$ $(z \in U)$. Now we set

$$\frac{z(D^\alpha f(z))'}{D^\alpha g(z)} = (1-\beta)p(z) + \beta$$

or

$$z(D^\alpha f(z))' = D^\alpha g(z)\{(1-\beta)p(z)+\beta\}, \tag{2.10}$$

where $p(z) = 1 + p_n z^n + \cdots$. From Lemma 2.2, (1.8) and (2.10), we have

$$\begin{aligned}
\frac{z(D^{\alpha+1} f(z))'}{D^{\alpha+1} g(z)} &= \frac{D^{\alpha+1}(zf'(z))}{D^{\alpha+1} g(z)} \\
&= \frac{\frac{1}{\alpha+1} z(D^\alpha(zf'(z)))' + \frac{\alpha}{\alpha+1} D^\alpha(zf'(z))}{\frac{1}{\alpha+1} z(D^\alpha g(z))' + \frac{\alpha}{\alpha+1} D^\alpha g(z)} \\
&= \frac{\frac{z(D^\alpha(zf'(z)))'}{D^\alpha g(z)} + \alpha \frac{D^\alpha(zf'(z))}{D^\alpha g(z)}}{\frac{z(D^\alpha g(z))'}{D^\alpha g(z)} + \alpha} \\
&= \frac{\frac{z(D^\alpha(zf'(z)))'}{D^\alpha g(z)} + \alpha((1-\beta)p(z)+\beta)}{(1-\gamma)h(z)+\gamma+\alpha}.
\end{aligned} \tag{2.11}$$

Differentiating both sides of (2.10), we have

$$(z(D^\alpha f(z))')' = (1-\beta)p'(z)D^\alpha g(z) + (D^\alpha g(z))'((1-\beta)p(z)+\beta)$$

Hence, from (1.8) we get

$$\begin{aligned}
&\frac{z(D^\alpha(zf'(z)))'}{D^\alpha g(z)} \\
&= (1-\beta)zp'(z) + ((1-\beta)p(z)+\beta)\frac{z(D^\alpha g(z))'}{D^\alpha g(z)} \\
&= (1-\beta)zp'(z) + ((1-\beta)p(z)+\beta)((1-\gamma)h(z)+\gamma)
\end{aligned} \tag{2.12}$$

From (2.11) and(2.12), we have

$$\frac{z(D^{\alpha+1} f(z))'}{D^{\alpha+1} g(z)} = (1-\beta)p(z) + \beta + \frac{(1-\beta)zp'(z)}{(1-\gamma)h(z)+\gamma+\alpha}$$

or

$$\begin{aligned} &Re\left((1-\beta)p(z)+\frac{(1-\beta)zp'(z)}{(1-\gamma)h(z)+\gamma+\alpha}\right) \\ &= Re\left(\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}g(z)}-\beta\right)>0. \end{aligned} \tag{2.13}$$

Define the function $\psi(u,v)$ by

$$\psi(u,v)=(1-\beta)u+\frac{(1-\beta)v}{(1-\gamma)h(z)+\gamma+\alpha}. \tag{2.14}$$

It is clear that the function $\psi(u,v)$ defined in $D=C\times C$ by (2.14) satisfies conditions (i) and (ii) of Lemma 2.1. To verify condition (iii),

$$Re\psi(iu_2,v_1)=\frac{(1-\beta)v_1\{(1-\gamma)h_1+\gamma+\alpha)\}}{\{(1-\gamma)h_1+\gamma+\alpha)\}^2+\{(1-\gamma)h_2\}^2}$$

where $h(z)=h_1+ih_2$ and $Reh(z)=h_1>0$. By putting $v\leq\frac{-n(1+u_2^2)}{2}$,

$$Re\psi(iu_2,v_1)\leq\frac{(1-\beta)n(1+u_2^2)\{(1-\gamma)h_1+\gamma+\alpha\}}{2\{(1-\gamma)h_1+\gamma+\alpha\}^2+\{(1-\gamma)h_2\}^2}<0.$$

Therefore, the function $\psi(u,v)$ satisfies the conditions in Lemma 2.1. This implies that $Rep(z)>0$ $(z\in U)$, which is equivalent to

$$Re\left(\frac{z(D^\alpha f(z))'}{D^\alpha g(z)}\right)>\gamma \quad (z\in U).$$

Hence $f\in\mathcal{K}_\alpha(n,\beta,\gamma)$.

From Theorem 2.5, (1.8) and the definition of $C^*_\alpha(n,\beta,\gamma)$, we have

COROLLARY 2.6. *For $\alpha\geq 0$, we have $C^*_{\alpha+1}(n,\beta,\gamma)\subset C^*_\alpha(n,\beta,\gamma)$*

Proof. We note that

$$f\in C^*(n,\beta,\gamma) \text{ if and only if } zf'\in\mathcal{K}(n,\beta,\gamma). \tag{2.15}$$

By Theorem 2.5 and (2.15), we have that

$$D^{\alpha+1}(zf'(z)) = z(D^{\alpha+1}f(z))' \in \mathcal{K}(n,\beta,\gamma)$$

if $f \in \mathcal{C}^*_{\alpha+1}(n,\beta,\gamma)$. Hence

$$zf' \in \mathcal{K}_{\alpha+1}(n,\beta,\gamma) \subset \mathcal{K}_{\alpha}(n,\beta,\gamma).$$

Therefore

$$z(D^{\alpha}f(z))' = D^{\alpha}(zf'(z)) \in \mathcal{K}(n,\beta,\gamma).$$

It follows that

$$f \in \mathcal{C}^*_{\alpha}(n,\beta,\gamma).$$

Next we define the integral operator $I_{n,c}(f)$ as

$$I_{n,c}(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{2.16}$$

for $f \in A(n)$. The operator $I_{1,c}$ was studied by Noor[7], Owa and Chen[10]. The operator $I_{1,m}$(when m is positive integers) was studied by Bernardi[1] and $I_{1,1}$ was investigated by Libera[3] and Livingston[4].

LEMMA 2.7. *If $f \in A(n)$, then for $\alpha > -1$, $D^{\alpha+1}I_{n,\alpha}(f) = D^{\alpha}f$.*

Proof. By simple calulating of (2.16), we get

$$I_{n,\alpha}(f) = z + \sum_{k=n+1}^{\infty}\frac{\alpha+1}{\alpha+k}a_k z^k.$$

From (1.7), we have

$$\begin{aligned} D^{\alpha+1}I_{n,\alpha}(f) &= z + \sum_{k=n+1}^{\infty}\frac{\prod_{j=1}^{k-1}(j+\alpha+1)}{(k-1)!}\frac{\alpha+1}{\alpha+k}a_k z^k \\ &= z + \sum_{k=n+1}^{\infty}\frac{\prod_{j=1}^{k-1}(j+\alpha)}{(k-1)!}a_k z^k = D^{\alpha}f. \end{aligned}$$

With the aid of Theorem 2.3 and Lemma 2.7, we have

THEOREM 2.8. *If $f \in \mathcal{S}^*_\alpha(n,\gamma)$ with $\alpha \geq 0, 0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $I_{n,\alpha}(f)$ also belongs to $\mathcal{S}^*_\alpha(n,\gamma)$.*

Proof. If $f \in \mathcal{S}^*_\alpha(n,\gamma)$, then $D^\alpha f \in \mathcal{S}^*(n,\gamma)$. By Lemma 2.7, we have

$$D^{\alpha+1} I_{n,\alpha}(f) \in \mathcal{S}^*(n,\gamma).$$

From Theorem 2.3,

$$I_{n,\alpha}(f) \in \mathcal{S}^*_{\alpha+1}(n,\gamma) \subset \mathcal{S}^*_\alpha(n,\gamma).$$

Using Theorem 2.5 and Lemma 2.7, we have the following.

THEOREM 2.9. *If $f \in \mathcal{K}_\alpha(n,\beta,\gamma)$ with $\alpha \geq 0, 0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $I_{n,\alpha}(f)$ also belongs to $\mathcal{K}_\alpha(n,\alpha,\beta)$.*

Finally, we state the similar results for the classes $\mathcal{C}^*_\alpha(n,\gamma)$ and $\mathcal{C}^*_\alpha(n,\beta,\gamma)$ from Theorem 2.8 and Theorem 2.9.

CORLLARY 2.10. *If $f \in \mathcal{C}^*_\alpha(n,\gamma)$ with $\alpha \geq 0, 0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $I_{n,\alpha}(f)$ also belongs to $\mathcal{C}^*_\alpha(n,\gamma)$.*

COROLLARY 2.11. *If $f \in \mathcal{C}^*_\alpha(n,\beta,\gamma)$ with $\alpha \geq 0, 0 \leq \beta < 1$ and $0 \leq \gamma < 1$, then $I_{n,\alpha}(f)$ also belongs to $\mathcal{C}^*_\alpha(n,\alpha,\beta)$.*

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